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ON I-STATISTICAL AND I-LACUNARY STATISTICAL CONVERGENCE OF WEIGHT g

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Abstract. In this paper, following a very recent and new approach of [1], we further generalize recently introduced summability methods in [6] (where ideals of $\mathbb{N}$ were used to extend certain important summability methods) and introduce new notions, namely, $\mathcal{I}$-statistical convergence of weight g and $\mathcal{I}$-lacunary statistical convergence of weight g, where $g : \mathbb{N} \to [0, \infty)$ is a function satisfying $g(n) \to \infty$ and $n/g(n) \nrightarrow 0$. We mainly investigate their relationship and also make some observations about these classes. Our results extend the recent results of [7].

1. Introduction

The idea of convergence of a real sequence was extended to statistical convergence by Fast [8] (see also Schoenberg [26]) as follows : If $\mathbb{N}$ denotes the set of natural numbers and $K \subset \mathbb{N}$ then $K(m,n)$ denotes the cardinality of the set $K \cap [m,n]$. The upper and lower natural density of the subset K is defined by

$$\overline{d}(K) = \limsup_{n\to\infty} \frac{K(1,n)}{n} \text{ and } \underline{d}(K) = \liminf_{n\to\infty} \frac{K(1,n)}{n}.$$

If $\overline{d}(K) = \underline{d}(K)$ then we say that the natural density of K exists and it is denoted simply by $d(K)$. Clearly $d(K) = \lim_{n\to\infty} \frac{K(1,n)}{n}$.

A sequence (x_n) of real numbers is said to be statistically convergent to L if for arbitrary $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \epsilon\}$ has natural density zero. Statistical convergence turned out to be one of the most active areas of research in summability theory after the works of Fridy [11] and Šalát [21]. For some very interesting investigations concerning statistical convergence one may consult the paper of Moricz [20] where more references on this important summability method

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can be found.

The idea of statistical convergence was further extended to $\mathcal{I}$-convergence in [13] using the notion of ideals of $\mathbb{N}$ with many interesting consequences. More investigations in this direction and more applications of ideals can be found in [5, 6, 15, 16, 24, 25] where many important references can be found.

In another direction, a new type of convergence called lacunary statistical convergence was introduced in [12] as follows. A lacunary sequence is an increasing integer sequence $\theta = (k_r) = (k_0 < k_1 < \dots)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$, as $r \to \infty$. Let $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$.

A sequence (x_n) of real numbers is said to be lacunary statistically convergent to L (or, S_θ-convergent to L) if for any $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0$$

where $|A|$ denotes the cardinality of $A \subset \mathbb{N}$. In [12] the relation between lacunary statistical convergence and statistical convergence was established among other things. More results on this convergence can be seen from [17, 22, 23].

Recently in [6, 24] we used ideals to introduce the concepts of $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence which naturally extend the notions of the above mentioned convergences.

On the other hand in [2, 3] a different direction was given to the study of statistical convergence where the notion of statistical convergence of order α, $0 < \alpha < 1$ was introduced by using the notion of natural density of order α (where n is replaced by n^α in the denominator in the definition of natural density). It was observed in [2] that the behaviour of this new convergence was not exactly parallel to that of statistical convergence and some basic properties were obtained. One can also see [4] for related works.

Very recently in has been shown in [1] that one can further extend the concept of natural or asymptotic density (as well as natural density of order α) by considering natural density of weight g where $g : \mathbb{N} \to [0, \infty)$ is a function with $\lim_{n\to\infty} g(n) = \infty$ and $\frac{n}{g(n)} \nrightarrow 0$.

In a natural way, in this paper we combine the approaches of [6] and [1] and introduce new and more general summability methods, namely, $\mathcal{I}$-statistical convergence of weight g and $\mathcal{I}$-lacunary statistical convergence of weight g. In this context it should be mentioned that the concept of lacunary statistical convergence of weight g (which happens to be a special case of $\mathcal{I}$-lacunary statistical convergence of weight g) has also not been studied till now. We mainly investigate their relationship and also make some observations about these classes. Our results in some way extend the recent results of [7].

