Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 12 Issue 1(2020), Pages 1-11.

ROUGH $\Delta\mathcal{I}_2$-STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES IN NORMED LINEAR SPACES

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Abstract. In this paper, we introduce rough $\Delta\mathcal{I}_2$-statistical convergence as an extension of rough convergence. We define the set of rough $\Delta\mathcal{I}_2$-statistical limit points of a sequence and analyze the results with proofs.

1. Introduction

The concept of convergence of a sequence of real numbers was independently extended to statistical convergence independently by Fast [10] and Schoenberg [37]. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [24] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers. Kostyrko et al. [25] studied the idea of $\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points.

The idea of $\mathcal{I}$-statistical convergence was introduced by Savaş and Das [27] as an extension of ideal convergence. Later on it was further investigated by Debnath and Debnath [11], Et et al. [12], Savaş and Gürdal [26] and many others.

The idea of rough convergence was first introduced by Phu [32] in finite-dimensional normed spaces. In another paper [33] related to this subject, Phu defined the rough continuity of linear operators and showed that every linear operator $f : X \to Y$ is r -continuous at every point $x \in X$ under the assumption $dimY < \infty$ and $r > 0$, where X and Y are normed spaces. In [34], Phu extended the results given in [32] to infinite-dimensional normed spaces. Aytar [3] studied the rough statistical convergence. Also, Aytar [4] studied that the rough limit set and the core of a real sequence. Pal et al. [9] generalized the idea of rough convergence into rough statistical convergence and rough ideal convergence. Recently, rough convergence of double sequences has been introduced by Malik and Maity [14] and investigated some basic properties of this type of convergence and also studied the relation between rough convergence and Pringsheim convergence for double sequences. In [15] rough statistical convergence of double sequences in finite dimensional normed linear spaces was studied and investigated some basic properties of this type of convergence rough statistical convergence of double sequences. Recently, Dündar and Çakan [5, 6, 7] introduced the notion of rough $\mathcal{I}$-convergence and the set of

2000 *Mathematics Subject Classification.* 40A05, 40A35.
Key words and phrases. Statistical convergence; $\mathcal{I}_2$-convergence; rough convergence.

Submitted November 23, 2019. Published January 22, 2020.
Communicated by Feyzi Basar.

rough $\mathcal{I}$-limit points of a sequence and studied the notions of rough convergence, $\mathcal{I}_2$-convergence and the sets of rough limit points and rough $\mathcal{I}_2$-limit points of a double sequence. Savaş et al. [28] introduced rough $\mathcal{I}$-statistical convergence as an extension of rough convergence. Rough convergence, rough statistical convergence and $\Delta\mathcal{I}$-convergence for difference sequences and for double difference sequences have been studied. For details, see ([17], [18], [19], [20], [21], [22], [23]).

In view of the recent applications of ideals in the theory of convergence of sequences, it seems very natural to extend the interesting concept of rough $\Delta\mathcal{I}_2$-statistical convergence of double seqeunces in normed linear spaces further by using ideals which we mainly do here.

2. Definitions and notations

In this section, we recall some definitions and notations, which form the base for the present study (See [1, 2, 3, 4, 5, 6, 7, 16, 32, 33, 34]).

During the preparation of the paper, let r be a nonnegative real number and $\mathbb{R}^n$ denotes the real n-dimensional space with the norm $\|.\|$. Consider a sequence $x = (x_k) \subset X = \mathbb{R}^n$.

The sequence $x = (x_k)$ is said to be r-convergent to x_*, denoted by $x_k \xrightarrow{r} x_*$ provided that

$$\forall \varepsilon > 0 \ \exists i_\varepsilon \in \mathbb{N} : \ \ k \geq i_\varepsilon \Rightarrow \|x_k - x_*\| < r + \varepsilon.$$

The set

$$\mathrm{LIM}^r x := \{x_* \in \mathbb{R}^n : x_k \xrightarrow{r} x_*\}$$

is called the r-limit set of the sequence $x = (x_k)$. A sequence $x = (x_i)$ is said to be r-convergent if $\mathrm{LIM}^r x \neq \emptyset$. In this case, r is called the convergence degree of the sequence $x = (x_k)$. For $r = 0$, we get the ordinary convergence. There are several reasons for this interest (see [32]).

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if
(i) $\emptyset \in \mathcal{I}$,
(ii) for each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$,
(iii) for each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.
An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is a filter in $\mathbb{N}$ if and only if
(i) $\emptyset \notin \mathcal{F}$,
(ii) for each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$,
(iii) for each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

If $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is a nontrivial ideal, then the family of sets

$$\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$$

is a filter of $\mathbb{N}$ and it is called the filter associated with the ideal $\mathcal{I}$.

Mursaleen et al. [31] defined $\mathcal{I}$-statistical cluster point of real number sequence.

