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APPLICATIONS OF NON-UNIQUE FIXED POINT THEOREM OF ĆIRIĆ TO NONLINEAR INTEGRAL EQUATIONS

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Abstract. In this paper we discuss the application of the non-unique fixed point theorem of Ćirić to nonlinear Fredholm integral equations. We establish an existence theorem for the solutions of such integral equations and apply the theorem to particular examples.

1. Introduction and Preliminaries

The problem of existence and uniqueness of solutions of differential or integral equations is among the most common applications of the fixed point theory. It is known that in the case of linear differential or integral equations, a unique solution exists under certain conditions of the coefficient functions. However, for the nonlinear equations the problem of existence and uniqueness of solutions does not have a trivial answer in general. Moreover, in many cases the nonlinear differential or integral equation may have more than one solutions [9].

In 1974, Ćirić [2] stated and proved a fixed point theorem for mappings which may have more than one fixed points. In his paper Ćirić also emphasized the importance of non-unique fixed points and the periodic points. Later other authors obtained more results related with non-unique fixed points [1],[3]-[8].

In this section we will recall some basic notions and theoretical results related with non-unique fixed points. In Section 2, we will state and prove an existence theorem for a certain class of nonlinear Fredholm integral equations of the second type. In the last section, we will apply the theorem to specific examples of nonlinear integral equation having more than one solution.

In what follows, we will first recall the notions of the orbital continuity and orbital completeness.

Let (X,d) be a metric space and $T : X \to X$ be a self mapping.

Definition 1.1. *(See* [2]*)*

(a) *For any* $x_0 \in X$ *the set* $O(x_0) = \{x_0, Tx_0, T^2x_0, \ldots, T^nx_0, \ldots\}$ *is called an orbit of* T.

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(b) *The map T is called orbitally continuous if*

$$\lim_{i\to\infty} T^{n_i}x = y \text{ implies } \lim_{i\to\infty} TT^{n_i}x = Ty.$$

(c) *The space (X,d) is called T-orbitally complete if every Cauchy sequence of type $\{T^{n_i}x\}_{i\in\mathbb{N}}$ converges to a limit in X.*

It is clear that a continuous map is orbitally continuous and a complete metric space (X,d) is T-orbitally complete. We next recall the fixed point theorem of Ćirić.

Theorem 1.2. [2] *Let T be an orbitally continuous self map on the T-orbitally complete metric space (X,d). If there exists $k\in[0,1)$ such that*

$$\min\{d(Tx,Ty),d(x,Tx),d(y,Ty)\}-\min\{d(x,Ty),d(y,Tx)\}\le kd(x,y), \quad (1.1)$$

for all $x,y\in X$, then for each $x_0\in X$ the sequence $\{T^nx_0\}$ converges to a fixed point of T.

Some generalizations of this theorem have been reported later by Achari [1], Pachpate [8] and Karapınar [5].

2. Nonlinear Fredholm integral equations with non-unique solutions

In this section we consider nonlinear Fredholm integral equations of the second type. This type equations are derived from boundary value problems associated with differential equations. Consider the Fredholm integral equation of the second type

$$x(t)=u(t)+\lambda\int_a^b K(s,t)F(s,x(s))ds \quad (2.1)$$

where $u:[a,b]\to\mathbb{R}$ is a given continuous function, $F:[a,b]\times C[a,b]\to\mathbb{R}$ is a given nonlinear function, $K:[a,b]\times[a,b]\to\mathbb{R}$ is the kernel, $\lambda\in\mathbb{R}$ is constant and $x\in C[a,b]$ is the unknown function.

Let d be the metric induced by $\|\cdot\|_\infty$, that is,

$$d(x,y)=\|x(t)-y(t)\|_\infty=\sup_{t\in[a,b]}|x(t)-y(t)|, \quad (2.2)$$

in other words, the usual metric in $C[a,b]$. Then the space $C[a,b]$ is a complete metric space with respect to $d(x,y)$ defined in (2.2).

Define $T:C[a,b]\to C[a,b]$ as

$$Tx(t)=u(t)+\lambda\int_a^b K(s,t)F(s,x(s))ds. \quad (2.3)$$

Then, a solution of the equation (2.1) is a fixed point of T. It is clear that T is continuous, and hence orbitally continuous mapping. The following existence theorem is inspired by the fixed point theorem of Ćirić [2].