2. Basic definitions and facts

The following definitions and notions will be needed in the sequel.

Definition 1. *A non-empty family $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to be an ideal of $\mathbb{N}$ if the following conditions hold:*
(a) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$,
(b) $A \in \mathcal{I},\ B \subset A$ imply $B \in \mathcal{I}$.

Definition 2. *A non-empty family $\mathcal{F} \subset 2^{\mathbb{N}}$ is said to be a filter of $\mathbb{N}$ if the following conditions hold:*
(a) $\phi \notin \mathcal{F}$,
(b) $A, B \in \mathcal{F}$ imply $A \cap B \in \mathcal{F}$,
(c) $A \in \mathcal{F},\ A \subset B$ imply $B \in \mathcal{F}$.

If $\mathcal{I}$ is a proper nontrivial ideal of $\mathbb{N}$ (i.e. $\mathbb{N} \notin \mathcal{I}$), then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \ \exists\ A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$ is a filter of $\mathbb{N}$. It is called the filter associated with the ideal $\mathcal{I}$. A proper ideal $\mathcal{I}$ is said to be admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Throughout $\mathcal{I}$ will stand for a proper admissible ideal of $\mathbb{N}$.

Definition 3. *([13], See also [15])*
(i) A sequence (x_n) of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in \mathcal{I}$.

(ii) A sequence (x_n) of elements of $\mathbb{R}$ is said to be $\mathcal{I}^*$-convergent to $L \in \mathbb{R}$ if there exists $M \in \mathcal{F}(\mathcal{I})$ such that $(x_n)_{n \in M}$ converges to L.

We now present the basis of our main discussions. Let $g : \mathbb{N} \to [0, \infty)$ be a function with $\lim_{n\to\infty} g(n) = \infty$. The upper density of weight g was defined in [1] by the formula
$$\overline{d}_g(A) = \limsup_{n\to\infty} \frac{A(1,n)}{g(n)}$$
for $A \subset \mathbb{N}$ where as before $A(1, n)$ denotes the cardinality of the set $A \cap [1, n]$. Then the family
$$\mathcal{I}_g = \{A \subset \mathbb{N} : \overline{d}_g(A) = 0\}$$
forms an ideal. It has been observed in [1] that $\mathbb{N} \in \mathcal{I}_g$ iff $\frac{n}{g(n)} \to 0$. So we additionally assume that $n/g(n) \nrightarrow 0$ so that $\mathbb{N} \notin \mathcal{I}_g$ and $\mathcal{I}_g$ is a proper admissible ideal of $\mathbb{N}$. The collection of all such weight functions g satisfying the above properties will be denoted by G. As a natural consequence we can introduce the following definition.

Definition 4. *A sequence (x_n) of real numbers is said to converge d_g-statistically to x if for any given $\varepsilon > 0$, $\overline{d}_g(A(\varepsilon)) = 0$ where $A(\varepsilon)$ is the set defined in Definition 3.*

3. Main Results

We first introduce our main definition.

Definition 5. *A sequence $x=(x_n)$ is said to be $\mathcal{I}-$statistically convergent of weight g to L or $\mathrm{S}(\mathcal{I})^g$ convergent to L if for each $\varepsilon>0$ and $\delta>0$*

$$\left\{n\in\mathbb{N}:\frac{1}{g(n)}\left|\left\{k\leq n:|x_k-L|\geq\varepsilon\right\}\right|\geq\delta\right\}\in\mathcal{I}.$$

In this case we write $x_k\to L\left(S(\mathcal{I})^g\right)$. The class of all sequences that are $\mathcal{I}$-statistically convergent of weight g will be denoted simply by $\mathrm{S}(\mathcal{I})^g$.