A real sequence $x = (x_k)$ is said to be Δ-ideal convergent to $x \in \mathbb{R}$ provided for each $\varepsilon > 0$

$$\{k \in \mathbb{N} : |\Delta x_k - x| \geq \varepsilon\} \in \mathcal{I}.$$

The set of all Δ-ideal convergent sequences is doneted by $c_{\mathcal{I}}(\Delta)$.

A real sequence $x = (x_k)$ is said to be $\Delta\mathcal{I}^*$-convergent to $x \in \mathbb{R}$, if there exists $M = \{m = m_i : m_i < m_{i+1}, i \in \mathbb{N}\}$ such that $M \in \mathcal{F}(\mathcal{I})$ and $\Delta{-}lim_{k\to\infty} x_{k_m} = x$. In this case, we write $\Delta\mathcal{I}^* - \lim x_k = x$.

Theorem 2.1. *If a $\mathcal{I}$-statistically bounded sequence has one cluster point then it is $\mathcal{I}$-statistically convergent.*

A sequence $x = (x_k)$ is said to be rough $\mathcal{I}$-convergent (r-$\mathcal{I}$-convergent) to x_* with the roughness degree r, denoted by $x_k \xrightarrow{r-\mathcal{I}} x_*$ provided that $\{k \in \mathbb{N} : \|x_k - x_*\| \geq r + \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$; or equivalently, if the condition

$$\mathcal{I} - \limsup \|x_k - x_*\| \leq r$$

is satisfied. In addition, we can write $x_k \xrightarrow{r-\mathcal{I}} x_*$ iff the inequality $\|x_k - x_*\| < r + \varepsilon$ holds for every $\varepsilon > 0$ and almost all k.

A double sequence $x = (x_{mn})_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ of real numbers is said to be bounded if there exists a positive real number M such that $|x_{mn}| < M$, for all $m, n \in \mathbb{N}$. That is

$$\|x\|_\infty = \sup_{m,n} |x_{mn}| < \infty.$$

A double sequence $x = (x_{mn})$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense (shortly, p-convergent to $L \in \mathbb{R}$), if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$, whenever $m, n > N_\varepsilon$. In this case, we write

$$\lim_{m,n\to\infty} x_{mn} = L.$$

We recall that a subset K of $\mathbb{N}\times\mathbb{N}$ is said to have natural density $d(K)$ if

$$d(K) = \lim_{m,n\to\infty} \frac{K(m,n)}{m.n},$$

where $K(m,n) = |\{(j,k) \in \mathbb{N}\times\mathbb{N} : j \leq m, k \leq n\}|$.

Throughout the paper we consider a sequence $x = (x_{mn})$ such that $(x_{mn}) \in \mathbb{R}^n$.

Let $x = (x_{mn})$ be a double sequence in a normed space $(X, \|.\|)$ and r be a non negative real number. x is said to be r-statistically convergent to ξ, denoted by $x \xrightarrow{r-st_2} \xi$, if for $\varepsilon > 0$ we have $d(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|x_{mn} - \xi\| \geq r + \varepsilon\}$. In this case, ξ is called the r-statistical limit of x.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N}\times\mathbb{N}$ is called strongly admissible if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$.

It is evident that a strongly admissible ideal is admissible also.

Throughout the paper we take $\mathcal{I}_2$ as a strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$.

Let (X, ρ) be a metric space A double sequence $x = (x_{mn})$ in X is said to be $\mathcal{I}_2$-convergent to $L \in X$, if for any $\varepsilon > 0$ we have $A(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \in \mathcal{I}_2$. In this case, we say that x is $\mathcal{I}_2$-convergent and we write

$$\mathcal{I}_2 - \lim_{m,n\to\infty} x_{mn} = L.$$

A double sequence $x = (x_{mn})$ is said to be rough convergent (r-convergent) to x_* with the roughness degree r, denoted by $x_{mn} \xrightarrow{r} x_*$ provided that

$$\forall\varepsilon > 0\ \ \exists k_\varepsilon \in \mathbb{N} : m, n \geq k_\varepsilon \Rightarrow \|x_{mn} - x_*\| < r + \varepsilon,$$

or equivalently, if

$$\limsup \|x_{mn} - x_*\| \leq r.$$

A double sequence $x = (x_{mn})$ is said to be r-$\mathcal{I}_2$-convergent to x_* with the roughness degree r, denoted by $x_{mn} \overset{r-\mathcal{I}_2}{\longrightarrow} x_*$ provided that

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|x_{mn} - x_*\| \geq r+\varepsilon\} \in \mathcal{I}_2, \tag{2.1}$$

for every $\varepsilon > 0$; or equivalently, if the condition

$$\mathcal{I}_2 - \limsup \|x_{mn} - x_*\| \leq r \tag{2.2}$$

is satisfied. In addition, we can write $x_{mn} \overset{r-\mathcal{I}_2}{\longrightarrow} x_*$ iff the inequality $\|x_{mn} - x_*\| < r+\varepsilon$ holds for every $\varepsilon > 0$ and almost all (m,n).

Now, we give the definition of $\mathcal{I}_2$-asymptotic density of $\mathbb{N}\times\mathbb{N}$.