Theorem 2.1. *Let $[a,b]$ be a finite interval and $x\in C[a,b]$. Assume that for any $x,y\in C[a,b]$, and $s\in[a,b]$ the function $F(s,x(s))$ satisfies the condition*

$$|F(s,x(s))-F(s,y(s))|\le|f(s)||x-y| \quad (2.4)$$

for all $s \in [0,1]$*, such that the integral*

$$\int_a^b |K(s,t)f(s)|^2 ds \le L^2 \tag{2.5}$$

is bounded where $0 < |\lambda| L\sqrt{b-a} < 1$*. Then, the map* T *defined in* (2.3) *has a fixed point, that is, the integral equation* (2.1) *has a solution in* $C[a,b]$*. Moreover, for any* $x_0 \in C[a,b]$*, the sequence* $\{T^n x_0\}$ *converges to a solution of* (2.1)*.*

Proof. We first define

$$M = \min\{d(Tx,Ty), d(x,Tx), d(y,Ty)\}, \tag{2.6}$$

$$N = \min\{d(x,Ty), d(y,Tx)\}. \tag{2.7}$$

The proof will be considered in three main cases.

Case 1.

Let

$$\begin{aligned} M &= d(Tx,Ty), \\ N &= \min\{d(x,Ty), d(y,Tx)\}. \end{aligned}$$

Then in both possibilities for N we have

$$\begin{aligned} M - N &= d(Tx,Ty) - \min\{d(x,Ty), d(y,Tx)\} \\ &\le d(Tx,Ty) = \sup_{t\in[a,b]} |Tx(t) - Ty(t)| \end{aligned}$$

where

$$|Tx(t) - Ty(t)| = |\lambda \int_a^b K(s,t)[F(s,x(s)) - F(s,y(s))]ds|.$$

By the condition (2.4) on the function $F(s,x(s))$, one can rewrite the above inequality as

$$|Tx(t) - Ty(t)| \le |\lambda| \int_a^b |K(s,t)f(s)||x(s) - y(s)|ds.$$

Using the Cauchy-Schwarz inequality for integrals

$$\int_a^b f(x)g(x)dx \le \left(\int_a^b f^2(x)dx\right)^{1/2} \left(\int_a^b g^2(x)dx\right)^{1/2},$$

we obtain

$$|Tx(t) - Ty(t)| \le |\lambda| \left(\int_a^b |K(s,t)f(s)|^2 ds\right)^{1/2} \left(\int_a^b |x(s) - y(s)|^2 ds\right)^{1/2}.$$

From the integrability condition (2.5) we deduce

$$|Tx(t) - Ty(t)| \le |\lambda| L \left(\int_a^b |x(s) - y(s)|^2 ds\right)^{1/2}.$$

Taking supremum over $[a,b]$ we get

$$d(Tx,Ty) \le |\lambda| L\sqrt{b-a}\, d(x,y),$$

whereupon

$$M - N \le |\lambda| L\sqrt{b-a}\, d(x,y).$$

We conclude that

$$d(Tx,Ty) - \min\{d(x,Ty), d(y,Tx)\} \leq kd(x,y),$$

where $k = |\lambda| L\sqrt{b-a}$. Then the condition (1.1) of Theorem 1.2 is satisfied for $0 < k = |\lambda| L\sqrt{b-a} < 1$. Therefore, the map T defined in (2.3) has a fixed point.

Case 2.
Let

$$M = d(x,Tx),$$
$$N = d(x,Ty).$$

Then, employing the triangle inequality of the metric with

$$d(x,Tx) \leq d(x,Ty) + d(Ty,Tx),$$

we obtain

$$M - N = d(x,Tx) - d(x,Ty) \leq d(Tx,Ty) = \sup_{t\in[a,b]} |Tx(t) - Ty(t)|,$$

where

$$|Tx(t) - Ty(t)| = |\lambda \int_a^b K(s,t)[F(s,x(s)) - F(s,y(s))]ds|.$$

As in the Case 1, by using the condition (2.4) and then Cauchy-Schwarz inequality and the condition (2.5), upon taking the supremum over $[a,b]$, the last inequality implies

$$d(Tx,Ty) \leq |\lambda| L\sqrt{b-a}d(x,y).$$

Hence,

$$d(x,Tx) - d(x,Ty) \leq kd(x,y),$$

where $k = \sqrt{b-a}|\lambda|L$. Then the condition (1.1) of the Theorem 1.2 holds for $0 < k = |\lambda| L\sqrt{b-a} < 1$. As a result, T has a fixed point. Note that the case

$$M = d(y,Ty),$$
$$N = d(y,Tx),$$

reduces to Case 2 if we interchange the roles of x and y.