Remark. *For $\mathcal{I}=\mathcal{I}_{fin}$, $\mathrm{S}(\mathcal{I})^g$ $-$convergence coincidences with statistical convergence of weight g. Further taking $g(n)=n^{\alpha}$, it reduces to $\mathcal{I}$ statistical convergence of order α [7].*

Example 1. *Let us consider the sequence (λ_n) where $\lambda_n=1$ for $n=1$ to 10 and $\lambda_n=n-10$ for $n>10$ and take $\mathcal{I}=\mathcal{I}_d$ (the ideal of density zero sets of $\mathbb{N}$) and let $A=\left\{1^2,2^2,3^2,4^2,...\right\}$. Choose g to be monotonically increasing.*

Define $x=(x_k)$ by

$$x_k=\begin{cases} k & for\ n-[\sqrt{g(\lambda_n)}]+1\leq k\leq n,\ n\notin A\\ k & for\ n-\lambda_n+1\leq k\leq n,\ n\in A\\ 0 & otherwise.\end{cases}$$

Then for every $\varepsilon>0$ $(0<\varepsilon<1)$, as

$$\frac{1}{g(\lambda_n)}\left|\left\{k\in I_n:|x_k|\geq\varepsilon\right\}\right|=\frac{\sqrt{g(\lambda_n)}}{g(\lambda_n)}\to 0$$

as $n\to\infty$ and $n\notin A$ where as usual, $I_n=[n-\lambda_n+1,n]$. So for any $\delta>0$,

$$\left\{n\in\mathbb{N}:\frac{1}{g(\lambda_n)}\left|\left\{k\in I_n:|x_k|\geq\varepsilon\right\}\right|\geq\delta\right\}\subset A\cup\{1,2,...,m_1\}$$

for some $m_1\in\mathbb{N}$. Now fix $\delta>0$. Note that $\lim\limits_{n\to\infty}\frac{n-\lambda_n}{g(n)}=\frac{10}{g(n)}\to 0$ and consequently we can choose $m_2\in\mathbb{N}$ such that $\frac{n-\lambda_n}{g(n)}<\frac{\delta}{2}$ for all $n\geq m_2$. We can now observe that for the above $\varepsilon>0$,

$$\begin{aligned}\frac{1}{g(n)}\left|\left\{k\leq n:|x_k|\geq\varepsilon\right\}\right| &= \frac{1}{g(n)}\left|\left\{k\leq n-\lambda_n:|x_k|\geq\varepsilon\right\}\right|+\frac{1}{g(n)}\left|\left\{k\in I_n:|x_k|\geq\varepsilon\right\}\right|\\ &\leq \frac{n-\lambda_n}{g(n)}+\frac{1}{g(n)}\left|\left\{k\in I_n:|x_k|\geq\varepsilon\right\}\right|\\ &\leq \frac{\delta}{2}+\frac{1}{g(n)}\left|\left\{k\in I_n:|x_k|\geq\varepsilon\right\}\right|\end{aligned}$$

for all $n\geq m_2$. Therefore

$$\begin{aligned}&\left\{n\in\mathbb{N}:\frac{1}{g(n)}\left|\left\{k\leq n:|x_k|\geq\varepsilon\right\}\right|\geq\delta\right\}\\ \subset\ &\left\{n\in\mathbb{N}:\frac{1}{g(\lambda_n)}\left|\left\{k\in I_n:|x_k|\geq\varepsilon\right\}\right|\geq\frac{\delta}{2}\right\}\cup\{1,2,3,...,m_2\}\\ \subset\ &A\cup\{1,2,3,...,m\}\end{aligned}$$

where $m = \max\{m_1, m_2\}$. *Clearly the set on the right hand side belongs to* $\mathcal{I}$ *and consequently the set on the left hand side also belongs to* $\mathcal{I}$. *This shows that* $x = (x_k)$ *is* $\mathcal{I}-$ *statistically convergent of weight* g *to* 0.

We now introduce the other main definition.

Definition 6. *Let* θ *be a lacunary sequence. A sequence* $x = (x_n)$ *is said to be* $\mathcal{I}-$ *lacunary statistically convergent of weight* g *to* L *or* $\mathrm{S}_\theta(\mathcal{I})^g$ *convergent to* L *if for each* $\varepsilon > 0$ *and* $\delta > 0$

$$\left\{r \in \mathbb{N} : \frac{1}{g(h_r)} \left|\{k \in I_r : |x_k - L| \geq \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}.$$

In this case we write $x_k \to L\left(S_\theta(\mathcal{I})^g\right)$. *The class of all* $\mathcal{I}-$ *lacunary sequences of weight* g *will be denoted simply by* $\mathrm{S}_\theta(\mathcal{I})^g$.

