Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 12 Issue 1(2020), Pages 51-70.

A DISPERSION INEQUALITY AND ACCUMULATED SPECTROGRAMS IN THE WEINSTEIN SETTING

KHALED HLEILI

Abstract. In this paper, we establish a quantitative version of Shapiro's mean dispersion theorem for the continuous wavelet transform. Next, we prove the boundedness and compactness properties of the localization operators associated with Weinstein wavelet transforms. Finally, we study the scalograms for the same wavelet transform.

1. Introduction

Time-frequency localization is an ongoing active topic of research in harmonic analysis. In [7], Daubechies introduced time-frequency localization operators obtained by restricting the integral in the inversion formula to a subset of $\mathbb{R}^2$. The eigenfunctions and eigenvalues of these operators have been studied in ([8],[9]). The study of the properties of time-frequency operators and its connection with other mathematical topics have been a continued topic of research, e.g. [1, 6, 16].

In [23], Shapiro studied the localization for an orthonormal sequence of functions and showed that if the means and the dispersions of an orthonormal sequence $(\phi_k)_k$ in $L^2(R^d)$ and their Fourier transforms $(\hat{\phi})_k$ are uniformly bounded, then the sequence $(\phi_k)_k$ is necessarily finite. In [18], Jaming and Powell used the Rayleigh-Ritz technique for estimating eigenvalues of operators to give a quantitative version of Shapiro's theorem. Recently, Malinnikova [19] obtained a quantitative multivariable version of Shapiro's theorem for generalized dispersions by showing that if $(\phi_k)_k$ is an orthonormal sequence in $L^2(R^d)$ then for every positive real number p, there exists a constant $C_{p,d}$ such that for every $n \in \mathbb{N}^*$

$$\sum_{k=1}^{n} \|x^{\frac{p}{2}}\phi_k\|_2^2 + \|y^{\frac{p}{2}}\hat{\phi_k}\|_2^2 \geqslant C_{p,d} n^{1+\frac{p}{2d}}.$$

Some other results on time-frequency localisation of orthonormal sequences have been recently obtained by Ghobber and Omri (see[11, 12, 14]).

2000 *Mathematics Subject Classification.* 42B10, 42C40, 42C25.

Key words and phrases. Weinstein operator; wavelet transform; Schapiro's theorem; Weinstein wavelet Scalograms.

Submitted May 7, 2019. Published February 10, 2020.
Communicated by Paul Mueller.

We consider the Weinstein operator (see[3, 4]) defined on $\mathbb{R}^n\times]0,+\infty[$ by

$$\Delta_W=\sum_{j=1}^{n+1}\frac{\partial^2}{\partial x_j^2}+\frac{2\alpha+1}{x_{n+1}}\frac{\partial}{\partial x_{n+1}}=\Delta_n+\ell_\alpha,\quad \alpha>\frac{-1}{2},$$

where Δ_n is the Laplacian operator in $\mathbb{R}^n$ and ℓ_α the Bessel operator with respect to the variable x_{n+1} defined by

$$\ell_\alpha=\frac{\partial^2}{\partial x_{n+1}^2}+\frac{2\alpha+1}{x_{n+1}}\frac{\partial}{\partial x_{n+1}},\quad \alpha>\frac{-1}{2}.$$

For $n>2$, the operator Δ_W is the Laplace-Beltrami operator on the Riemanian space $\mathbb{R}^n\times]0,+\infty[$ equipped with the metric [3]

$$ds^2=x_{n+1}^{\frac{2(2\alpha+1)}{n-1}}\sum_{i=1}^{n+1}dx_i^2.$$

The Weinstein operator Δ_W has several applications in pure and applied Mathematics especially in Fluid Mechanics (see e.g. [5, 24]). The harmonic analysis associated with the Weinstein operator is studied by Ben Nahia and Ben Salem (see [3, 4]). In particular the authors have introduced and studied the generalized Fourier transform associated with the Weinstein operator. This transform is called the Weinstein transform.

In this paper, we prove a quantitative version of Shapiro's mean dispersion theorem for the continuous wavelet transform. Next, we establish the boundedness and compactness properties of the localization operators and we study the scalograms for the continuous wavelet transform.

This paper is arranged as follows. In section 2, we recall some harmonic analysis results related to the Weinstein operator and we express wavelet transform associated with the Weinstein operator, which was introduced in [21]. In section 3, we will show an analogue of Shapiro's umbrella theorem for the continuous wavelet transforms. In Section 4, we introduce the wavelet localization operators in the setting of the Weinstein operator, more precisely some properties of the localization wavelet operators are established. In the last section, we study the scalograms associated with the Weinstein continuous wavelet transform.