A subset $K \subset \mathbb{N}\times\mathbb{N}$ is said to be have $\mathcal{I}_2$-asymptotic density $d_{\mathcal{I}_2}(K)$ if

$$d_{\mathcal{I}_2}(K) = \mathcal{I}_2 - \lim_{m,n\to\infty} \frac{|K(m,n)|}{m.n},$$

where $K(m,n) = \{(j,k) \in \mathbb{N}\times\mathbb{N} : j \leq m, k \leq n; (j,k) \in K\}$ and $|K(m,n)|$ denotes number of elements of the set $K(m,n)$.

A double sequence $x = \{x_{jk}\}$ of real numbers is $\mathcal{I}_2$-statistically convergent to ε, and we write $x \overset{\mathcal{I}_2-st}{\rightarrow} \xi$, provied that for any $\varepsilon > 0$ and $\delta > 0$

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N}: \frac{1}{mn}\left|\{(j,k) : \|x_{jk} - \xi\| \geq \varepsilon,\ j \leq m, k \leq n\}\right| \geq \delta\right\} \in \mathcal{I}_2.$$

Let $x = \{x_{jk}\}$ be a double sequence in a normed linear space $(X, \|.\|)$ and r be a non negative real number. Then x is said to be rough $\mathcal{I}_2$-statistical convergent to ξ or r-$\mathcal{I}_2$-statistical convergent to ξ if for any $\varepsilon > 0$ and $\delta > 0$

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N}: \frac{1}{mn}\left|\{(j,k)\,,\ j \leq m, k \leq n : \|x_{jk} - \xi\| \geq r+\varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}_2.$$

In this case, ξ is called the rough $\mathcal{I}_2$-statistical limit of $x = \{x_{jk}\}$ and we denote it by $x \overset{r-\mathcal{I}_2-st}{\longrightarrow} \xi$.

3. Main results

Definition 3.1. *Let (Δx_{kl}) be a double sequence in a normed linear space $(X, \|.\|)$ and r be a nonnegative real number. Then, (Δx_{kl}) is said to be rough $\mathcal{I}_2$-convergent to x_* or r-$\mathcal{I}_2$-convergent to x_* if for any $\varepsilon > 0$*

$$\{(k,l) \in \mathbb{N}\times\mathbb{N} : \|\Delta x_{kl} - x_*\| \geq r+\varepsilon\} \in \mathcal{I}_2.$$

In this case x_* is called the r-$\mathcal{I}_2$-limit of (Δx_{kl}) and we denote it by $\Delta x \overset{r-\mathcal{I}_2}{\longrightarrow} x_*$. Here r is called roughness degree.

Definition 3.2. *A double sequence (Δx_{kl}) in X is said to be rough $\mathcal{I}_2$-statistical convergent to x_* or r-$\mathcal{I}_2$-statistical convergent to x_*, denoted by $\Delta x \overset{r-\mathcal{I}_2-st}{\longrightarrow} x_*$, provided that*

$$\left\{(n,m) \in \mathbb{N}\times\mathbb{N} : \frac{1}{nm}\left|\{k \leq n, l \leq m : \|\Delta x_{kl} - x_*\| \geq r+\varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}_2$$

for any $\varepsilon > 0$ and $\delta > 0$, or equivalently we can say $\mathcal{I}_2$-st $\limsup \|\Delta x_{kl} - x_*\| \leq r$. If we take $r = 0$, we obtain the notion $\Delta\mathcal{I}_2$-statistical convergence.

For instance assume that the sequence (Δy_{kl}) is rough $\mathcal{I}_2$-statistical convergent which can not be calculated exactly, one has to do with an approximated sequence (Δx_{kl}) satisfying $\|\Delta x_{kl} - \Delta y_{kl}\| \leq r$ for all k, l. Then, rough $\mathcal{I}_2$-statistical convergence of sequence (Δx_{kl}) is not assured, but as the inclusion,

$$\left\{(n,m) \in \mathbb{N} \times \mathbb{N} : \tfrac{1}{nm} \left|\{k \leq n, l \leq m : \|\Delta x_{kl} - x_*\| \geq r + \varepsilon\}\right| \geq \delta\right\}$$

$$\subseteq \left\{(n,m) \in \mathbb{N} \times \mathbb{N} : \tfrac{1}{nm} \left|\{k \leq n, l \leq m : \|\Delta y_{kl} - x_*\| \geq \varepsilon\}\right| \geq \delta\right\}$$

holds, so the sequence (Δx_{kl}) is r-$\mathcal{I}_2$-statistically convergent.

