Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://bmathaa.org
Volume 10 Issue 4(2018), Pages 31-44.

CALDERON-REPRODUCING FORMULA FOR THE CONTINUOUS WAVELET TRANSFORM RELATED TO THE WEINSTEIN OPERATOR

KHALED HLEILI, MANEL HLEILI

Abstract. In this paper, we introduce the continuous wavelet transform associated with the Weinstein operator and we prove for this transform a reproducing inversion formulas of Calderón's type. Next, we study the extremal functions associated to the continuous wavelet transform.

1. Introduction

We consider the Weinstein operator (see[1, 2]) defined on $\mathbb{R}^n\times]0,+\infty[$ by

$$\Delta_W = \sum_{j=1}^{n+1} \frac{\partial^2}{\partial x_j^2} + \frac{2\alpha+1}{x_{n+1}}\frac{\partial}{\partial x_{n+1}} = \Delta_n + \ell_\alpha, \quad \alpha > \frac{-1}{2},$$

where Δ_n is the Laplacian operator in $\mathbb{R}^n$ and ℓ_α the Bessel operator with respect to the variable x_{n+1} defined by

$$\ell_\alpha = \frac{\partial^2}{\partial x_{n+1}^2} + \frac{2\alpha+1}{x_{n+1}}\frac{\partial}{\partial x_{n+1}}, \quad \alpha > \frac{-1}{2}.$$

For $n > 2$, the operator Δ_W is the Laplace-Beltrami operator on the Riemanian space $\mathbb{R}^n\times]0,+\infty[$ equipped with the metric [1]

$$ds^2 = x_{n+1}^{\frac{2(2\alpha+1)}{n-1}} \sum_{i=1}^{n+1} dx_i^2.$$

The Weinstein operator Δ_W has several applications in pure and applied Mathematics especially in Fluid Mechanics (see e.g. [3, 18]). The harmonic analysis associated with the Weinstein operator is studied by Ben Nahia and Ben Salem (see [1, 2]). In particular the authors have introduced and studied the generalized Fourier transform associated with the Weinstein operator. This transform is called the Weinstein transform. Our investigation in the present work consists to study the extremal functions for wavelet transforms associated with the Weinstein operator; this transform was first introduced by Grossmann and Morlet [7] and became

2000 *Mathematics Subject Classification.* 42B10, 42C40.

Key words and phrases. Weinstein operator; wavelet transform; inversion formulas; extremal functions.

Submitted February 6, 2018. Published October 31, 2018.
Communicated by Mircea Ivan.

an active field of research, due to the fact that wavelet analysis has applications in the diverse subjects of communication, seismic data, signal and image processing [4, 5].
We denote by $L^p(d\nu_\alpha), 1 \leqslant p \leqslant +\infty$, the Lebesgue space constituted of measurable functions f on $\mathbb{R}^{n+1}_+$ such that

$$\|f\|_{\nu_\alpha,p} = \left(\int_{\mathbb{R}^{n+1}_+} |f(x)|^p d\nu_\alpha(x)\right)^{\frac{1}{p}} < +\infty, 1 \leqslant p < +\infty,$$

$$\|f\|_{\infty,p} = \text{ess} \sup_{x\in\mathbb{R}^{n+1}_+} |f(x)| < +\infty,$$

where $d\nu_\alpha$ is the measure defined on $\mathbb{R}^{n+1}_+$ by

$$d\nu_\alpha(x) = \frac{x_{n+1}^{2\alpha+1}}{(2\pi)^{\frac{n}{2}} 2^\alpha \Gamma(\alpha+1)} dx.$$

For $f \in L^1(d\nu_\alpha)$ the Weinstein transform is defined by

$$\forall \lambda \in \mathbb{R}^{n+1}_+, \quad \mathscr{F}_W(f)(\lambda) = \int_{\mathbb{R}^{n+1}_+} f(x)\Lambda(-x,\lambda)d\nu_\alpha(x),$$

where $\Lambda(-x,\lambda)$ denotes the Weinstein kernel.
The Weinstein translation operators τ_x, $x = (x', x_{n+1}) \in \mathbb{R}^{n+1}_+$ are defined on $L^2(d\nu_\alpha)$ by

$$\mathscr{F}_W(\tau_x(f))(\lambda) = \Lambda(\lambda, x)\mathscr{F}_W(f)(\lambda), \quad x, \lambda \in \mathbb{R}^{n+1}_+. \tag{1.1}$$

(For more details see the next section).
Let ψ be an admissible wavelet in $L^2(d\nu_\alpha)$. The generalized continuous wavelet transform T_ψ associated with the Weinstein operator is defined for a function f in $L^p(d\nu_\alpha), p = 1, 2$ and for all $(a, x) \in]0, +\infty[\times\mathbb{R}^{n+1}_+$ by

$$T_\psi(f)(a, x) = \int_{\mathbb{R}^{n+1}_+} f(y)\overline{\psi_{a,x}(y)}d\nu_\alpha(y).$$

where

$$\psi_{a,x}(y) = \tau_{-x}(a^{\alpha+\frac{n+2}{2}}\psi(a.))(y), \quad y \in \mathbb{R}^{n+1}_+.$$

We study some of its properties, and we prove reproducing inversion formulas for this transform.
Let σ be a positive function on $\mathbb{R}^{n+1}_+$ satisfying :

$$\sigma(x) \geqslant 1, \quad x \in \mathbb{R}^{n+1}_+ \tag{1.2}$$

and

$$\frac{1}{\sigma} \in L^1(d\nu_\alpha). \tag{1.3}$$

We define the space $\Omega_\sigma(\mathbb{R}^{n+1}_+)$, by

$$\Omega_\sigma(\mathbb{R}^{n+1}_+) = \{f \in L^2(d\nu_\alpha), \sqrt{\sigma}\mathscr{F}_W(f) \in L^2(d\nu_\alpha)\}. \tag{1.4}$$