2. Preliminaries

In order to set up basic and standard notation we briefly overview the Weinstein operator and related harmonic analysis. Main references are [3, 4].

2.1. **Harmonic analysis associated with the Weinstein transform.** In the following we denote by

- $\mathbb{R}^{n+1}_+=\mathbb{R}^n\times[0,+\infty[$.
- $x=(x_1,...,x_n,x_{n+1})=(x',x_{n+1})\in\mathbb{R}^{n+1}_+$.
- $-x=(-x',x_{n+1})$
- $\mathcal{C}_e(\mathbb{R}^{n+1})$, the space of continuous functions on $\mathbb{R}^{n+1}$, even with respect to the last variable.
- $S_e(\mathbb{R}^{n+1})$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^{n+1}$, even with respect to the last variable.

• $L^p(d\nu_\alpha), 1 \leqslant p \leqslant +\infty$, the Lebesgue space constituted of measurable functions f on $\mathbb{R}_+^{n+1}$ such that

$$\|f\|_{\nu_\alpha,p} = \left(\int_{\mathbb{R}_+^{n+1}} |f(x)|^p d\nu_\alpha(x)\right)^{\frac{1}{p}} < +\infty, 1 \leqslant p < +\infty,$$

$$\|f\|_{\infty,p} = \text{ess} \sup_{x\in\mathbb{R}_+^{n+1}} |f(x)| < +\infty.$$

• $d\nu_\alpha$ is the measure defined on $\mathbb{R}_+^{n+1}$ by

$$d\nu_\alpha(x) = x_{n+1}^{2\alpha+1}dx = x_{n+1}^{2\alpha+1}dx'dx_{n+1}.$$

We consider the Weinstein operator Δ_W defined on $\mathbb{R}_+^{n+1}$ by

$$\Delta_W = \sum_{j=1}^{n+1} \frac{\partial^2}{\partial x_j^2} + \frac{2\alpha+1}{x_{n+1}} \frac{\partial}{\partial x_{n+1}}, \quad \alpha > \frac{-1}{2}.$$

Then

$$\Delta_W = \Delta_n + \ell_\alpha,$$

where Δ_n is the Laplacian operator in $\mathbb{R}^n$ and ℓ_α the Bessel operator with respect to the variable x_{n+1} defined by

$$\ell_\alpha = \frac{\partial^2}{\partial x_{n+1}^2} + \frac{2\alpha+1}{x_{n+1}} \frac{\partial}{\partial x_{n+1}}.$$

The Weinstein kernel Λ is given by

$$\forall (x,\lambda) \in \mathbb{R}^{n+1} \times \mathbb{C}^{n+1}, \quad \Lambda(x,\lambda) = j_\alpha(\lambda_{n+1}x_{n+1})e^{i\langle\lambda',x'\rangle},$$

where j_α is the spherical Bessel function defined by

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\Gamma(\alpha+1+k)} (\frac{z}{2})^{2k}, \ z \in \mathbb{C}.$$

The Weinstein kernel satisfies the following properties:
(i) For all $z, t \in \mathbb{C}^{n+1}$, we have

$$\Lambda(z,t) = \Lambda(t,z), \quad \Lambda(z,0) = 1 \quad \text{and} \quad \Lambda(\lambda z,t) = \Lambda(z,\lambda t), \forall \lambda \in \mathbb{C}.$$

(ii)

$$\forall x, y \in \mathbb{R}^{n+1}, \quad |\Lambda(x,y)| \leqslant 1.$$

2.2. **The Fourier-Weinstein transform.**

Definition 2.1. The Weinstein transform is given for $f \in L^1(d\nu_\alpha)$ by

$$\forall \lambda \in \mathbb{R}_+^{n+1}, \quad \mathscr{F}_W(f)(\lambda) = \int_{\mathbb{R}_+^{n+1}} f(x)\Lambda(-x,\lambda)d\nu_\alpha(x).$$

Some basic properties of this transform are as follows. For the proofs, we refer [3, 4].