In general the rough $\mathcal{I}_2$-statistical limit of a sequence may not be unique for the roughness degree $r > 0$. We define the set of all rough $\mathcal{I}_2$-statistical limit of (Δx_{kl}) with

$$\mathcal{I}_2 - st - LIM^r\left(\Delta x_{kl}\right) = \left\{x_* \in X : \Delta x_{kl} \overset{r-\mathcal{I}_2-st}{\longrightarrow} x_*\right\}.$$

The double sequence (Δx_{kl}) is said to be r-$\mathcal{I}_2$-statistically convergent provided $\mathcal{I}_2$-$-st - LIM^r\left(\Delta x_{kl}\right) \neq \emptyset$. It is clear that if $\mathcal{I}_2$-st-$\mathrm{LIM}^r_{(\Delta x_k)} \neq \emptyset$ for a sequence (Δx_{kl}), then we have

$$\mathcal{I}_2 - st - LIM^r\left(\Delta x_{kl}\right) = \left[\mathcal{I}_2 - st - \limsup\left(\Delta x_{kl}\right) - r,\ \mathcal{I}_2 - st - \liminf\left(\Delta x_{kl}\right) - r\right].$$

Definition 3.3. *A double sequence (Δx_{kl}) is said to be $\mathcal{I}_2$-statistical bounded if there is a number K such that*

$$\left\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{nm} \left|\{k \leq n, l \leq m : \|\Delta x_{kl}\| > K\}\right| > \delta\right\} \in \mathcal{I}_2$$

for any $\delta > 0$.

Theorem 3.4. *For a sequence (Δx_{kl}),*

$$diam\left(\mathcal{I}_2 - st - \mathrm{LIM}^r_{(\Delta x_{kl})}\right) \leq 2r.$$

In general $diam\left(\mathcal{I}_2 - st - \mathrm{LIM}^r\left(\Delta x_{kl}\right)\right)$ has no smaller bound.

Proof. Assume that $diam\left(\mathcal{I}_2 - st - LIM^r\left(\Delta x_{kl}\right)\right) > 2r$. Then, there exist $y, z \in \mathcal{I}_2 - st - LIM^r\left(\Delta x_{kl}\right)$ such that $\|y - z\| > 2r$. Now, we select $\varepsilon > 0$ so that $\varepsilon < \frac{\|y-z\|}{2} - r$ and $\delta > 0$. Since $y, z \in \mathcal{I}_2 - st - LIM^r\left(\Delta x_{kl}\right)$, for every $\varepsilon > 0$ and $\delta > 0$, we have

$$A_1\left(\varepsilon, \delta\right) = \left\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{nm} \left|\{k \leq n, l \leq m : \|\Delta x_{kl} - y\| \geq r + \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}_2$$

and

$$A_2\left(\varepsilon, \delta\right) = \left\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{nm} \left|\{k \leq n, l \leq m : \|\Delta x_{kl} - z\| \geq r + \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}_2.$$

Hence, $P = \mathbb{N} \times \mathbb{N} \backslash \left(A_1\left(\varepsilon, \delta\right) \cup A_2\left(\varepsilon, \delta\right)\right) \in \mathcal{F}\left(\mathcal{I}_2\right)$. So, $P \neq \emptyset$. Let $p, q \in P$. Then for infinitely many $k \leq p$, $l \leq q$

$$\|y - z\| \leq \|\Delta x_{kl} - y\| + \|\Delta x_{kl} - z\| < 2\left(r + \varepsilon\right) < 2\left(r + \frac{\|y - z\|}{2} - r\right) = \|y - z\|,$$

which is contradiction. Thus,

$$diam\left(\mathcal{I}_2 - st - \mathrm{LIM}^r_{(\Delta x_{kl})}\right) \leq 2r.$$

To prove the second part, consider a double sequence (Δx_{kl}) such that $\mathcal{I}_2$-st-$\lim \Delta x_{kl} = x_*$. Let $\varepsilon > 0$ and $\delta > 0$ then by the definition of $\Delta\mathcal{I}_2$-statistical convergence

$$A(\varepsilon,\delta) = \left\{(n,m) \in \mathbb{N}\times\mathbb{N} : \frac{1}{nm}\left|\{k \le n, l \le m : \|\Delta x_{kl} - x_*\| \ge \varepsilon\}\right| < \delta\right\} \in \mathcal{F}(\mathcal{I}_2).$$

Let $p,q \in A(\varepsilon,\delta)$ then

$$\frac{1}{pq}\left|\{k \le p, l \le q : \|\Delta x_{kl} - x_*\| \ge \varepsilon\}\right| < \delta$$

i.e., for maximum $k \le p$, $l \le q$, $\|\Delta x_{kl} - x_*\| < \varepsilon$.