The space $\Omega_\sigma(\mathbb{R}^{n+1}_+)$ provided with inner product

$$\langle f, g\rangle_\sigma = \int_{\mathbb{R}^{n+1}_+} \sigma(x)\mathscr{F}_W(f)(x)\overline{\mathscr{F}_W(g)(x)}d\nu_\alpha(x), \tag{1.5}$$

and the norm $\|f\|_\sigma = \sqrt{\langle f, f\rangle_\sigma}$, is a Hilbert space.
We Define the measure γ_α on $]0, +\infty[\times\mathbb{R}_+^{n+1}$ by

$$d\gamma_\alpha(a, x) = a^{2\alpha+n+1}da \otimes d\nu_\alpha(x),$$

and $L^p(d\gamma_\alpha), p \in [1, +\infty]$, the Lebesgue space on $]0, +\infty[\times\mathbb{R}_+^{n+1}$ with respect to the measure γ_α with the L^p-norm denoted by $\|.\|_{\gamma_\alpha,p}$.
Building on the ideas of Matsuura et al. [9], Saitoh [11, 13] and Yamada et al. [19], and using the theory of reproducing kernels [10], we give best approximation of the mapping T_ψ on the Hilbert space $\Omega_\sigma(\mathbb{R}_+^{n+1})$. More precisely, for all $\rho > 0, h \in L^2(d\gamma_\alpha)$, the infimum

$$\inf_{f\in\Omega_\sigma}\{\rho\|f\|_\sigma^2 + \|h - T_\psi(f)\|^2_{\gamma_\alpha,2}\},$$

is attained at one function $f^*_{\rho,h}$, called the extremal function and given by

$$f^*_{\rho,h}(y) = \frac{1}{a^{\alpha+\frac{n+2}{2}}}\int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}}\int_{\mathbb{R}_+^{n+1}} h(a, x)\left(\overline{\frac{\mathscr{F}_W(\psi)(\frac{\lambda}{a})\Lambda(-\lambda, y)\Lambda(\lambda, x)}{C_\psi + \rho\sigma(\lambda)}}\right)$$
$$\times\, d\nu_\alpha(\lambda)d\gamma_\alpha(a, x),$$

The extremal functions are studied in several directions [14, 15].
This paper is organized as follows, in the second section we recall some harmonic analysis results related to the Weinstein operator and its associated Fourier transform $\mathscr{F}_W$. In the third section we define and study the continuous wavelet transform T_ψ and we prove an improved version of the so-called Calderón's reproducing formula. The last section of this paper is devoted to give an application of the theory of reproducing kernels to the Tikhonov regularization, which gives the best approximation of the wavelet transform T_ψ.

2. Preliminaries

In order to set up basic and standard notation we briefly overview the Weinstein operator and related harmonic analysis. Main references are [1, 2].

In the following we denote by
- $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times [0, +\infty[$.
- $x = (x_1, ..., x_n, x_{n+1}) = (x', x_{n+1}) \in \mathbb{R}_+^{n+1}$.
- $-x = (-x', x_{n+1})$
- $\mathcal{C}_e(\mathbb{R}^{n+1})$, the space of continuous functions on $\mathbb{R}^{n+1}$, even with respect to the last variable.
- $S_e(\mathbb{R}^{n+1})$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^{n+1}$, even with respect to the last variable.

We consider the Weinstein operator Δ_W defined on $\mathbb{R}_+^{n+1}$ by

$$\Delta_W = \sum_{j=1}^{n+1}\frac{\partial^2}{\partial x_j^2} + \frac{2\alpha+1}{x_{n+1}}\frac{\partial}{\partial x_{n+1}}, \quad \alpha > \frac{-1}{2}.$$

Then

$$\Delta_W = \Delta_n + \ell_\alpha,$$

where Δ_n is the Laplacian operator in $\mathbb{R}^n$ and ℓ_α the Bessel operator with respect to the variable x_{n+1} defined by

$$\ell_\alpha = \frac{\partial^2}{\partial x_{n+1}^2} + \frac{2\alpha+1}{x_{n+1}}\frac{\partial}{\partial x_{n+1}}.$$

The Weinstein kernel Λ is given by

$$\forall (x,\lambda) \in \mathbb{R}^{n+1} \times \mathbb{C}^{n+1}, \quad \Lambda(x,\lambda) = j_\alpha(\lambda_{n+1}x_{n+1})e^{i\langle \lambda', x'\rangle},$$

where j_α is the spherical Bessel function defined by

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\alpha+1+k)}(\frac{z}{2})^{2k},\ z \in \mathbb{C}.$$

The Weinstein kernel satisfies the following properties:
(i) For all $z, t \in \mathbb{C}^{n+1}$, we have

$$\Lambda(z,t) = \Lambda(t,z), \quad \Lambda(z,0) = 1 \quad \text{and} \quad \Lambda(\lambda z, t) = \Lambda(z, \lambda t), \forall \lambda \in \mathbb{C}.$$

(ii)

$$\forall x, y \in \mathbb{R}^{n+1}, \quad |\Lambda(x,y)| \leqslant 1. \tag{2.1}$$

Definition 2.1. The Weinstein transform is given for $f \in L^1(d\nu_\alpha)$ by

$$\forall \lambda \in \mathbb{R}_+^{n+1}, \quad \mathscr{F}_W(f)(\lambda) = \int_{\mathbb{R}_+^{n+1}} f(x)\Lambda(-x,\lambda)d\nu_\alpha(x).$$

Some basic properties of this transform are as follows. For the proofs, we refer [1, 2].

• For every $f \in L^1(d\nu_\alpha)$, the function $\mathscr{F}_W(f)$ is continuous on $\mathbb{R}_+^{n+1}$ and we have

$$\|\mathscr{F}_W(f)\|_{\nu_\alpha,\infty} \leqslant \|f\|_{\nu_\alpha,1}.$$

• Let $f \in L^1(d\nu_\alpha)$ such that $\mathscr{F}_W(f) \in L^1(d\nu_\alpha)$, then for almost every $x \in \mathbb{R}_+^{n+1}$

$$f(x) = \int_{\mathbb{R}_+^{n+1}} \mathscr{F}_W(f)(\lambda)\Lambda(\lambda,x)d\nu_\alpha(\lambda). \tag{2.2}$$

• For $f \in S_e(\mathbb{R}^{n+1})$, if we define

$$\overline{\mathscr{F}_W}(f)(y) = \mathscr{F}_W(f)(-y),$$

then

$$\mathscr{F}_W\overline{\mathscr{F}_W} = \overline{\mathscr{F}_W}\mathscr{F}_W = Id.$$

• For all $f, g \in S_e(\mathbb{R}^{n+1})$, we have

$$\int_{\mathbb{R}_+^{n+1}} f(x)\overline{g(x)}d\nu_\alpha(x) = \int_{\mathbb{R}_+^{n+1}} \mathscr{F}_W(f)(\lambda)\overline{\mathscr{F}_W(g)(\lambda)}d\nu_\alpha(\lambda). \tag{2.3}$$

• The Weinstein transform $\mathscr{F}_W(f)$ is a topological isomorphism from $S_e(\mathbb{R}^{n+1})$ onto itself and for all $f \in S_e(\mathbb{R}^{n+1})$

$$\int_{\mathbb{R}_+^{n+1}} |f(x)|^2 d\nu_\alpha(x) = \int_{\mathbb{R}_+^{n+1}} |\mathscr{F}_W(f)(\lambda)|^2 d\nu_\alpha(\lambda). \tag{2.4}$$