• For every $f \in L^1(d\nu_\alpha)$, the function $\mathscr{F}_W(f)$ is continuous on $\mathbb{R}_+^{n+1}$ and we have

$$\|\mathscr{F}_W(f)\|_{\nu_\alpha,\infty} \leqslant \|f\|_{\nu_\alpha,1}.$$

• Let $f \in L^1(d\nu_\alpha)$ such that $\mathscr{F}_W(f) \in L^1(d\nu_\alpha)$, then for almost every $x \in \mathbb{R}_+^{n+1}$

$$f(x) = C_{\alpha,n} \int_{\mathbb{R}_+^{n+1}} \mathscr{F}_W(f)(\lambda)\Lambda(\lambda,x)d\nu_\alpha(\lambda),$$

where

$$C_{\alpha,n} = \frac{1}{\pi^n 4^{\alpha+\frac{n}{2}}(\Gamma(\alpha+1))^2}.$$

• For all $f, g \in S_e(\mathbb{R}^{n+1})$, we have

$$\int_{\mathbb{R}_+^{n+1}} f(x)\overline{g(x)}d\nu_\alpha(x) = C_{\alpha,n}\int_{\mathbb{R}_+^{n+1}} \mathscr{F}_W(f)(\lambda)\overline{\mathscr{F}_W(g)(\lambda)}d\nu_\alpha(\lambda).$$

• The Weinstein transform $\mathscr{F}_W(f)$ is a topological isomorphism from $S_e(\mathbb{R}^{n+1})$ onto itself and for all $f \in S_e(\mathbb{R}^{n+1})$

$$\int_{\mathbb{R}_+^{n+1}} |f(x)|^2 d\nu_\alpha(x) = C_{\alpha,n}\int_{\mathbb{R}_+^{n+1}} |\mathscr{F}_W(f)(\lambda)|^2 d\nu_\alpha(\lambda).$$

2.3. **The translation operator associated with the Weinstein operator.** The translation operator τ_x, $x = (x', x_{n+1}) \in \mathbb{R}_+^{n+1}$ associated with the Weinstein operator Δ_W is defined for $f \in \mathcal{C}_e(\mathbb{R}^{n+1})$ which is even with respect to the last variable and for all $y = (y', y_{n+1}) \in \mathbb{R}_+^{n+1}$ by

$$\tau_x(f)(y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^\pi f(x'+y', \sqrt{x_{n+1}^2 + y_{n+1}^2 + 2x_{n+1}y_{n+1}\cos\theta})\sin^{2\alpha}(\theta)d\theta. \tag{2.1}$$

In particular for all $x, y \in \mathbb{R}_+^{n+1}$ we have $\tau_x(f)(y) = \tau_y(f)(x)$ and $\tau_0(f) = f$. Moreover for all $L^p(d\nu_\alpha), 1 \leqslant p \leqslant +\infty$, the function $x \longmapsto \tau_x(f)$ belongs to $L^p(d\nu_\alpha)$ and we have

$$\|\tau_x(f)\|_{\nu_\alpha,p} \leqslant \|f\|_{\nu_\alpha,p}.$$

For $f \in L^p(d\nu_\alpha)$, $p = 1$ or 2

$$\mathscr{F}_W(\tau_x(f))(\lambda) = \Lambda(\lambda, x)\mathscr{F}_W(f)(\lambda), \quad x, \lambda \in \mathbb{R}_+^{n+1}.$$

By using the generalized translation, we define the generalized convolution product $f *_W g$ of functions $f, g \in L^1(d\nu_\alpha)$ as follows

$$f *_W g(x) = \int_{\mathbb{R}_+^{n+1}} \tau_{-x}(\check{f})(y)g(y)d\nu_\alpha(y), \quad x \in \mathbb{R}_+^{n+1}, \tag{2.2}$$

where $-x = (-x', x_{n+1})$ and $\check{f}(y) = \check{f}(y', y_{n+1}) = f(-y', y_{n+1})$.
This convolution is commutative and associative. Then (see e.g. [3]), if $1 \leqslant p, q, r \leqslant +\infty$ are such $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$, the function $f *_W g$ belongs to $L^r(d\nu_\alpha)$ and we have the following Young's inequality

$$\|f *_W g\|_{\nu_\alpha,r} \leqslant \|f\|_{\nu_\alpha,p}\|g\|_{\nu_\alpha,q}.$$

This then allows us to define $f *_W g$ for $f \in L^p(d\nu_\alpha)$ and $g \in L^q(d\nu_\alpha)$. Moreover for $f \in L^1(d\nu_\alpha)$ and $g \in L^q(d\nu_\alpha), q = 1 \,\text{or}\, 2$, we have

$$\mathscr{F}_W(f *_W g) = \mathscr{F}_W(f)\mathscr{F}_W(g).$$

Moreover, if f and g are in $L^2(d\nu_\alpha)$, then $f *_W g$ belongs to $\mathcal{C}_{e,0}(\mathbb{R}^{n+1})$ consisting of continuous functions h on $\mathbb{R}^{n+1}$, even with respect to the last variable, such that $\lim_{|x| \longrightarrow +\infty} h(x) = 0$ and we have

$$f *_W g = \mathscr{F}_W^{-1}(\mathscr{F}_W(f)\mathscr{F}_W(g)).$$

Thus, for every $f, g \in L^2(d\nu_\alpha)$, the function $f *_W g$ belongs to $L^2(d\nu_\alpha)$ if and only if $\mathscr{F}_W(f)\mathscr{F}_W(g)$ belongs to $L^2(d\nu_\alpha)$ and in this case, we have

$$\mathscr{F}_W(f *_W g) = \mathscr{F}_W(f)\mathscr{F}_W(g).$$

2.4. **Basic Weinstein Wavelet Theory.** In this subsection, we recall some results introduced and proved by authors in [10].