Now, for each

$$y \in \overline{B}_r(x_*) = \{y \in X : \|y - x_*\| \le r\}$$

we have

$$\|\Delta x_{kl} - y\| \le \|\Delta x_{kl} - x_*\| + \|x_* - y\| \le \|\Delta x_{kl} - x_*\| + r < r + \varepsilon$$

for maxium $k \le p \in A(\varepsilon,\delta)$, $l \le q \in A(\varepsilon,\delta)$ i.e.,

$$\left\{(n,m) \in \mathbb{N}\times\mathbb{N} : \frac{1}{nm}\left|\{k \le n, l \le m : \|\Delta x_{kl} - y\| \ge r+\varepsilon\}\right| < \delta\right\} \supseteq A(\varepsilon) \in \mathcal{F}(\mathcal{I}_2)$$

which shows that $y \in \mathcal{I}_2 - st - LIM^r(\Delta x_{kl})$ and consequently, $\mathcal{I}_2 - st - LIM^r(\Delta x_{kl}) = \overline{B}_r(x_*)$. This shows that in general upper bound $2r$ of the diameter of the set $\mathcal{I}_2 - st - LIM^r(\Delta x_{kl})$ can not be decreased any more. $\square$

Theorem 3.5. *A double sequence (Δx_{kl}) is $\mathcal{I}_2$-st-bounded if and only if there exists a non-negative real number r such that $\mathcal{I}_2 - st - \mathrm{LIM}^r(\Delta x_{kl}) \neq \emptyset$.*

Proof. Let (Δx_{kl}) be $\mathcal{I}_2$-st-bounded double sequence. Then, there exists a positive real number K such that

$$P = \left\{(n,m) \in \mathbb{N}\times\mathbb{N} : \frac{1}{nm}\left|\{k \le n, l \le m : \|\Delta x_{kl}\| > K\}\right| > \delta\right\} \in \mathcal{I}_2.$$

Let $\overline{r} = \sup\{\|\Delta x_{kl}\| \text{ for almost } k \le t,\ l \le s \in M = \mathbb{N}\times\mathbb{N}\backslash P\}$. The set $\mathcal{I} - st - LIM^{\overline{r}}(\Delta x_{kl})$ contains the origin of X and so $\mathcal{I}_2 - st - LIM^{\overline{r}}(\Delta x_{kl}) \neq \emptyset$.

Conversely, assume that $\mathcal{I}_2 - st - LIM^r(\Delta x_{kl}) \neq \emptyset$ for some $r \ge 0$. Then, there exists $x_* \in \mathcal{I}_2 - st - LIM^r(\Delta x_{kl})$ i.e.,

$$\left\{(n,m) \in \mathbb{N}\times\mathbb{N} : \frac{1}{nm}\left|\{k \le n, l \le m : \|\Delta x_{kl} - x_*\| \ge r+\varepsilon\}\right| \ge \delta\right\} \in \mathcal{I}_2,$$

for each $\varepsilon > 0$ and $\delta > 0$. This implies that (Δx_{kl}) is $\mathcal{I}_2$-st-bounded. $\square$

Theorem 3.6. *The set $\mathcal{I}_2 - st - \mathrm{LIM}^r(\Delta x_{kl})$ of a sequence (Δx_{kl}) is a closed set.*

Proof. If $\mathcal{I}_2 - st - LIM^r(\Delta x_{kl}) = \emptyset$, then there is nothing to prove.

Assume that $\mathcal{I}_2 - st - LIM^r(\Delta x_{kl}) \neq \emptyset$. Now, consider a sequence (Δy_{kl}) in $\mathcal{I}_2 - st - LIM^r(\Delta x_{kl})$ with $\lim_{k,l\to\infty} \Delta y_{kl} = y_*$. Choose $\varepsilon > 0$. Then, there exists $n_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that

$$\|\Delta y_{kl} - y_*\| < \frac{\varepsilon}{2},$$

for all $k,l \ge n_{\frac{\varepsilon}{2}}$.

Let $(n_0, m_0) \in \mathbb{N} \times \mathbb{N}$ such that $y_{n_0 m_0} \in (\Delta y_{kl}) \subseteq \mathcal{I}_2 - st - LIM^r (\Delta x_{kl})$. So,

$$A = \left\{ (n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{nm} \left| \left\{ k \leq n, l \leq m : \|\Delta x_{kl} - y_{n_0 m_0}\| \geq r + \frac{\varepsilon}{2} \right\} \right| < \delta \right\} \in \mathcal{F}(\mathcal{I}_2).$$

Let $p, q \in A$. Then,

$$\frac{1}{pq} \left| \left\{ k \leq p, l \leq q : \|\Delta x_{kl} - y_{n_0 m_0}\| \geq r + \frac{\varepsilon}{2} \right\} \right| < \delta,$$

i.e., for maximum $k \leq p, l \leq q$, $\|\Delta x_{kl} - y_{n_0 m_0}\| < r + \frac{\varepsilon}{2}$.