The translation operator τ_x, $x = (x', x_{n+1}) \in \mathbb{R}_+^{n+1}$ associated with the Weinstein operator Δ_W is defined for $f \in \mathcal{C}_e(\mathbb{R}^{n+1})$ which is even with respect to the last variable and for all $y = (y', y_{n+1}) \in \mathbb{R}_+^{n+1}$ by

$$\tau_x(f)(y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^\pi f(x'+y', \sqrt{x_{n+1}^2 + y_{n+1}^2 + 2x_{n+1}y_{n+1}\cos\theta}) \sin^{2\alpha}(\theta) d\theta.$$

In particular for all $x, y \in \mathbb{R}_+^{n+1}$ we have $\tau_x(f)(y) = \tau_y(f)(x)$ and $\tau_0(f) = f$. Moreover for all $L^p(d\nu_\alpha), 1 \leqslant p \leqslant +\infty$, the function $x \longmapsto \tau_x(f)$ belongs to $L^p(d\nu_\alpha)$ and we have

$$\|\tau_x(f)\|_{\nu_\alpha,p} \leqslant \|f\|_{\nu_\alpha,p}. \tag{2.5}$$

By using the generalized translation, we define the generalized convolution product $f *_W g$ of functions $f, g \in L^1(d\nu_\alpha)$ as follows

$$f *_W g(x) = \int_{\mathbb{R}_+^{n+1}} \tau_{-x}(\check{f})(y) g(y) d\nu_\alpha(y), \quad x \in \mathbb{R}_+^{n+1},$$

where $-x = (-x', x_{n+1})$ and $\check{f}(y) = \check{f}(y', y_{n+1}) = f(-y', y_{n+1})$.
This convolution is commutative and associative. Then (see e.g. [1]), if $1 \leqslant p, q, r \leqslant +\infty$ are such $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$, the function $f *_W g$ belongs to $L^r(d\nu_\alpha)$ and we have the following Young's inequality

$$\|f *_W g\|_{\nu_\alpha,r} \leqslant \|f\|_{\nu_\alpha,p} \|g\|_{\nu_\alpha,q}.$$

This then allows us to define $f *_W g$ for $f \in L^p(d\nu_\alpha)$ and $g \in L^q(d\nu_\alpha)$. Moreover for $f \in L^1(d\nu_\alpha)$ and $g \in L^q(d\nu_\alpha), q = 1 \,\text{or}\, 2$, we have

$$\mathscr{F}_W(f *_W g) = \mathscr{F}_W(f)\mathscr{F}_W(g).$$

Moreover, if f and g are in $L^2(d\nu_\alpha)$, then $f *_W g$ belongs to $\mathcal{C}_{e,0}(\mathbb{R}^{n+1})$ consisting of continuous functions h on $\mathbb{R}^{n+1}$, even with respect to the last variable, such that $\lim\limits_{|x| \longrightarrow +\infty} h(x) = 0$ and we have

$$f *_W g = \mathscr{F}_W^{-1}(\mathscr{F}_W(f)\mathscr{F}_W(g)).$$

Thus, for every $f, g \in L^2(d\nu_\alpha)$, the function $f *_W g$ belongs to $L^2(d\nu_\alpha)$ if and only if $\mathscr{F}_W(f)\mathscr{F}_W(g)$ belongs to $L^2(d\nu_\alpha)$ and in this case, we have

$$\mathscr{F}_W(f *_W g) = \mathscr{F}_W(f)\mathscr{F}_W(g).$$

3. Generalized continuous Wavelet transform associated with the Weinstein operator

In this section we recall some results introduced and proved by Gasmi, Ben Mohamed and Bettaibi in [6].

Definition 3.1. Let $\psi \in L^2(d\nu_\alpha)$ be a nonzero function, we say that ψ is an admissible wavelet associated to the Weinstein operator if

$$0 < C_\psi = \int_0^{+\infty} \left|\mathscr{F}_W(\psi)(\frac{\lambda}{a})\right|^2 \frac{da}{a} < +\infty. \tag{3.1}$$

Let a be a nonnegative real number. The dilation operator D_a of a measurable function ψ, is defined by

$$D_a(\psi)(x) = a^{\alpha+\frac{n+2}{2}}\psi(ax), \quad x \in \mathbb{R}_+^{n+1}.$$

Then, we have immediately the following properties:
(1) For all $\psi \in L^2(d\nu_\alpha)$

$$\|D_a(\psi)\|_{\nu_\alpha,2} = \|\psi\|_{\nu_\alpha,2}.$$

(2) For all $x \in \mathbb{R}_+^{n+1}$

$$D_a\tau_x = \tau_{\frac{x}{a}} D_a.$$

(3)

$$\mathscr{F}_W D_a = D_{\frac{1}{a}} \mathscr{F}_W. \tag{3.2}$$

Proposition 3.2. *For all $\psi \in L^p(d\nu_\alpha), 1 \leqslant p \leqslant +\infty$ and $(a, x) \in]0, +\infty[\times\mathbb{R}_+^{n+1}$, the function $\psi_{a,x}$ defined by*

$$\psi_{a,x}(y) = \tau_{-x}(D_a(\psi))(y), \quad y \in \mathbb{R}_+^{n+1}, \tag{3.3}$$

belongs to $L^p(d\nu_\alpha)$ and we have

$$\|\psi_{a,x}\|_{\nu_\alpha,p} \leqslant a^{(\alpha+\frac{n+2}{2})-\frac{2\alpha+n+2}{p}}\|\psi\|_{\nu_\alpha,p}. \tag{3.4}$$

Proof. The case $p = +\infty$ is trivial. Let $1 \leqslant p < +\infty$, from (2.5), we get

$$\begin{aligned}
\|\psi_{a,x}\|^p_{\nu_\alpha,p} &= \|\tau_{-x}(D_a\psi)\|^p_{\nu_\alpha,p} \\
&\leqslant \int_{\mathbb{R}_+^{n+1}} |D_a(\psi)(y)|^p d\nu_\alpha(y) \\
&= a^{p(\alpha+\frac{n+2}{2})-(2\alpha+n+2)}\|\psi\|^p_{\nu_\alpha,p}.
\end{aligned}$$

$\square$

Definition 3.3. Let ψ be an admissible wavelet in $L^2(d\nu_\alpha)$. The generalized continuous wavelet transform T_ψ associated with the Weinstein operator is defined for a function f in $L^p(d\nu_\alpha), p = 1, 2$ and for all $(a, x) \in]0, +\infty[\times\mathbb{R}_+^{n+1}$ by

$$T_\psi(f)(a, x) = \int_{\mathbb{R}_+^{n+1}} f(y)\overline{\psi_{a,x}(y)}d\nu_\alpha(y). \tag{3.5}$$

We have the following expressions of the transform T_ψ
(i) For every $f \in L^p(d\nu_\alpha), p = 1, 2,$

$$T_\psi(f)(a, x) = f * D_a(\overline{\check{\psi}})(x). \tag{3.6}$$

(ii) For every $f \in L^2(d\nu_\alpha)$

$$T_\psi(f)(a, x) = \langle f, \psi_{a,x}\rangle_{\nu_\alpha}. \tag{3.7}$$

Theorem 3.4. *Let ψ be an admissible wavelet in $L^2(d\nu_\alpha)$. We have*
(i) Palncherel's formula for T_ψ: For every $f \in \Omega_\sigma(\mathbb{R}_+^{n+1})$,

$$\int_{\mathbb{R}_+^{n+1}} |f(x)|^2 d\nu_\alpha(x) = \frac{1}{C_\psi}\int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}} |T_\psi(f)(a, x)|^2 d\gamma_\alpha(a, x), \tag{3.8}$$

where $\Omega_\sigma(\mathbb{R}_+^{n+1})$ is the space defined by relation (1.4).