Definition 2.2. A Weinstein wavelet on $\mathbb{R}_+^{n+1}$ is a measurable function g on $\mathbb{R}_+^{n+1}$ satisfying, for almost all $\lambda \in \mathbb{R}_+^{n+1}$, the condition

$$0 < C_g = \int_0^{+\infty} \left|\mathscr{F}_W(g)(a\lambda)\right|^2 \frac{da}{a} < +\infty.$$

Let $a > 0$ and g be a measurable function. we consider the function g_a defined by

$$\forall s \in \mathbb{R}_+^{n+1}, \quad g_a(s) = \frac{1}{a^{2\alpha+n+2}} g(\frac{s}{a}).$$

Proposition 2.3. *For all $g \in L^p(d\nu_\alpha), 1 \leqslant p \leqslant +\infty$ and $(a, x) \in]0, +\infty[\times\mathbb{R}_+^{n+1}$, the function g_a belongs to $L^p(d\nu_\alpha)$ and we have*

$$\|g_a\|_{\nu_\alpha,p} \leqslant a^{(2\alpha+n+2)\frac{1-p}{p}} \|g\|_{\nu_\alpha,p}.$$

Definition 2.4. Let g be a Weinstein Wavelet on $\mathbb{R}_+^{n+1}$ in $L^2(d\nu_\alpha)$. The Weinstein continuous Wavelet transform on $\mathbb{R}_+^{n+1}$ is defined for regular functions $f \in \mathbb{R}_+^{n+1}$ by

$$S_g^W(f)(a, x) = \int_{\mathbb{R}_+^{n+1}} f(y)\overline{g_{a,x}(y)} d\nu_\alpha(y),$$

where $g_{a,x}, a > 0$ are the family of Weinstein Wavelets on $\mathbb{R}_+^{n+1}$ in $L^2(d\nu_\alpha)$ given by

$$\forall y \in \mathbb{R}_+^{n+1}, \quad g_{a,x}(y) = a^{\alpha+1+\frac{n}{2}} \tau_{-x}(g_a)(y),$$

and τ_{-x} are the Weinstein translation operators given by the relation (2.1).

The Weinstein continuous Wavelet transform can be also be written in the form

$$S_g^W(f)(a, x) = a^{\alpha+1+\frac{n}{2}} f * \overline{\breve{g}_a}(x) == \langle f, g_{a,x}\rangle_{\nu_\alpha}, \tag{2.3}$$

where $*$ is the Weinstein convolution product given by (2.2) and $\langle ., .\rangle_{\nu_\alpha}$ is the usual inner product in the Hilbert space $L^2(d\nu_\alpha)$.

Remark 2.5. Let $g \in L^p(d\nu_\alpha)$ and $p \in [1, +\infty]$, we have

$$\forall (a, x) \in]0, +\infty[\times\mathbb{R}_+^{n+1}, \ \|g_{a,x}\|_{\nu_\alpha,p} \leqslant a^{(2\alpha+n+2)(\frac{1}{p}-\frac{1}{2})} \|g\|_{\nu_\alpha,p}. \tag{2.4}$$

Define the measure γ_α on $]0, +\infty[\times\mathbb{R}_+^{n+1}$ by

$$d\gamma_\alpha(a, x) = a^{2\alpha+n+3} da \otimes d\nu_\alpha(x),$$

and $L^p(d\gamma_\alpha), p \in [1, +\infty]$, the Lebesgue space on $]0, +\infty[\times\mathbb{R}_+^{n+1}$ with respect to the measure γ_α with the L^p-norm denoted by $\|.\|_{\gamma_\alpha,p}$.

Theorem 2.6. *(Orthogonality relation) Let g be a Weinstein Wavelet on $\mathbb{R}_+^{n+1}$ in $L^2(d\nu_\alpha)$. Then for every functions f_1 and f_2 in $L^2(d\nu_\alpha)$, we have*

$$\langle f_1, f_2\rangle_{\nu_\alpha} = \frac{1}{C_g}\langle S_g^W(f_1), S_g^W(f_2)\rangle_{\gamma_\alpha}, \tag{2.5}$$

where $\langle ., .\rangle_{\gamma_\alpha}$ is the usual inner product in the Hilbert space $L^2(d\gamma_\alpha)$.