Case 3.
In the last case we assume that

$$M = d(x,Tx),$$
$$N = d(y,Tx).$$

Since $M = d(x,Tx)$ is minimum then $d(x,Tx) \leq d(y,Ty)$ and we have

$$M - N = d(x,Tx) - d(y,Tx) \leq d(y,Ty) - d(y,Tx). \tag{2.8}$$

Hence, one can rewrite the inequality (2.8) as

$$\begin{aligned} M - N &= d(x,Tx) - d(y,Tx) \leq d(y,Ty) - d(y,Tx) \\ &\leq d(Tx,Ty) = \sup_{t\in[a,b]} |Tx(t) - Ty(t)|, \end{aligned}$$

by using the triangle inequality of the metric. Using the condition (2.4) and then Cauchy-Schwarz inequality and the condition (2.5) of the theorem, we take the supremum over $[a,b]$ and deduce the inequality

$$d(x,Tx) - d(y,Tx) \leq kd(x,y),$$

where $k = |\lambda| L\sqrt{b-a}$. Hence, the contraction condition (1.1) holds for $0 < k = |\lambda| L\sqrt{b-a} < 1$. We conclude that the map T has a fixed point. Note that the case

$$M = d(y,Ty),$$
$$N = d(x,Ty),$$

reduces to Case 3 if we interchange the roles of x and y. This completes the proof. $\square$

3. Examples

In this section we apply the existence theorem to specific examples. We will consider both homogeneous and nonhomogeneous Fredholm integral equations of the second kind, that is, both the cases when the function u in (2.1) is the zero function and a nonzero function. In the first two examples the kernel K is separable and hence, the equations can be solved using the direct computation method [9]. In the last example we will consider an equation with a nonseparable kernel.

Example 3.1. *Consider the nonlinear homogeneous Fredholm equation of the second kind*

$$x(t) = \lambda \int_0^1 stx^2(s)ds, \qquad t \in [0,1]. \tag{3.1}$$

We will first find the exact solutions of equation. Following the direct computation method, we first define

$$\alpha = \int_0^1 sx^2(s)ds. \tag{3.2}$$

Then $x(t) = \lambda\alpha t$. Substitution of x in (3.2) leads to

$$\alpha = (\lambda\alpha)^2 \int_0^1 s^3 ds = \frac{(\lambda\alpha)^2}{4}.$$

Solving this quadratic equation for α gives $\alpha_1 = 0$ and $\alpha_2 = \dfrac{4}{\lambda^2}$. Then, for the computed values of α we get two solutions of the form $x_1(t) = 0$ and $x_2(t) = \dfrac{4}{\lambda}t$. Notice that the first solution $x_1(t) = 0$ is the trivial solution. Then, the Fredholm equation (3.1) has non-unique solution if $\lambda \neq 0$.

Now, we will check the conditions of the existence theorem of the previous section for this example. Observe that the function $F(s,x(s)) = x^2(s)$ and the kernel $K(s,t) = st$ satisfy the conditions

$$|F(s,x(s)) - F(s,y(s))| = |x+y||x-y| \leq (|x|+|y|)|x-y| \leq 2c|x-y|,$$

for $|x| \leq c$ and

$$\int_0^1 |K(s,t)f(s)|^2 ds \leq 4c^2 \int_0^1 (st)^2 ds \leq \frac{4c^2}{3} = L^2, \qquad \text{for} \quad t \in [0,1].$$

Hence, according to the Theorem 2.1, for $|\lambda|\frac{2c}{\sqrt{3}} < 1$, *i.e.,* $|\lambda| < \frac{\sqrt{3}}{2c}$, *the given Fredholm equation* (3.1) *has solution. In fact, by direct computation we obtained that two solutions exist for any nonzero value of the constant* λ.

In the next example we will consider a nonhomogeneous Fredholm integral equation with a separable kernel.