Lemma 3.5. *Let ψ be an admissible wavelet in $L^2(d\nu_\alpha)$. For every $f \in \Omega_\sigma(\mathbb{R}_+^{n+1})$, the function $T_\psi(f)$ belongs to $L^p(d\gamma_\alpha), p \in [2, +\infty]$ and we have*

$$\|T_\psi(f)\|_{\gamma_\alpha,p} \leqslant C_\psi^{\frac{1}{p}} \|\psi\|_{\nu_\alpha,2}^{1-\frac{2}{p}} \|f\|_{\nu_\alpha,2}.$$

Proof. • For $p = 2$, the Plancherel theorem for the continuous wavelet transform (3.8) gives

$$\|T_\psi(f)\|_{\gamma_\alpha,2} = \sqrt{C_\psi} \|f\|_{\nu_\alpha,2}.$$

• For $p = +\infty$, from relations (3.4) and (3.7), we have

$$\begin{aligned} |T_\psi(f)(a,x)| &\leqslant \|\psi_{a,x}\|_{\nu_\alpha,2} \|f\|_{\nu_\alpha,2} \\ &\leqslant \|\psi\|_{\nu_\alpha,2} \|f\|_{\nu_\alpha,2}, \end{aligned}$$

so,

$$\|T_\psi(f)\|_{\gamma_\alpha,\infty} \leqslant \|\psi\|_{\nu_\alpha,2} \|f\|_{\nu_\alpha,2}.$$

We get the result from the Riesz-Thorin theorem [16, 17]. □

Lemma 3.6. *Let ψ be an admissible wavelet in $L^2(d\nu_\alpha)$ such that $\mathscr{F}_W(\psi) \in L^\infty(d\nu_\alpha)$. Then, for every $0 < \varepsilon < \delta < \infty$, the function*

$$\mathcal{K}_{\varepsilon,\delta}(\lambda) = \frac{1}{C_\psi} \int_\varepsilon^\delta |\mathscr{F}_W(\psi)(\frac{\lambda}{a})|^2 \frac{da}{a},$$

belongs to $L^2(d\nu_\alpha)$ and we have

$$\|\mathcal{K}_{\varepsilon,\delta}\|_{\nu_\alpha,2}^2 \leqslant \ln(\frac{\delta}{\varepsilon}) \frac{\delta^{2\alpha+n+2} - \varepsilon^{2\alpha+n+2}}{C_\psi^2(2\alpha+n+2)} \|\psi\|_{\nu_\alpha,2}^2 \|\mathscr{F}_W(\psi)\|_{\nu_\alpha,\infty}^2.$$

Proof. Using Hölder's inequality for the measure $\dfrac{da}{a}$, we get for every $\lambda \in \mathbb{R}_+^{n+1}$

$$|\mathcal{K}_{\varepsilon,\delta}(\lambda)|^2 \leqslant \frac{1}{C_\psi^2} \ln(\frac{\delta}{\varepsilon}) \int_\varepsilon^\delta |\mathscr{F}_W(\psi)(\frac{\lambda}{a})|^4 \frac{da}{a}.$$

By a change of variable $\lambda = \mu a$, we obtain

$$\begin{aligned} \|\mathcal{K}_{\varepsilon,\delta}\|_{\nu_\alpha,2}^2 &\leqslant \frac{1}{C_\psi^2} \ln(\frac{\delta}{\varepsilon}) \int_\varepsilon^\delta \left[\int_{\mathbb{R}_+^{n+1}} |\mathscr{F}_W(\psi)(\frac{\lambda}{a})|^4 d\nu_\alpha(\lambda) \right] \frac{da}{a} \\ &\leqslant \frac{1}{C_\psi^2} \ln(\frac{\delta}{\varepsilon}) \frac{\delta^{2\alpha+n+2} - \varepsilon^{2\alpha+n+2}}{2\alpha+n+2} \|\mathscr{F}_W(\psi)\|_{\nu_\alpha,2}^2 \|\mathscr{F}_W(\psi)\|_{\nu_\alpha,\infty}^2. \end{aligned}$$

Now, relation (2.4) gives the desired result. □

In the following we establish reproducing inversion formula of Calderón's type for the mapping T_ψ.

Theorem 3.7. Calderón's formula. *Let ψ be an admissible wavelet in $L^2(d\nu_\alpha)$ such that $\mathscr{F}_W(\psi) \in L^\infty(d\nu_\alpha)$. Then for every $f \in L^2(d\nu_\alpha)$ and $0 < \varepsilon < \delta < \infty$, the function*

$$f^{\varepsilon,\delta}(s) = \frac{1}{C_\psi} \int_\varepsilon^\delta \int_{\mathbb{R}_+^{n+1}} T_\psi(f)(a,x) \psi_{a,x}(s) d\gamma_\alpha(a,x),$$

belongs to $L^2(d\nu_\alpha)$ and satisfies

$$\lim_{(\varepsilon,\delta) \longrightarrow (0^+,+\infty)} \|f^{\varepsilon,\delta} - f\|_{\nu_\alpha,2} = 0. \tag{3.9}$$

Proof. By (1.1), (2.3), (3.3) and (3.6), we get

$$\begin{aligned}\int_{\mathbb{R}_+^{n+1}} & T_\psi(f)(a,x)\psi_{a,x}(s)d\nu_\alpha(x)\\ &= \int_{\mathbb{R}_+^{n+1}} f *_W D_a(\overline{\check{\psi}})(x)\overline{\tau_{-s}(D_a(\overline{\check{\psi}}))}(x)d\nu_\alpha(x)\\ &= \int_{\mathbb{R}_+^{n+1}} \mathscr{F}_W(f)(\lambda)|\mathscr{F}_W(D_a(\psi))(\lambda)|^2\Lambda(\lambda,s)d\nu_\alpha(\lambda).\end{aligned}$$