Choose an $n_0, m_0 > n_{\frac{\varepsilon}{2}}$ we get

$$\|\Delta x_{kl} - y_*\| \leq \|\Delta x_{kl} - y_{n_0 m_0}\| + \|y_{n_0 m_0} - y_*\| < r + \varepsilon$$

for maximum $k \leq p \in A$, $l \leq q \in A$. That is

$$\left\{ (n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{nm} \left| \left\{ k \leq n, l \leq m : \|\Delta x_{kl} - y_*\| \geq r + \varepsilon \right\} \right| < \delta \right\} \supseteq A \in \mathcal{F}(\mathcal{I}_2).$$

Hence, $y_* \in \mathcal{I}_2 - st - LIM^r (\Delta x_{kl})$ and so, $\mathcal{I}_2 - st - LIM^r (\Delta x_{kl})$ is a closed set. □

Theorem 3.7. *The set $\mathcal{I}_2 - st - \mathrm{LIM}^r (\Delta x_{kl})$ of a double sequence (Δx_{kl}) is convex.*

Proof. Let $y_0, y_1 \in \mathcal{I}_2 - st - LIM^r (\Delta x_{kl})$ and $\varepsilon > 0$ and $\delta > 0$ be given. Let

$$A_0(\varepsilon, \delta) = \left\{ (n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{nm} \left| \left\{ k \leq n, l \leq m : \|\Delta x_{kl} - y_0\| \geq r + \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}_2$$

and

$$A_1(\varepsilon, \delta) = \left\{ (n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{nm} \left| \left\{ k \leq n, l \leq m : \|\Delta x_{kl} - y_1\| \geq r + \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}_2.$$

$M = \mathbb{N} \times \mathbb{N} \setminus (A_1(\varepsilon, \delta) \cup A_2(\varepsilon, \delta)) \in \mathcal{F}(\mathcal{I}_2)$ and so M must be a infinite set. Let $(t, s) \in M$ then, $d(K_1) = 0$, where

$$K_1 = \{ k \leq t, l \leq s : \|\Delta x_{kl} - y_0\| \geq r + \varepsilon \}$$

and $d(K_2) = 0$, where

$$K_2 = \{ k \leq t, l \leq s : \|\Delta x_{kl} - y_1\| \geq r + \varepsilon \}.$$

Now for each $(k, l) \in K_1^c \cap K_2^c$ and each $\lambda \in [0, 1]$,

$$\|\Delta x_{kl} - [(1-\lambda) y_0 + \lambda y_1]\| = \|(1-\lambda)(\Delta x_{kl} - y_0) + \lambda (\Delta x_{kl} - y_1)\| < r + \varepsilon.$$

Since, $d(K_1^c \cap K_2^c) = 1$, we get

$$\frac{1}{ts} \left| \left\{ k \leq t, l \leq s : \|\Delta x_{kl} - [(1-\lambda) y_0 + \lambda y_1]\| \geq r + \varepsilon \right\} \right| < \delta.$$

Therefore,

$$\left\{ (n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{nm} \left| \left\{ k \leq n, l \leq m : \|\Delta x_{kl} - [(1-\lambda) y_0 + \lambda y_1]\| \geq r + \varepsilon \right\} \right| < \delta \right\} \supseteq M \in \mathcal{F}(\mathcal{I}_2),$$

which shows that $(1-\lambda) y_0 + \lambda y_1 \in \mathcal{I}_2 - st - LIM^r (\Delta x_{kl})$ for any $\lambda \in [0, 1]$. Hence, the set $\mathcal{I}_2 - st - LIM^r (\Delta x_{kl})$ is convex. □

Definition 3.8. *An element $c \in X$ is called $\mathcal{I}_2$-statistical cluster point of a sequence (Δx_{kl}) if for each $\varepsilon > 0$ and $\delta > 0$ the set*

$$\left\{ (n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{nm} \left| \{ k \leq n, l \leq m : \|\Delta x_{kl} - c\| \geq \varepsilon \} \right| < \delta \right\} \notin \mathcal{I}_2.$$

The set of all $\mathcal{I}_2$-statistical cluster points of (Δx_{kl}) will be denoted by $\mathcal{I}_2$-$S\big(\Gamma_{(\Delta x_{kl})}\big)$.

Theorem 3.9. *Let (Δx_{kl}) be a double sequence, then for an arbitrary $c \in \mathcal{I}_2$-$S\big(\Gamma_{(\Delta x_{kl})}\big)$, $\|x_* - c\| \leq r$ for all $x_* \in \mathcal{I}_2 - st - \mathrm{LIM}^r (\Delta x_{kl})$.*

Proof. If possible suppose that there exists $c \in \Delta\mathcal{I}_2$-S$(\Gamma_x)$ and $x_* \in \mathcal{I}_2 - st - LIM^r (\Delta x_{kl})$ such that $\|x_* - c\| > r$. Let $\varepsilon = \frac{\|x_* - c\| - r}{2}$. Then, we have

$$M = \left\{ (n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{nm} \left| \{ k \leq n, l \leq m : \|\Delta x_{kl} - c\| \geq \varepsilon \} \right| < \delta \right\} \notin \mathcal{I}_2.$$

Now, we define the set

$$K = \left\{ (n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{nm} \left| \{ k \leq n, l \leq m : \|\Delta x_{kl} - x_*\| \geq r + \varepsilon \} \right| \geq \delta \right\}.$$

Let $t, s \in M$, i.e.,

$$\frac{1}{ts} \left| \{ k \leq t, l \leq s : \|\Delta x_{kl} - c\| \geq \varepsilon \} \right| < \delta.$$