Corollary 2.7. *Let g be a Weinstein Wavelet on $\mathbb{R}_+^{n+1}$ in $L^2(d\nu_\alpha)$.*
(i) Then the normalized Weinstein continuous wavelet transform $\frac{1}{\sqrt{C_g}}S_g^W$ is an isometry from $L^2(d\nu_\alpha)$ into a subspace of $L^2(d\gamma_\alpha)$. In particular, we have the following Plancherel formula

$$\int_{\mathbb{R}_+^{n+1}} |f(x)|^2 d\nu_\alpha(x) = \frac{1}{C_g}\int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}} |S_g^W(f)(a,x)|^2 d\gamma_\alpha(a,x).$$

The adjoint of the wavelet transform is

$$(S_g^W)^*(h)(x) = \frac{1}{C_g}\int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}} h(a,y)g_{a,y}(x)d\gamma_\alpha(a,y), \quad x\in\mathbb{R}_+^{n+1}.$$

inverts the Weinstein continuous wavelet transform on its range.
(ii) For all $f\in L^1(d\nu_\alpha)\cap L^2(d\nu_\alpha)$ such that $\mathscr{F}_W(f)\in L^1(d\nu_\alpha)$, we have

$$f = (S_g^W)^*(S_g^W(f)). \tag{2.6}$$

Proposition 2.8. *Let g be a Weinstein Wavelet on $\mathbb{R}_+^{n+1}$ in $L^2(d\nu_\alpha)$. Then, $S_g^W(L^2(d\nu_\alpha))$ is a reproducing kernel Hilbert space in $L^2(d\gamma_\alpha)$ with kernel*

$$\mathcal{K}_g((a',x');(a,x)) = \frac{1}{C_g}\int_{\mathbb{R}_+^{n+1}} g_{a',x'}(y)\overline{g_{a,x}(y)}d\nu_\alpha(y) \tag{2.7}$$

The kernel $\mathcal{K}_g$ is bounded and we have

$$\forall (a,x),(a',x')\in]0,+\infty[\times\mathbb{R}_+^{n+1},\quad |\mathcal{K}_g((a',x');(a,x))| \leqslant \frac{\|g\|_{\nu_\alpha,2}^2}{C_g}.$$

Notation We denote by:
•P_g the orthogonal projection from $L^2(d\gamma_\alpha)$ onto $S_g^W(L^2(d\nu_\alpha))$.
•$P_\Sigma F = \chi_\Sigma F$, $F\in L^2(d\gamma_\alpha)$, the orthogonal projection from $L^2(d\gamma_\alpha)$ onto the subspace of functions of $L^2(d\gamma_\alpha)$ supported in a subset $\Sigma\subset]0,+\infty[\times\mathbb{R}_+^{n+1}$ satisfying

$$0 < \gamma_\alpha(\Sigma) = \int\int_\Sigma d\gamma_\alpha(a,y) < \infty,$$

where χ_Σ is the characteristic function of Σ.
•$\ell^p(\mathbb{N})$, $1\leqslant p\leqslant\infty$, the set of all infinite sequences of real (or complex) numbers $x=(x_j)_{j\in\mathbb{N}}$, such that
$\|x\|_p^p = \sum_{j=1}^{+\infty}|x_j|^p < \infty$, $1\leqslant p<\infty$.
$\|x\|_\infty = \sup_{j\in\mathbb{N}}|x_j| < \infty$.
•$\mathcal{B}(L^p(d\nu_\alpha))$, $1\leqslant p\leqslant\infty$, the space of bounded operators from $L^p(d\nu_\alpha)$ into itself.

Definition 2.9. (i) The singular values $(s_n(K))_{n\in\mathbb{N}}$ of a compact operator K in $\mathcal{B}(L^p(d\nu_\alpha))$ are the eigenvalues of the positive self-adjoint operator $|K| = \sqrt{K*K}$.