Now, using Fubini-Tonnelli's theorem, (2.1), (3.2), Lemma 3.6 and Hölder's inequality, we have

$$\begin{aligned}\frac{1}{C_\psi}\int_\varepsilon^\delta & \left|\int_{\mathbb{R}_+^{n+1}} T_\psi(f)(a,x)\psi_{a,x}(s)d\nu_\alpha(x)\right| a^{2\alpha+n+1}da\\ &\leqslant \int_{\mathbb{R}_+^{n+1}} |\mathscr{F}_W(f)(\lambda)|\left(\frac{1}{C_\psi}\int_\varepsilon^\delta |\mathscr{F}_W(\psi)(\frac{\lambda}{a})|^2\frac{da}{a}\right)d\nu_\alpha(\lambda)\\ &= \int_{\mathbb{R}_+^{n+1}} |\mathscr{F}_W(f)(\lambda)|\mathcal{K}_{\varepsilon,\delta}(\lambda)d\nu_\alpha(\lambda)\\ &\leqslant \frac{\ln(\frac{\delta}{\varepsilon})\sqrt{\delta^{2\alpha+n+2}-\varepsilon^{2\alpha+n+2}}}{C_\psi\sqrt{2\alpha+n+2}}\|f\|_{\nu_\alpha,2}\|\psi\|_{\nu_\alpha,2}\|\mathscr{F}_W(\psi)\|_{\nu_\alpha,\infty}.\end{aligned}$$

Then, from Fubini's theorem and (2.2), we obtain

$$\begin{aligned}f^{\varepsilon,\delta}(s) &= \int_{\mathbb{R}_+^{n+1}} \mathscr{F}_W(f)(\lambda)\mathcal{K}_{\varepsilon,\delta}(\lambda)\Lambda(\lambda,s)d\nu_\alpha(\lambda)\\ &= \mathscr{F}_W^{-1}(\mathscr{F}_W(f)\mathcal{K}_{\varepsilon,\delta})(s).\end{aligned}$$

On the other hand, from relation (3.1), the function $\mathcal{K}_{\varepsilon,\delta}$ belongs to $L^\infty(d\nu_\alpha)$, from this fact and (2.4), the function $f^{\varepsilon,\delta}\in L^2(d\nu_\alpha)$, and we have

$$\mathscr{F}_W(f^{\varepsilon,\delta}) = \mathscr{F}_W(f)\mathcal{K}_{\varepsilon,\delta}.$$

Using the previous result and (2.4), we get

$$\|f^{\varepsilon,\delta}-f\|_{\nu_\alpha,2}^2 = \int_{\mathbb{R}_+^{n+1}} |\mathscr{F}_W(f)(\lambda)|^2(\mathcal{K}_{\varepsilon,\delta}(\lambda)-1)^2d\nu_\alpha(\lambda).$$

The relation (3.9) follows from

$$\lim_{(\varepsilon,\delta)\longrightarrow(0^+,+\infty)}\mathcal{K}_{\varepsilon,\delta}(\lambda)=1,$$

and the dominated convergence theorem. □

4. The extremal function related to the continuous wavelet transform

The main result of this section can be stated as follows.

Proposition 4.1. *Let ψ be an admissible wavelet in $L^2(d\nu_\alpha)$. For every $f\in\Omega_\sigma(\mathbb{R}_+^{n+1})$, the operators T_ψ are bounded linear operators from $\Omega_\sigma(\mathbb{R}_+^{n+1})$ into $L^p(d\gamma_\alpha), p\in[2,+\infty]$, and we have*

$$\|T_\psi(f)\|_{\gamma_\alpha,p}\leqslant C_\psi^{\frac{1}{p}}\|\psi\|_{\nu_\alpha,2}^{1-\frac{2}{p}}\|f\|_\sigma.$$

Proof. Let $f \in \Omega_\sigma(\mathbb{R}_+^{n+1})$. According to Lemma 3.5, the operator $T_\psi(f)$ belongs to $L^p(d\gamma_\alpha), p \in [2, +\infty]$, and

$$\|T_\psi(f)\|_{\gamma_\alpha,p} \leqslant C_\psi^{\frac{1}{p}} \|\psi\|_{\nu_\alpha,2}^{1-\frac{2}{p}} \|f\|_{\nu_\alpha,2}.$$

By (1.2), we have $\|f\|_\sigma^2 \geqslant \int_{\mathbb{R}_+^{n+1}} |\mathscr{F}_W(f)(x)|^2 d\nu_\alpha(x)$, and (2.4) gives the result. □

Definition 4.2. Let $\rho > 0$ and let ψ be an admissible wavelet in $L^2(d\nu_\alpha)$, we denote by $\langle , \rangle_{\sigma,\rho}$ the inner product defined on the space $\Omega_\sigma(\mathbb{R}_+^{n+1})$ by

$$\langle f, g\rangle_{\sigma,\rho} = \int_{\mathbb{R}_+^{n+1}} (C_\psi + \rho\sigma(x))\, \mathscr{F}_W(f)(x)\overline{\mathscr{F}_W(g)(x)} d\nu_\alpha(x), \tag{4.1}$$

and the norm $\|f\|_{\sigma,\rho} = \sqrt{\langle f, f\rangle_{\sigma,\rho}}$.

Proposition 4.3. *Let ψ be an admissible wavelet in $L^2(d\nu_\alpha)$. Then the Hilbert space $(\Omega_\sigma(\mathbb{R}_+^{n+1}), \langle ., .\rangle_{\sigma,\rho})$ has the following reproducing Kernel*

$$K_{\sigma,\rho}(x, y) = \int_{\mathbb{R}_+^{n+1}} \frac{\Lambda(\lambda, x)\Lambda(-\lambda, y)}{C_\psi + \rho\sigma(\lambda)} d\nu_\alpha(\lambda), \tag{4.2}$$

that is
(i) For every $y \in \mathbb{R}_+^{n+1}$, the function $x \longmapsto K_{\sigma,\rho}(x, y)$ belongs to $\Omega_\sigma(\mathbb{R}_+^{n+1})$.
(ii) For every $f \in \Omega_\sigma(\mathbb{R}_+^{n+1})$, and $y \in \mathbb{R}_+^{n+1}$, we have the reproducing property,

$$\langle f, K_{\sigma,\rho}(., y)\rangle_{\sigma,\rho} = f(y).$$

Proof. From relations (2.1), (1.2) and (1.3), the function

$$\Psi_y : \lambda \longmapsto \frac{\Lambda(-\lambda, y)}{C_\psi + \rho\sigma(\lambda)},$$

belongs to $L^1(d\nu_\alpha) \cap L^2(d\nu_\alpha)$. Then, the function $K_{\sigma,\rho}$ is well defined and by (2.2), we have

$$K_{\sigma,\rho}(x, y) = \mathscr{F}_W^{-1}(\Psi_y)(x), \quad x \in \mathbb{R}_+^{n+1}.$$