Below we examine the inclusion between different classes arising out of different weight functions g.

Theorem 3.1. *Let* $g_1, g_2 \in G$ *be such that there exist* $M > 0$ *and* $j_0 \in \mathbb{N}$ *such that* $\frac{g_1(n)}{g_2(n)} \leq M$ *for all* $n \geq j_0$. *Then* $S(\mathcal{I})^{g_1} \subset S(\mathcal{I})^{g_2}$.

Proof. For any $\varepsilon > 0$,

$$\begin{aligned} \frac{\left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right|}{g_2(n)} &= \frac{g_1(n)}{g_2(n)} \cdot \frac{\left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right|}{g_1(n)} \\ &\leq M \cdot \frac{\left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right|}{g_1(n)}. \end{aligned}$$

for $n \geq j_0$. Hence for any $\delta > 0$,

$$\begin{aligned} &\left\{n \in \mathbb{N} : \frac{\left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right|}{g_2(n)} \geq \delta\right\} \\ \subset\; &\left\{n \in \mathbb{N} : \frac{\left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right|}{g_1(n)} \geq \frac{\delta}{M}\right\} \cup \{1, 2, ..., j_0\}. \end{aligned}$$

If $x = (x_n) \in \mathrm{S}(\mathcal{I})^{g_1}$ then the set on the right hand side belongs to the ideal $\mathcal{I}$ and so the set on the left hand side also belongs to $\mathcal{I}$. This shows that $S(\mathcal{I})^{g_1} \subset S(\mathcal{I})^{g_2}$. □

Similar inclusion relations hold for $\mathrm{S}_\theta(\mathcal{I})^g$ also. It is easy to check that both $\mathrm{S}(\mathcal{I})^g$ and $\mathrm{S}_\theta(\mathcal{I})^g$ are linear subspaces of the space of all real sequences. Next we present a topological characterization of these spaces. As both the proofs are similar, we give the detailed proof for the class $\mathrm{S}_\theta(\mathcal{I})^g$ only.

Theorem 3.2. $\mathrm{S}_\theta(\mathcal{I})^g \cap \ell_\infty$ *is a closed subset of* ℓ_∞ *where* ℓ_∞ *is the Banach space of all bounded real sequences endowed with the sup norm where* $g \in G$ *is such that* $\frac{g(n)}{n} \leq M$ *for all* $n \geq j_0$ *for some* $M > 0$ *and* $j_0 \in \mathbb{N}$.

Proof. Suppose that $(x^n) \subseteq \mathrm{S}_\theta(\mathcal{I})^g \cap \ell_\infty$ is a convergent sequence and it converges to $x \in \ell_\infty$. We need to show that $x \in \mathrm{S}_\theta(\mathcal{I})^g \cap \ell_\infty$. First note that $\mathrm{S}_\theta(\mathcal{I})^g \subseteq \mathrm{S}_\theta(\mathcal{I})$ (by Theorem 3.1 and in view of the choice of the function g) and it is known that $\mathrm{S}_\theta(\mathcal{I}) \cap \ell_\infty$ is closed in ℓ_∞ (see Theorem 1 [6]) which implies that $x \in \mathrm{S}_\theta(\mathcal{I}) \cap \ell_\infty$. Again as $x^n \in \mathrm{S}_\theta(\mathcal{I})^g \subseteq \mathrm{S}_\theta(\mathcal{I})$ $\forall\ n = 1, 2, 3, ...,$ x^n is $\mathcal{I}-$ statistically convergent to some number L_n for $n = 1, 2, 3, ...$ Then following the arguments of Theorem 2.13 [7] we can show that the sequence of real numbers (L_n) is convergent to some number L. We now show that $x \to L\,(\mathrm{S}_\theta(\mathcal{I})^g)$. Fix $\varepsilon > 0$ and choose a strictly decreasing sequence of positive real numbers (ε_n) converging to 0. Choose $m \in \mathbb{N}$ such that $\varepsilon_m < \frac{\varepsilon}{4}$, $\|x - x^m\|_\infty < \frac{\varepsilon}{4}$, $|L_m - L| < \frac{\varepsilon}{4}$. Then