Example 3.2. *Consider the integral equation*

$$x(t) = -t + \lambda \int_0^1 st[x^2(s) + 3x(s)]ds, \qquad t \in [0,1]. \tag{3.3}$$

Applying again the direct computation method we define

$$\alpha = \int_0^1 s[x^2(s) + 3x(s)]ds. \tag{3.4}$$

Then we have $x(t) = (-1+\lambda\alpha)t$. *Now we substitute* x *in* (3.4) *which leads to*

$$\alpha = \int_0^1 [(-1+\lambda\alpha)^2 s^3 + (-1+\lambda\alpha)3s^2]ds = \frac{1}{4}(-1+\lambda\alpha)^2 + (-1+\lambda\alpha).$$

Upon solving this quadratic equation for α *we obtain*

$$\alpha_1 = \frac{(2-\lambda) - \sqrt{(\lambda-2)^2+3}}{\lambda^2},$$

and

$$\alpha_2 = \frac{(2-\lambda) + \sqrt{(\lambda-2)^2+3}}{\lambda^2}.$$

We insert these values in $x(t)$ *which gives the solutions*

$$x_1(t) = t\left(-2 + \frac{1}{\lambda}(2 - \sqrt{(\lambda-2)^2+3})\right),$$

and

$$x_2(t) = t\left(-2 + \frac{1}{\lambda}(2 + \sqrt{(\lambda-2)^2+3})\right),$$

that is, the given equation has two solutions provided that $\lambda \neq 0$.

For this example, we can see that the function $F(s,x(s)) = x^2(s) + 3x(s)$ *and the kernel* $K(s,t) = st$ *satisfy the conditions*

$$\begin{aligned} |F(s,x(s)) - F(s,y(s))| &= |x(s)+y(s)+3||x(s)-y(s)| \\ &\leq (|x(s)|+|y(s)|+3)|x(s)-y(s)| \\ &\leq (2c+3)|x(s)-y(s)|, \end{aligned}$$

for $|x| \leq c$ *and*

$$\int_0^1 |K(s,t)f(s)|^2 ds \leq (2c+3)^2 \int_0^1 (st)^2 ds \leq \frac{(2c+3)^2}{3} = L^2, \qquad \text{for} \quad t \in [0,1].$$

Then from the Theorem 2.1 it follows that for $|\lambda|\frac{2c+3}{\sqrt{3}} < 1$, *or equivalently,* $|\lambda| < \frac{\sqrt{3}}{2c+3}$, *the Fredholm equation given in* (3.3) *has non-unique solution. We have confirmed by direct computation that this equation has two solutions for any nonzero value of the constant* λ.

In the first two examples we considered the case of separable kernel so that we could solve the equations exactly by using the direct computation method. In the last example we will consider an integral equation with a nonseparable kernel.

Example 3.3. *Consider the Fredholm integral equation*

$$x(t) = t^2 + \lambda \int_1^2 \frac{1}{s+t} \frac{x^2(s)}{x^2(s)+1} ds, \qquad t \in [1,2], \tag{3.5}$$

and assume that $\lambda \neq 0$. *The direct computation method is not applicable, hence the exact solution cannot be obtained. We will check the conditions of the Theorem 2.1. The function* $F(s,x(s)) = \dfrac{x^2(s)}{x^2(s)+1}$ *and the kernel* $K(s,t) = \dfrac{1}{s+t}$ *satisfy the conditions*

$$\begin{aligned} |F(s,x(s)) - F(s,y(s))| &= \left| \frac{x^2(s)}{x^2(s)+1} - \frac{y^2(s)}{y^2(s)+1} \right| \\ &\leq \frac{|x(s)+y(s)||x(s)-y(s)|}{(x^2(s)+1)(y^2(s)+1)} \\ &\leq 2c|x(s)-y(s)|, \qquad \text{for} \quad |x| \leq c \end{aligned}$$

and

$$\int_1^2 |K(s,t)f(s)|^2 ds \leq 4c^2 \int_1^2 \frac{1}{(t+s)^2} ds \leq c^2 = L^2, \qquad \text{for} \quad t \in [1,2].$$

Then from the Theorem 2.1 it follows that for $|\lambda| c < 1 \Leftrightarrow 0 < |\lambda| < \dfrac{1}{c}$, *the given Fredholm equation* (3.5) *has solution.*

4. Conclusion

The aim of the paper is to underline the importance of non-unique fixed point by expressing an application with concrete examples. The main result of the paper, that is Theorem 2.1, gives conditions for existence of solutions for a class of nonlinear Fredholm integral equations. It is confirmed by the examples that integral equations from this class have non-unique solutions. As a future study, we suggest studying the application of some numerical methods to find approximately the non-unique solutions of this type of equations.

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