By (2.4), it follows that the function $K_{\sigma,\rho}(., y)$, belongs to $L^2(d\nu_\alpha)$ and we have

$$\mathscr{F}_W(K_{\sigma,\rho}(., y))(x) = \Psi_y(x), \quad x \in \mathbb{R}_+^{n+1}. \tag{4.3}$$

Then by (2.1), (1.3) and (4.3), we obtain

$$\|K_{\sigma,\rho}(., y)\|_\sigma^2 \leqslant \frac{1}{\rho^2} \|\frac{1}{\sigma}\|_{\nu_\alpha,1}. \tag{4.4}$$

This proves that for every $y \in \mathbb{R}_+^{n+1}$, the function $K_{\sigma,\rho}(., y)$ belongs to $\Omega_\sigma(\mathbb{R}_+^{n+1})$.
(ii) From (4.1) and (4.3), we obtain

$$\langle f, K_{\sigma,\rho}(., y)\rangle_{\sigma,\rho} = \int_{\mathbb{R}_+^{n+1}} \mathscr{F}_W(f)(\lambda)\Lambda(\lambda, y) d\nu_\alpha(\lambda).$$

On the other hand, from relation (1.3) the function $\frac{1}{\sqrt{\sigma}}$ belongs to $L^2(d\nu_\alpha)$. Hence for every $f \in \Omega_\sigma(\mathbb{R}_+^{n+1})$, the function $\mathscr{F}_W(f)$ belongs to $L^1(d\nu_\alpha)$. From this result and (2.2), we obtain

$$\langle f, K_{\sigma,\rho}(., y)\rangle_{\sigma,\rho} = f(y).$$

This completes the proof of the proposition. □

Theorem 4.4. *Let ψ be an admissible wavelet in $L^2(d\nu_\alpha)$ and $a > 0$, for every $h \in L^2(d\gamma_\alpha)$ and for every $\rho > 0$, there exists a unique function $f^*_{\rho,h}$, where the infimum*

$$\inf_{f\in\Omega_\sigma}\{\rho\|f\|^2_\sigma + \|h - T_\psi(f)\|^2_{\gamma_\alpha,2}\}, \tag{4.5}$$

is attained. Moreover the extremal function $f^*_{\rho,h}$ is given by

$$f^*_{\rho,h}(y) = \int_0^{+\infty}\int_{\mathbb{R}^{n+1}_+} h(a,x)\overline{V_{\sigma,\rho}(x,y)}d\gamma_\alpha(a,x),$$

where $V_{\sigma,\rho}(x,y) = \dfrac{1}{a^{\alpha+\frac{n+2}{2}}}\displaystyle\int_{\mathbb{R}^{n+1}_+}\frac{\mathscr{F}_W(\psi)(\frac{\lambda}{a})\Lambda(-\lambda,y)\Lambda(\lambda,x)}{C_\psi + \rho\sigma(\lambda)}d\nu_\alpha(\lambda).$

Proof. The existence and unicity of the extremal function $f^*_{\rho,h}$ satisfying relation (4.5) is given by [8, 9, 12]. On the other hand from Propositions 4.1 and 4.3, we have

$$f^*_{\rho,h}(y) = \langle h, T_\psi(K_{\sigma,\rho}(.,y))\rangle_{\gamma_\alpha}, \tag{4.6}$$

where $\langle,\rangle_{\gamma_\alpha}$ denoted the inner product of $L^2(d\gamma_\alpha)$, and $K_{\sigma,\rho}$ is the kernel given by relation (4.2). From (2.3), (1.1), (3.3), (3.5) and (4.3), we obtain

$$\begin{aligned}T_\psi(K_{\sigma,\rho}(.,y))(a,x) &= \int_{\mathbb{R}^{n+1}_+}\mathscr{F}_W(K_{\sigma,\rho}(.,y))(s)\overline{\mathscr{F}_W(\psi_{a,x})(s)}d\nu_\alpha(s)\\ &= \int_{\mathbb{R}^{n+1}_+}\frac{\Lambda(s,x)\Lambda(-s,y)}{C_\psi + \rho\sigma(s)}\overline{\mathscr{F}_W(D_a(\psi))(s)}d\nu_\alpha(s).\end{aligned}$$

Now, using (3.2), we get

$$T_\psi(K_{\sigma,\rho}(.,y))(a,x) = \frac{1}{a^{\alpha+\frac{n+2}{2}}}\int_{\mathbb{R}^{n+1}_+}\frac{\Lambda(s,x)\Lambda(-s,y)}{C_\psi + \rho\sigma(s)}\overline{\mathscr{F}_W(\psi)(\frac{s}{a})}d\nu_\alpha(s).$$

This clearly yields the result. $\square$

Theorem 4.5. *Let $\rho > 0$ and let ψ be an admissible wavelet in $L^2(d\nu_\alpha)$ and $K_{\sigma,\rho}(.,y) \in L^1(d\nu_\alpha)$. For every $h \in L^2(d\gamma_\alpha)$, we have*

(i)

$$|f^*_{\rho,h}(y)| \leqslant \frac{\sqrt{C_\psi}}{\rho}\|h\|_{\gamma_\alpha,2}\|\frac{1}{\sigma}\|^{\frac{1}{2}}_{\nu_\alpha,1}.$$

(ii)

$$f^*_{\rho,h}(y) = \int_{\mathbb{R}^{n+1}_+}\frac{\Lambda(x,y)}{C_\psi + \rho\sigma(x)} \times\left(\int_0^{+\infty}\mathscr{F}_W(h(a,.))(x)\mathscr{F}_W(\psi)(\frac{x}{a})a^{\alpha+\frac{n}{2}}da\right)d\nu_\alpha(x).$$

(iii)

$$\mathscr{F}_W(f^*_{\rho,h})(x) = \frac{1}{C_\psi + \rho\sigma(x)}\left(\int_0^{+\infty}\mathscr{F}_W(h(a,.))(x)\mathscr{F}_W(\psi)(\frac{x}{a})a^{\alpha+\frac{n}{2}}da\right).$$

(iv) $\|f^*_{\rho,h}\|_\sigma \leqslant \dfrac{\sqrt{C_\psi}}{\rho}\|h\|_{\gamma_\alpha,2}.$

Proof. (i) According to Proposition 4.3, the function $x \longmapsto K_{\sigma,\rho}(x,y)$ belongs to $\Omega_\sigma(\mathbb{R}_+^{n+1})$. Now, using (4.6) and Proposition 4.1, we have

$$\begin{aligned}|f^*_{\rho,h}(y)| &\leqslant \|h\|_{\gamma_\alpha,2}\|T_\psi(K_{\sigma,\rho}(.,y))\|_{\gamma_\alpha,2}\\ &= \sqrt{C_\psi}\|h\|_{\gamma_\alpha,2}\|K_{\sigma,\rho}(.,y)\|_\sigma.\end{aligned}$$