$$\begin{aligned} \frac{1}{g(h_r)}\left|\{k \in I_r : |x_k - L| \geq \varepsilon\}\right| &\leq \frac{1}{g(h_r)}\left|\{k \in I_r : |x_k^m - L_m| + \|x_k - x_k^m\|_\infty + |L_m - L| \geq \varepsilon\}\right| \\ &\leq \frac{1}{g(h_r)}\left|\left\{k \in I_r : |x_k^m - L_m| + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \geq \varepsilon\right\}\right| \\ &\leq \frac{1}{g(h_r)}\left|\left\{k \in I_r : |x_k^m - L_m| \geq \frac{\varepsilon}{2}\right\}\right|. \end{aligned}$$

Consequently

$$\begin{aligned} & \left\{r \in \mathbb{N} : \frac{1}{g(h_r)}\left|\{k \in I_r : |x_k - L| \geq \varepsilon\}\right| < \delta\right\} \\ \supseteq\ & \left\{r \in \mathbb{N} : \frac{1}{g(h_r)}\left|\left\{k \in I_r : |x_k^m - L_m| \geq \frac{\varepsilon}{2}\right\}\right| < \delta\right\} \in \mathcal{F}(\mathcal{I}). \end{aligned}$$

Hence

$$\left\{r \in \mathbb{N} : \frac{1}{g(h_r)}\left|\{k \in I_r : |x_k - L| \geq \varepsilon\}\right| < \delta\right\} \in \mathcal{F}(\mathcal{I})$$

and so

$$\left\{r \in \mathbb{N} : \frac{1}{g(h_r)}\left|\{k \in I_r : |x_k - L| \geq \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}.$$

This shows that $x \to L\,(S_\theta(\mathcal{I})^g)$. □

Theorem 3.3. *$S(I)^g \cap \ell_\infty$ is a closed subset of ℓ_∞ where $g \in G$ is such that $\frac{g(n)}{n} \leq M$ for all $n \geq j_0$, for some $M > 0$ and $j_0 \in \mathbb{N}$.*

Remark. *Theorem 2.13 and Theorem 2.14* [7] *are special cases of Theorem 3.2 and Theorem 3.3 respectively.*

Next we establish a result in line of Theorem 2 [6] and Theorem 2.17 [7] for $\mathcal{I}$-lacunary statistical convergence of weight g. For this we introduce the following definition.

Definition 7. *Let θ be a lacunary sequence. A sequence $x = (x_n)$ is said to be $N_\theta(\mathcal{I})^g-$convergent to L if for any $\varepsilon > 0$,*

$$\left\{r \in \mathbb{N} : \frac{1}{g(h_r)} \sum_{k \in I_r} |x_k - L| \geq \varepsilon\right\} \in \mathcal{I}.$$

It is denoted by $x_k \to L\,(N_\theta(\mathcal{I})^g)$ and the class of all such sequences is denoted simply by $N_\theta(\mathcal{I})^g$.

Theorem 3.4. *Let $\theta = (k_r)$ be a Lacunary sequence. Then*

(a) *$x_k \to L\,(N_\theta(\mathcal{I})^g) \Rightarrow x_k \to L\,(S_\theta(\mathcal{I})^g)$, and*

(b) *$N_\theta(\mathcal{I})^g$ is a proper subset of $S_\theta(\mathcal{I})^g$.*

Proof. (1)

(a) The proof similar to the proof of Theorem 2.17 (a) [7] and so is omitted.