Hence, for maximum $k \leq t, l \leq s$, $\|x_{kl} - c\| \leq \varepsilon$. Now

$$\|\Delta x_{kl} - x_*\| \geq \|x_* - c\| - \|\Delta x_{kl} - c\| > r + \varepsilon,$$

for all $k \leq t \in M$, $l \leq s \in M$. Therefore, $K \supseteq M$ implies that $K \notin \mathcal{I}_2$, which contradicts the fact that $x_* \in \mathcal{I}_2 - st - LIM^r (\Delta x_{kl})$. Thus, $\|x_* - c\| \leq r$ for all $x_* \in \mathcal{I}_2 - st - LIM^r (\Delta x_{kl})$ and $c \in \mathcal{I}_2$-S$\big(\Gamma_{(\Delta x_{kl})}\big)$. $\square$

Theorem 3.10. *A double sequence (Δx_{kl}) is rough $\mathcal{I}_2$-statistical convergent to x_* if and only if $\mathcal{I}_2 - st - \mathrm{LIM}^r (\Delta x_{kl}) = \overline{B}_r (x_*)$.*

Proof. The necessary part of the theorem is already proved in the 2nd part of *Theorem 2*. For the sufficiency, let $\mathcal{I}_2 - st - LIM^r (\Delta x_{kl}) = \overline{B}_r (x_*) \neq \emptyset$. Thus, the sequence (Δx_{kl}) is $\mathcal{I}_2$-statistically bounded. Suppose that (Δx_{kl}) has another $\Delta\mathcal{I}_2$-statistical cluster point x'_* different from x_*. The point

$$\overline{x}_* = x_* + \frac{r}{\|x_* - x'_*\|} (x_* - x'_*)$$

satisfies, $\|\overline{x}_* - x'_*\| > r$. Since, $x'_* \in \mathcal{I}_2 - S\left(\Gamma_{(\Delta x_{kl})}\right)$, by *Theorem 6*, $\overline{x}_* \notin \mathcal{I}_2 - st - LIM^r (\Delta x_{kl})$. But this is impossible as

$$\|\overline{x}_* - x'_*\| = r \text{ and } \mathcal{I}_2 - st - LIM^r (\Delta x_{kl}) = \overline{B}_r (x_*).$$

Therefore x_* is the unique $\mathcal{I}_2$-statistical cluster point of (Δx_{kl}). Also, (Δx_{kl}) is $\mathcal{I}_2$-bounded. So, by *Theorem 1*, (Δx_{kl}) is rough $\mathcal{I}_2$-statistical convergent to x_*. $\square$

Theorem 3.11. *Let $r > 0$. Then, a double sequence (Δx_{kl}) is rough $\mathcal{I}_2$-statistical convergent to x_* if and only if there exists a double sequence (Δy_{kl}) such that $\mathcal{I}_2 - st - \mathrm{LIM}^r (\Delta y_{kl}) = x_*$ and $\|\Delta x_{kl} - \Delta y_{kl}\| \leq r$, for all $(k,l) \in \mathbb{N} \times \mathbb{N}$.*

Proof. Necessity: Let $\Delta x \overset{r-\mathcal{I}_2-st}{\longrightarrow} x_*$. Then, we have,

$$\mathcal{I}_2 - st \limsup \|\Delta x_{kl} - x_*\| \leq r. \tag{1}$$

Now we define,

$$\Delta y_{kl} := \begin{cases} x_*, & \text{if } \|\Delta x_{kl} - x_*\| \leq r, \\ \Delta x_{kl} + r\frac{x_* - \Delta x_{kl}}{\|\Delta x_{kl} - x_*\|}, & \text{otherwise.} \end{cases}$$

Then,

$$\|\Delta y_{kl} - x_*\| = \begin{cases} 0, & \text{if } \|\Delta x_{kl} - x_*\| \leq r, \\ \\ \|\Delta x_{kl} - x_*\| - r, & \text{otherwise} \end{cases} \tag{2}$$

by the definition of y_{kl} we have $\|\Delta x_{kl} - \Delta y_{kl}\| \leq r$, for all $(k,l) \in \mathbb{N} \times \mathbb{N}$. Now by (1) and (2) we get,

$$\left\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{nm} \left|\{k \leq n, l \leq m : \|\Delta y_{kl} - x_*\| \geq \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}_2$$

which implies that $\mathcal{I}_2 - st - LIM^r(\Delta y_{kl}) = x_*$.