Then, by (4.4), we obtain

$$|f^*_{\rho,h}(y)| \leqslant \frac{\sqrt{C_\psi}}{\rho}\|h\|_{\gamma_\alpha,2}\|\frac{1}{\sigma}\|^{\frac{1}{2}}_{\nu_\alpha,1}.$$

(ii) From relation (4.6), we have

$$f^*_{\rho,h}(y) = \int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}} h(a,x)\overline{T_\psi(K_{\sigma,\rho}(.,y))(a,x)}d\gamma_\alpha(a,x).$$

Since

$$\begin{aligned}&\int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}} \left|h(a,x)\overline{T_\psi(K_{\sigma,\rho}(.,y))(a,x)}\right|d\gamma_\alpha(a,x)\\ &\qquad\leqslant \|h\|_{\gamma_\alpha,2}\|T_\psi(K_{\sigma,\rho}(.,y))\|_{\gamma_\alpha,2}.\end{aligned}$$

On the other hand, the function $x \longmapsto K_{\sigma,\rho}(x,y)$ belongs to $\Omega_\sigma(\mathbb{R}_+^{n+1}) \cap L^1(d\nu_\alpha)$, then by Fubini's theorem, (2.3), (3.2), (3.6) and (4.3), we obtain

$$\begin{aligned}f^*_{\rho,h}(y) &= \int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}} h(a,x)\overline{T_\psi(K_{\sigma,\rho}(.,y))(a,x)}d\gamma_\alpha(a,x)\\ &= \int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}} \mathscr{F}_W(h(a,.))(x)\overline{\mathscr{F}_W(T_\psi(K_{\sigma,\rho}(.,y))(a,.))(x)}d\nu_\alpha(x)a^{2\alpha+n+1}da.\\ &= \int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}} \mathscr{F}_W(h(a,.))(x)\overline{\mathscr{F}_W(K_{\sigma,\rho}(.,y))(x)}\overline{\mathscr{F}_W(D_a(\overline{\check{\psi}}))(x)}d\nu_\alpha(x)a^{2\alpha+n+1}da.\\ &= \int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}} \frac{\Lambda(x,y)\mathscr{F}_W(\psi)(\frac{x}{a})}{a^{\alpha+\frac{n+2}{2}}(C_\psi+\rho\sigma(x))}\mathscr{F}_W(h(a,.))(x)d\nu_\alpha(x)a^{2\alpha+n+1}da.\end{aligned}$$

Using Hölder's inequality, (1.2), (1.3) and (3.1), we have

$$\begin{aligned}&\int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}} \left|\frac{\Lambda(x,y)\mathscr{F}_W(\psi)(\frac{x}{a})}{a^{\alpha+\frac{n+2}{2}}(C_\psi+\rho\sigma(x))}\mathscr{F}_W(h(a,.))(x)\right|d\nu_\alpha(x)a^{2\alpha+n+1}da\\ &\qquad\leqslant \frac{\sqrt{C_\psi}}{\rho}\|h\|_{\gamma_\alpha,2}\|\frac{1}{\sigma}\|^{\frac{1}{2}}_{\nu_\alpha,1},\end{aligned}$$

then, by Fubini's theorem we deduce that

$$\begin{aligned}f^*_{\rho,h}(y) &= \int_{\mathbb{R}_+^{n+1}}\frac{\Lambda(x,y)}{C_\psi+\rho\sigma(x)}\\ &\quad\times\left(\int_0^{+\infty}\mathscr{F}_W(h(a,.))(x)\frac{\mathscr{F}_W(\psi)(\frac{x}{a})}{a^{\alpha+\frac{n+2}{2}}}a^{2\alpha+n+1}da\right)d\nu_\alpha(x).\end{aligned}$$

(iii) The function,
$x \longrightarrow \dfrac{1}{C_\psi + \rho\sigma(x)} \left(\int_0^{+\infty} \mathscr{F}_W(h(a,.))(x) \mathscr{F}_W(\psi)(\frac{x}{a}) a^{\alpha+\frac{n}{2}} da \right)$ belongs to $L^1(d\nu_\alpha) \cap L^2(d\nu_\alpha)$. Then from relation (2.2), we have

$$f^*_{\rho,h}(y) = \mathscr{F}_W^{-1} \left(\frac{1}{C_\psi + \rho\sigma(.)} \left(\int_0^{+\infty} \mathscr{F}_W(h(a,.))(.) \mathscr{F}_W(\psi)(\frac{.}{a}) a^{\alpha+\frac{n}{2}} da \right) \right)(y).$$

By (2.4), it follows that the function $f^*_{\rho,h}$ belongs to $L^2(d\nu_\alpha)$ and

$$\mathscr{F}_W(f^*_{\rho,h})(x) = \frac{1}{C_\psi + \rho\sigma(x)} \left(\int_0^{+\infty} \mathscr{F}_W(h(a,.))(x) \mathscr{F}_W(\psi)(\frac{x}{a}) a^{\alpha+\frac{n}{2}} da \right).$$

(iv) From Hölder's inequality, (1.2) and (3.1), we have

$$\begin{aligned} |\mathscr{F}(f^*_{\rho,h})(x)|^2 &\leqslant \frac{1}{\big(C_\psi + \rho\sigma(x)\big)^2} \left(\int_0^{+\infty} |\mathscr{F}_W(h(a,.))(x)|^2 a^{2\alpha+n+1} da \right) \\ &\leqslant \frac{C_\psi}{\rho^2\sigma(x)} \left(\int_0^{+\infty} |\mathscr{F}_W(h(a,.))(x)|^2 a^{2\alpha+n+1} da \right), \end{aligned}$$

thus, applying (1.5) and (2.4), we obtain $\|f^*_{\rho,h}\|_\sigma \leqslant \dfrac{\sqrt{C_\psi}}{\rho} \|h\|_{\gamma_\alpha,2}$.
This completes the proof of the theorem. □

Corollary 4.6. *Let $\rho > 0$ and let ψ be an admissible wavelet in $L^2(d\nu_\alpha)$. For every $f \in \Omega_\sigma(\mathbb{R}^{n+1}_+) \cap L^1(d\nu_\alpha)$ and $h = T_\psi(f)$, the extremal function $f^*_{\rho,T_\psi(f)}$ satisfies the following properties*
(i) $\mathscr{F}_W(f^*_{\rho,T_\psi(f)})(x) = \dfrac{C_\psi \mathscr{F}_W(f)(x)}{C_\psi + \rho\sigma(x)}$.
(ii) $\|f^*_{\rho,T_\psi(f)}\|_\sigma \leqslant \dfrac{C_\psi}{\rho} \|f\|_\sigma$.