(ii) For $1 \leqslant p \leqslant \infty$, the Schatten class S_p is the space of all compact operators whose singular values lie in $\ell^p(\mathbb{N})$. The space S_p is equipped with the norm

$$\|K\|_{S_p} = \left(\sum_{k=1}^{+\infty} |s_k|^p\right)^{\frac{1}{p}}.$$

We note that the space S_2 is the space of Hilbert-Schmidt operators, and S_1 is the space of trace class operators.
The trace of an operator K in S_1 is defined by

$$Tr(K) = \sum_{n=1}^{+\infty} \langle K v_n, v_n\rangle_{\nu_\alpha} \tag{2.8}$$

where $(v_n)_n$ is any orthonormal basis of $L^2(d\nu_\alpha)$.
If K is positive, then

$$\|K\|_{S_1} = Tr(K). \tag{2.9}$$

Moreover, a compact operator K on the Hilbert space $L^2(d\nu_\alpha)$ is Hilbert-Schmidt, then the positive operator K^*K is in the space of trace class and

$$\|K\|_{HS}^2 = \|K^*K\|_{S_1} = Tr(K^*K) = \sum_{n=1}^{+\infty} \|Kv_n\|_{\nu_\alpha,2}^2,$$

for any orthonormal basis $(v_n)_n$ of $L^2(d\nu_\alpha)$.
We define $S_\infty = \mathcal{B}(L^2(d\nu_\alpha))$, equipped with the norm

$$\|K\|_{S_\infty} = \sup_{g \in L^2(d\nu_\alpha);\, \|g\|_{\nu_\alpha,2}=1} \|Kg\|_{\nu_\alpha,2}.$$

3. Mean dispersion theorem for the S_g^W

In this section, we express an uncertainty principle by means of the generalized time-phase dispersion of S_g^W. For this, let p be a positive real number, g be a Weinstein wavelet and $f \in L^2(d\nu_\alpha)$, we define the generalized pth-time-phase dispersion of $S_g^W(f)$ by

$$\varrho_p(S_g^W(f)) = \left(\int_0^{+\infty} \int_{\mathbb{R}_+^{n+1}} |(\frac{1}{a}, x)|^p |S_g^W(f)(a,x)|^2 d\gamma_\alpha(a,x)\right)^{\frac{1}{p}}.$$

Theorem 3.1. [20] *Let g be a Weinstein wavelet and let Σ be a subset of $]0,+\infty[\times\mathbb{R}_+^{n+1}$ such that $0 < \gamma_\alpha(\Sigma) < +\infty$.*
(i) If $\|P_\Sigma P_g\| < 1$, then for every f in $L^2(d\nu_\alpha)$, we have

$$\sqrt{C_g}\|f\|_{\nu_\alpha,2} \leqslant \frac{1}{\sqrt{1 - \|P_\Sigma P_g\|^2}} \|\chi_\Sigma^c S_g^W(f)\|_{\gamma_\alpha,2}.$$

(ii) $P_\Sigma P_g$ is a Hilbert-Schmidt operator and we have

$$\|P_\Sigma P_g\|^2 \leqslant \frac{\gamma_\alpha(\Sigma)}{C_g} \|g\|_{\nu_\alpha,2}^2.$$

Remark 3.2. The orthogonal projector $P_g : L^2(d\gamma_\alpha) \longrightarrow S_g^W(L^2(d\nu_\alpha))$ is given by $S_g^W(S_g^W)^*$. Explicitly, it is an integral operator

$$P_gF(a,y) = \int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}} F(a',y')\mathcal{K}_g((a,y);(a',y'))d\gamma_\alpha(a',y'),\ (a,y)\in]0,+\infty[\times\mathbb{R}_+^{n+1}. \tag{3.1}$$

Clearly, the kernel is hermitian symmetric

$$\overline{\mathcal{K}_g((a',y');(a,y))} = \mathcal{K}_g((a,y);(a',y')), \tag{3.2}$$

and has the reproducing property

$$F(a,y) = \langle F, \mathcal{K}_g((.,.);(a,y))\rangle_{\gamma_\alpha}, \quad F\in S_g^W(L^2(d\nu_\alpha)).$$

Theorem 3.3. *Let g be a Weinstein wavelet, $(\phi_\beta)_{\beta\in\mathbb{N}^{n+1}}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$ and Σ be a measurable subset of $]0,+\infty[\times\mathbb{R}_+^{n+1}$ such that $0<\gamma_\alpha(\Sigma)<+\infty$. Then for every non-empty subset $K\subset\mathbb{N}^{n+1}$, we have*

$$\sum_{\beta\in K}(1-\|\chi_{\Sigma^c}S_g^W(\phi_\beta)\|_{\gamma_\alpha,2}) \leqslant \frac{\gamma_\alpha(\Sigma)}{C_g}\|g\|^2_{\nu_\alpha,2}.$$