(b) To prove that inclusion $N_\theta(\mathcal{I})^g \subset S_\theta(\mathcal{I})^g$ is proper, let θ be given and define x_k to be $1, 2, ..., \left[\sqrt{g(h_r)}\right]$ at first $\left[\sqrt{g(h_r)}\right]$ integers in I_r and $x_k = 0$ otherwise for all $r = 1, 2, 3......$ Now for any $\varepsilon > 0$,

$$\frac{1}{g(h_r)} |\{k \in I_r : |x_k - 0| \geq \varepsilon\}| \leq \frac{\left[\sqrt{g(h_r)}\right]}{g(h_r)}$$

and consequently for any $\delta > 0$, we get

$$\left\{r \in \mathbb{N} : \frac{1}{g(h_r)} |\{k \in I_r : |x_k - 0| \geq \varepsilon\}| \geq \delta\right\} \subseteq \left\{r \in \mathbb{N} : \frac{\left[\sqrt{g(h_r)}\right]}{g(h_r)} \geq \delta\right\}.$$

Note that the set on the right hand side is a finite set and so is a member of $\mathcal{I}$. Thus $x_k \to 0\,(S_\theta(\mathcal{I})^g)$. Again observe that

$$\frac{1}{g(h_r)} \sum_{k \in I_r} |x_k - 0| = \frac{1}{g(h_r)} \frac{\left[\sqrt{g(h_r)}\right]\left(\left[\sqrt{g(h_r)}\right] + 1\right)}{2}.$$

Hence

$$\begin{aligned}\left\{r \in \mathbb{N} : \frac{1}{g(h_r)} \sum_{k \in I_r} |x_k - 0| \geq \frac{1}{4}\right\} &= \left\{r \in \mathbb{N} : \frac{\left[\sqrt{g(h_r)}\right]\left(\left[\sqrt{g(h_r)}\right] + 1\right)}{g(h_r)} \geq \frac{1}{2}\right\} \\ &= \{p, p+1, p+2, ...\}\end{aligned}$$

for some $p \in \mathbb{N}$ which evidently belongs to $\mathcal{F}(\mathcal{I})$ as $\mathcal{I}$ is admissible. Therefore $x_k \nrightarrow 0\,(N_\theta(\mathcal{I})^g)$

□

Remark. *In Theorem 2* [6] *it was further proved that*

(c) $x \in \ell_\infty$ *and* $x_k \to L\,(S_\theta(\mathcal{I})) \Rightarrow x_k \to L\,(N_\theta(\mathcal{I}))$,
(d) $S_\theta(\mathcal{I}) \cap \ell_\infty = N_\theta(\mathcal{I}) \cap \ell_\infty$

It is not clear whether these results hold for any $g \in G$ and we leave it as an open problem.

In the remaining results we intend to investigate the relationship between the two new convergence methods introduced above, namely, $\mathcal{I}-$statistical and $\mathcal{I}-$Lacunary statistical convergence of weight g.

Theorem 3.5. *For any Lacunary sequence θ, $\mathcal{I}-$statistical convergence of weight g implies $\mathcal{I}-$Lacunary statistical convergence of weight g if*

$$\liminf_r \frac{g(h_r)}{g(k_r)} > 1.$$

Proof. Since $\liminf_r \frac{g(h_r)}{g(k_r)} > 1$, so we can find a $H > 1$ such that for sufficiently large r we have

$$\frac{g(h_r)}{g(k_r)} \geq H.$$

Since $x_k \to L\left(S(\mathcal{I})^g\right)$, hence for every $\varepsilon > 0$ and sufficiently large r we have

$$\begin{aligned}\frac{1}{g\left(k_r\right)}\left|\left\{k \leq k_r : \left|x_k - L\right| \geq \varepsilon\right\}\right| &\geq \frac{1}{g\left(k_r\right)}\left|\left\{k \in I_r : \left|x_k - L\right| \geq \varepsilon\right\}\right| \\ &\geq H \cdot \frac{1}{g\left(h_r\right)}\left|\left\{k \in I_r : \left|x_k - L\right| \geq \varepsilon\right\}\right|.\end{aligned}$$