Sufficiency: Assume that the given condition holds. For any $\varepsilon > 0$ and $\delta > 0$ the set

$$M(\varepsilon, \delta) = \left\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{nm} \left|\{k \leq n, l \leq m : \|\Delta y_{kl} - x_*\| \geq \varepsilon\}\right| < \delta\right\} \in \mathcal{I}_2$$

and $\|\Delta x_{kl} - \Delta y_{kl}\| \leq r$, for all $(k,l) \in \mathbb{N} \times \mathbb{N}$. Therefore,

$$\|\Delta x_{kl} - x_*\| = \|\Delta x_{kl} - \Delta y_{kl}\| + \|\Delta y_{kl} - x_*\| < r + \varepsilon,$$

for maximum $k \leq t \in M^c(\varepsilon, \delta), l \leq s \in M^c(\varepsilon, \delta)$. This shows that

$$\left\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{nm} \left|\{k \leq n, l \leq m : \|\Delta x_{kl} - x_*\| \geq r + \varepsilon\}\right| < \delta\right\} \supseteq M^c(\varepsilon, \delta) \in \mathcal{F}(\mathcal{I}_2)$$

and so, r-$\mathcal{I}_2$-$st \lim \Delta x_{kl} = x_*$. □

Corollary 3.12. *If $(X, \|.\|)$ is a strictly convex space, (Δx_{kl}) is a double sequence in X and there exists $y_1, y_2 \in \mathcal{I}_2 - st - \mathrm{LIM}^r_{(\Delta x_{kl})}$ such that $\|y_1 - y_2\| = 2r$, then this sequence is rough $\mathcal{I}_2$-statistically convergent to $\frac{y_1+y_2}{2}$.*

The proof is straightforward and so is omitted.

Theorem 3.13. *(i) If $c \in \Gamma_{(\Delta x_{kl})}(\mathcal{I}_2)$, then $\mathcal{I}_2 - st - \mathrm{LIM}^r_{(\Delta x_{kl})} \subseteq \overline{B}_r(c)$.*

(ii) $\mathcal{I}_2$-st-$\mathrm{LIM}^r_{(\Delta x_{kl})} = \bigcap\limits_{c \in \Gamma_{(\Delta x_{kl})}(\mathcal{I}_2)} \overline{B}_r(c) = \left\{x_* \in \mathbb{R}^n : \Gamma_{(\Delta x_{kl})}(\mathcal{I}_2) \subseteq \overline{B}_r(x_*)\right\}$.

Proof. (i) If $x_* \in \mathcal{I}_2 - st - LIM^r(\Delta x_{kl})$ and $c \in \Gamma_{(\Delta x_{kl})}(\mathcal{I}_2)$, then $\|x_* - c\| \leq r$. Hence the result follows.

(ii) By (i) we can write

$$\mathcal{I}_2 - st - LIM^r(\Delta x_{kl}) \subseteq \bigcap_{c \in \Gamma_{(\Delta x_{kl})}(\mathcal{I}_2)} \overline{B}_r(c).$$

Assume that $y \in \bigcap\limits_{c \in \Gamma_{(\Delta x_{kl})}(\mathcal{I}_2)} \overline{B}_r(c)$. We have $\|y - c\| \leq r$ for all $c \in \Gamma_{(\Delta x_{kl})}(\mathcal{I}_2)$ and so

$$\Gamma_{(\Delta x_{kl})}(\mathcal{I}_2) \subseteq \overline{B}_r(x_*).$$

Then, clearly

$$\bigcap_{c\in\Gamma_{(\Delta x_{kl})}(\mathcal{I}_2)} \overline{B}_r(c) = \left\{x_* \in \mathbb{R}^n : \Gamma_{(\Delta x_{kl})}(\mathcal{I}_2) \subseteq \overline{B}_r(x_*)\right\}.$$

If possible let $y \notin \mathcal{I}_2 - st - LIM^r(\Delta x_{kl})$. Then, there exists an $\varepsilon > 0$ such that

$$K = \left\{(n,m) \in \mathbb{N}\times\mathbb{N} : \frac{1}{nm}\left|\{k \leq n, l \leq m : \|\Delta x_{kl} - y\| \geq r+\varepsilon\}\right| < \delta\right\} \notin \mathcal{I},$$

which implies the existence of an $\mathcal{I}_2$-cluster point c of the sequence (Δx_{kl}) with $\|y-c\| \geq r+\varepsilon$. Hence

$$\Gamma_{(\Delta x_{kl})}(\mathcal{I}_2) \subseteq \overline{B}_r(y) \ \ and\ y \notin \left\{x_* \in \mathbb{R}^n : \Gamma_{(\Delta x_{kl})}(\mathcal{I}_2) \subseteq \overline{B}_r(x_*)\right\}.$$

Finally the fact that $y \in \mathcal{I}_2 - st - LIM^r(\Delta x_{kl})$ follows from the observation that

$$y \in \left\{x_* \in \mathbb{R}^n : \Gamma_{(\Delta x_{kl})}(\mathcal{I}_2) \subseteq \overline{B}_r(x_*)\right\}.$$

□

4. Conclusion

The rough convergence have been recently studied by several authors. In view of the recent applications of ideals in the theory of convergence of sequences, it seems very natural to extend the interesting concept of rough $\Delta\mathcal{I}_2$-statistical convergence further by using ideals which we mainly do here and investigate some properties of this new type convergence. So, we have extended some well-known results.

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