Proof. Let $(e_\beta)_{\beta\in\mathbb{N}^{n+1}}$ be an orthonormal basis of $L^2(d\gamma_\alpha)$, since $P_\Sigma P_g$ is a Hilbert-Schmidt operator satisfying $\|P_\Sigma P_g\|^2_{HS}\leqslant\frac{\gamma_\alpha(\Sigma)}{C_g}\|g\|^2_{\nu_\alpha,2}$, we deduce that the positive operator $P_gP_\Sigma P_g$ satisfies

$$\sum_{\beta\in\mathbb{N}^{n+1}}\langle P_gP_\Sigma P_g e_\beta, e_\beta\rangle_{\gamma_\alpha} = \sum_{\beta\in\mathbb{N}^{n+1}}\|P_\Sigma P_g e_\beta\|^2_{\gamma_\alpha,2} = \|P_\Sigma P_g\|^2_{HS} < +\infty,$$

which means according to Gohberg et al. [15, p. 63], that $P_gP_\Sigma P_g$ is a trace class operator with

$$Tr(P_gP_\Sigma P_g) = \|P_\Sigma P_g\|^2_{HS} \leqslant \frac{\gamma_\alpha(\Sigma)}{C_g}\|g\|^2_{\nu_\alpha,2}.$$

Actually since $(\phi_\beta)_{\beta\in K}$ is an orthonormal sequence in $L^2(d\nu_\alpha))$, then by (2.5) we deduce that $(S_g^W(\phi_\beta))_{\beta\in K}$ is an orthonormal sequence in $L^2(d\gamma_\alpha)$ and therefore again by Gohberg et al. [15, p. 63], we get that

$$\begin{aligned}\sum_{\beta\in K}\langle P_\Sigma S_g^W(\phi_\beta), S_g^W(\phi_\beta)\rangle_{\gamma_\alpha} &= \sum_{\beta\in K}\langle P_gP_\Sigma P_g S_g^W(\phi_\beta), S_g^W(\phi_\beta)\rangle_{\gamma_\alpha}\\ &\leqslant Tr(P_gP_\Sigma P_g),\end{aligned} \tag{3.3}$$

and by combining (5.5) and (3.3), we obtain

$$\sum_{\beta\in K}\langle P_\Sigma S_g^W(\phi_\beta), S_g^W(\phi_\beta)\rangle_{\gamma_\alpha} \leqslant \frac{\gamma_\alpha(\Sigma)}{C_g}\|g\|^2_{\nu_\alpha,2}. \tag{3.4}$$

Now, by Cauchy-Schwarz inequality, we have for every $\beta\in K$

$$\begin{aligned}\langle P_\Sigma S_g^W(\phi_\beta), S_g^W(\phi_\beta)\rangle_{\gamma_\alpha} &= 1-\langle P_{\Sigma^c} S_g^W(\phi_\beta), S_g^W(\phi_\beta)\rangle_{\gamma_\alpha}\\ &\geqslant 1-\|\chi_{\Sigma^c}S_g^W(\phi_\beta)\|_{\gamma_\alpha,2},\end{aligned}$$

and (3.4) completes the proof of the theorem. □

Definition 3.4. Let $0<\eta<1$ and Σ be a measurable subset of $]0,+\infty[\times\mathbb{R}_+^{n+1}$. Let g be a Weinstein wavelet and $f\in L^2(d\nu_\alpha)$ be a nonzero function. We say that S_g^W is η-time- frequency concentrated in Σ, if

$$\|\chi_{\Sigma^c}S_g^W(f)\|_{\gamma_\alpha,2}\leqslant\eta\|S_g^W(f)\|_{\gamma_\alpha,2}.$$

Proposition 3.5. *Let η and ρ be positive real numbers such that $\eta<\frac{1}{\sqrt{C_g}}$, $B_\rho=\{(a,x)\in]0,+\infty[\times\mathbb{R}_+^{n+1},\ \frac{1}{a^2}+|x|^2\leqslant\rho^2\}$ and let g be a Weinstein wavelet. Let $\Omega\subset\mathbb{N}^{n+1}$ be a non-empty subset and $(\phi_\beta)_{\beta\in\mathbb{N}^{n+1}}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$. If $S_g^W(\phi_\beta)$ is η-time-frequency concentrated in the set B_ρ, then Ω is finite and*

$$Card(\Omega)\leqslant\frac{\|g\|^2_{\nu_\alpha,2}}{C_g(1-\eta\sqrt{C_g})}\frac{\pi^{\frac{n}{2}}\Gamma(\alpha+1)\Gamma(\frac{2\alpha+n+4}{2})}{4\Gamma(2\alpha+n+4)}\rho^{4\alpha+2n+6}.$$

Proof. According to Theorem 3.3, we have

$$\sum_{\beta\in K}(1-\|\chi_{B_\rho^c}S_g^W(\phi_\beta)\|_{\gamma_\alpha,2})\leqslant\frac{\gamma_\alpha(B_\rho)}{C_g}\|g\|^2_{\nu_\alpha,2}. \tag{3.5}$$