Then for any $\delta > 0$ we get

$$\begin{aligned}&\left\{r \in \mathbb{N} : \frac{1}{g\left(h_r\right)}\left|\left\{k \in I_r : \left|x_k - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \\ \subseteq &\left\{r \in \mathbb{N} : \frac{1}{g\left(k_r\right)}\left|\left\{k \leq k_r : \left|x_k - L\right| \geq \varepsilon\right\}\right| \geq H\delta\right\} \in \mathcal{I}.\end{aligned}$$

This shows that $x_k \to L\left(S_\theta(\mathcal{I})^g\right)$. □

For the next result as in [6] we assume that the lacunary sequence θ satisfies the condition that for any set $C \in \mathcal{F}(\mathcal{I})$

$$\bigcup \left\{n : k_{r-1} < n < k_r, r \in C\right\} \in \mathcal{F}(\mathcal{I}).$$

Theorem 3.6. *For a lacunary sequence θ satisfying the above condition, $\mathcal{I}-$Lacunary statistical convergence of weight g implies $\mathcal{I}-$statistical convergence of weight g (where $g(n) \neq n$), if* $\sup \sum_{i=1}^{r} \frac{g(h_i)}{g(k_{r-1})} = B(say) < \infty$ *where g is further assumed to be monotonically increasing.*

Proof. Suppose that $x_k \to L\left(S_\theta(\mathcal{I})^g\right)$. For $\varepsilon, \delta, \delta_1 > 0$ define the sets

$$C = \left\{r \in \mathbb{N} : \frac{1}{g\left(h_r\right)}\left|\left\{k \in I_r : \left|x_k - L\right| \geq \varepsilon\right\}\right| < \delta\right\}$$

and

$$T = \left\{n \in \mathbb{N} : \frac{1}{g\left(n\right)}\left|\left\{k \leq n : \left|x_k - L\right| \geq \varepsilon\right\}\right| < \delta_1\right\}.$$

From our assumption it follows that $C \in \mathcal{F}(\mathcal{I})$, the dual filter of $\mathcal{I}$. Also note that

$$A_j = \frac{1}{g\left(h_j\right)}\left|\left\{k \in I_j : \left|x_k - L\right| \geq \varepsilon\right\}\right| < \delta$$

for all $j \in C$. Let $n \in \mathbb{N}$ be such that $k_{r-1} < n < k_r$ for some $r \in C$. Now

$$\begin{aligned}
& \frac{1}{g(n)} \left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right| \\
\leq \; & \frac{1}{g(k_{r-1})} \left|\{k \leq k_r : |x_k - L| \geq \varepsilon\}\right| \\
= \; & \frac{1}{g(k_{r-1})} \left|\{k \in I_1 : |x_k - L| \geq \varepsilon\}\right| + \cdots + \frac{1}{g(k_{r-1})} \left|\{k \in I_r : |x_k - L| \geq \varepsilon\}\right| \\
= \; & \frac{g(n_1)}{g(k_{n-1})} \cdot \frac{1}{g(h_1)} \left|\{k \in I_1 : |x_k - L| \geq \varepsilon\}\right| + \\
& \frac{g(h_2)}{g(k_{r-1})} \cdot \frac{1}{g(h_2)} \left|\{k \in I_2 : |x_k - L| \geq \varepsilon\}\right| \\
+ \; & ... + \frac{g(h_r)}{g(k_{r-1})} \cdot \frac{1}{g(h_r)} \left|\{k \in I_r : |x_k - L| \geq \varepsilon\}\right| \\
= \; & \frac{g(h_1)}{g(k_{r-1})} \cdot A_1 + \frac{g(h_2)}{g(k_{r-1})} \cdot A_2 + ... + \frac{g(h_r)}{g(k_{r-1})} \cdot A_r
\end{aligned}$$

Now choosing $\delta_1 = \frac{\delta}{B}$ and since $\bigcup \{n : k_{r-1} < h < k_r, r \in C\} \subset T$ where $C \in \mathcal{F}(\mathcal{I})$ it follows from our assumption on θ that the set T also belongs to $\mathcal{F}(\mathcal{I})$ and this complete the proof of the theorem □

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