Proof. Part (i) follows directly from (3.1), (3.2), (3.6) and Theorem 4.5 (iii).
Proposition 4.1 and Theorem 4.5 (iv) gives the result of (ii). □

Proposition 4.7. *Let $f \in \Omega_\sigma(\mathbb{R}^{n+1}_+) \cap L^1(d\nu_\alpha)$ and $\rho > 0$. The extremal function $f^*_{\rho,T_\psi(f)}$ satisfies*

$$\lim_{\rho \longrightarrow 0^+} \|f^*_{\rho,T_\psi(f)} - f\|_\sigma = 0.$$

Moreover, $\{f^*_{\rho,T_\psi(f)}\}_{\rho>0}$ converges uniformly to f as $\rho \longrightarrow 0^+$.

Proof. From Corollary 4.6 (i), we have

$$\mathscr{F}_W(f^*_{\rho,T_\psi(f)} - f)(x) = \frac{-\rho\sigma(x)}{C_\psi + \rho\sigma(x)} \mathscr{F}_W(f)(x). \tag{4.7}$$

Consequently,

$$\|f^*_{\rho,T_\psi(f)} - f\|^2_\sigma = \int_{\mathbb{R}^{n+1}_+} \frac{\rho^2\sigma^3(x)}{\big(C_\psi + \rho\sigma(x)\big)^2} |\mathscr{F}_W(f)(x)|^2 d\nu_\alpha(x).$$

Using the dominated convergence theorem and the fact that
$\dfrac{\rho^2\sigma^3(x)}{\big(C_\psi + \rho\sigma(x)\big)^2} |\mathscr{F}_W(f)(x)|^2 \leqslant \sigma(x) |\mathscr{F}_W(f)(x)|^2$,

we deduce that

$$\lim_{\rho \longrightarrow 0^+} \|f^*_{\rho,T_\psi(f)} - f\|_\sigma = 0.$$

On the other hand, by relations (1.2) and (1.3), the function $\mathscr{F}_W(f) \in L^1(d\nu_\alpha) \cap L^2(d\nu_\alpha)$, then from (2.2) and (4.7), we have

$$f^*_{\rho,T_\psi(f)}(y) - f(y) = \int_{\mathbb{R}^{n+1}_+} \frac{-\rho\sigma(x)}{C_\psi + \rho\sigma(x)} \mathscr{F}_W(f)(x)\Lambda(x,y)d\nu_\alpha(x).$$

Again, by dominated convergence theorem and the fact that

$$\frac{\rho\sigma(x)}{C_\psi + \rho\sigma(x)} |\mathscr{F}_W(f)(x)\Lambda(x,y)| \leqslant |\mathscr{F}_W(f)(x)|,$$

we deduce that

$$\lim_{\rho \longrightarrow 0^+} \|f^*_{\rho,T_\psi(f)} - f\|_{\nu_\alpha,\infty} = 0.$$

Which ends the proof. □

Acknowledgments. The authors wish to acknowledge the approval and the support of this research study by the Grant No. SCI-2017-1-7-F-7032 from the Deanship of the Scientific Research in Northern Border University (N.B.U.), Arar, KSA.

References

[1] Z. Ben Nahia and N. Ben Salem, *Spherical harmonics and applications associated with the Weinstein operator*, Potential Theory- Proceedings of the ICPT f **94** (1994) 223–241.

[2] Z. Ben Nahia and N. Ben Salem, *On a mean value property associated with the Weinstein operator*, Potential Theory - Proceedings of the ICPT **94** (1994) 243–253.

[3] M. Brelot, *Equation de Weinstein et potentiels de Marcel Riesz*, Hirsch F, Mokobodzki G, editors. Seminaire de Thorie du Potentiel, No. 3 (Paris, 1976/1977). Lecture Notes in Mathematics, Vol. **681**. Berlin: Springer, (1978) 18–38.

[4] C.K. Chui, *An Introduction of Wavelets, vol.* **38**, Academic Press Professional Inc., San Diego 1992.

[5] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF Regional Conf. Ser. in Appl. Math., vol. **61** 1992. ISBN: 978-0-89871-274-2.

[6] A. Gasmi, H. Ben Mohamed and N. Bettaibi, *Inversion of Weinstein intertwining operator and its dual using Weinstein wavelets*, An. Stiint. Univ. Ovidius Constanta Ser. Mat. **24** , no. 1, (2016) 289–307.

[7] A. Grossmann and J. Morlet, *Decomposition of Hardy functions into square integrable wavelets of constant schape*, SIAM. J. Math. Anal. **15** (1984) 723–736.

[8] G.S. Kimeldorf and G. Wahba, *Some results on Tchebycheffian spline functions*, J. Math. Anal. Appl. **33** (1971) 82–95.

[9] T. Matsuura, S. Saitoh and D.D. Trong, *inversion formulas in heat conduction multidimensional spaces*, J. Inverse III-posed Problems **13** (2005) 479–493.

[10] S. Saitoh, *Hilbert spaces induced by Hilbert space valued functions*, Proc. Amer. Math. Soc. **89** (1983) 74-78.

[11] S. Saitoh, *The Weierstrass transform and an isometry in the heat equation*, Appl. Anal. **16** (1983) 1-6.

[12] S. Saitoh, *Approximate real inversion formulas of the Gaussian convolution*, Appl. Anal. **83** (2004) 727–733.

[13] S. Saitoh, *Best approximation, Tikhonov regularization and reproducing kernels*, Kodai Math. J. **28** (2005) 359-367.

[14] F. Soltani, *Inversion formulas in the Dunkl-type heat conduction on* $\mathbb{R}^d$, Appl. Anal. **84** (2005) 541–553.

[15] F. Soltani, *Best approximation formulas for the Dunkl* L^2*-multiplier operarors on* $\mathbb{R}^d$, Rocky Mountain J. Math **42** (2012) 305–328.

[16] E.M. Stein, *Interpolation of linear operators*, Trans. Am. Math. Soc. **83** (1956) 482–492.

[17] E.M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, Princeton 1971.

[18] A. Weinstein, *Singular partial differential equations and their applications*, Fluid dynamics and applied mathematics. New York: Gordon and Breach, (1962) 29–49.

[19] M. Yamada, T. Matsuura and S. Saitoh, *Representations of inverse functions by the integral transform with the sign kernel*, Frac. Calc. Appl. Anal. **2** (2007) 161–168.

Khaled Hleili
Faculty of Science, Department of Mathematics, Northern Border university, Arar, Saudi Arabia
E-mail address: `khaled.hleili@gmail.com`

Manel Hleili
University of Tunis El Manar, Faculty of sciences of Tunis, Tunis, Tunisia
E-mail address: `Hleili.manel@outlook.com`