However for every $\beta\in\Omega$, $\|\chi_{B_\rho^c}S_g^W(\phi_\beta)\|_{\gamma_\alpha,2}\leqslant\eta\sqrt{C_g}$ and

$$\gamma_\alpha(B_\rho)=\frac{\pi^{\frac{n}{2}}\Gamma(\alpha+1)\Gamma(\frac{2\alpha+n+4}{2})}{4\Gamma(2\alpha+n+4)}\rho^{4\alpha+2n+6}. \tag{3.6}$$

By combining (3.5) and (3.6), we obtain

$$Card(\Omega)\leqslant\frac{\|g\|^2_{\nu_\alpha,2}}{C_g(1-\eta\sqrt{C_g})}\frac{\pi^{\frac{n}{2}}\Gamma(\alpha+1)\Gamma(\frac{2\alpha+n+4}{2})}{4\Gamma(2\alpha+n+4)}\rho^{4\alpha+2n+6},$$

which means that Ω is finite. □

Corollary 3.6. *Let A and p be positive real numbers and g be a Weinstein wavelet. Let $\Omega\subset\mathbb{N}^{n+1}$ be a non-empty subset and $(\phi_\beta)_{\beta\in\mathbb{N}^{n+1}}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$. If the sequence $(\varrho_p(S_g^W(\phi_\beta))_{\beta\in K}$ is uniformly bounded by A, then Ω is finite and*

$$Card(\Omega)\leqslant\frac{\|g\|^2_{\nu_\alpha,2}}{C_g}\frac{\pi^{\frac{n}{2}}2^{\frac{2}{p}(4\alpha+2n+6)}\Gamma(\alpha+1)\Gamma(\frac{2\alpha+n+4}{2})}{2\Gamma(2\alpha+n+4)}A^{4\alpha+2n+6}.$$

Proof. Assume that $\varrho_p(S_g^W(\phi_\beta)\leqslant A$, then for every $\beta\in\Omega$ we have

$$\begin{aligned}\|\chi_{B^c_{A2^{\frac{2}{p}}}}S_g^W(\phi_\beta)\|^2_{\gamma_\alpha,2}&\leqslant\int\int_{|(\frac{1}{a},x)|\geqslant A2^{\frac{2}{p}}}|(\frac{1}{a},x)|^{-p}|(\frac{1}{a},x)|^p|S_g^W(\phi_\beta)(a,x)|^2d\gamma_\alpha(a,x)\\&\leqslant\frac{(\varrho_p(S_g^W(\phi_\beta)))^p}{4A^p}\leqslant\frac{1}{4}.\end{aligned}$$

This means that for every $\beta\in\Omega$, $S_g^W(\phi_\beta)$ is $\frac{1}{2\sqrt{C_g}}$-concentrated in the set $B_{A2^{\frac{2}{p}}}$ and by Proposition 3.5, we deduce that Ω is finite and

$$Card(\Omega)\leqslant\frac{\|g\|^2_{\nu_\alpha,2}}{C_g}\frac{\pi^{\frac{n}{2}}2^{\frac{2}{p}(4\alpha+2n+6)}\Gamma(\alpha+1)\Gamma(\frac{2\alpha+n+4}{2})}{2\Gamma(2\alpha+n+4)}A^{4\alpha+2n+6}.$$

□

Lemma 3.7. *Let p be a positive real number and g be a Weinstein wavelet. If $(\phi_\beta)_{\beta\in\mathbb{N}^{n+1}}$ is an orthonormal sequence in $L^2(d\nu_\alpha)$, then there exists $j_0 \in \mathbb{Z}$ such that*

$$\forall \beta \in \mathbb{N}^{n+1}, \quad \varrho_p(S_g^W(\phi_\beta) \geqslant 2^{j_0}.$$

Proof. For every $j \in \mathbb{Z}$, let

$$P_j = \{\beta \in \mathbb{N}^{n+1};\ \varrho_p(S_g^W(\phi_\beta) \in [2^{j-1}, 2^j[\},$$

then for every $\beta \in P_j$, we have

$$\int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}} |(\frac{1}{a}, x)|^p |S_g^W(\phi_\beta)(a,x)|^2 d\gamma_\alpha(a,x) \leqslant 2^{jp},$$

hence

$$\int\int_{|(\frac{1}{a},x)|\geqslant 2^{j+\frac{2}{p}}} |(\frac{1}{a}, x)|^p |S_g^W(\phi_\beta)(a,x)|^2 d\gamma_\alpha(a,x) \leqslant \frac{(\varrho_p(S_g^W(\phi_\beta)))^p}{2^{pj}4} \leqslant \frac{1}{4}.$$

By the last inequality, we deduce that for every $\beta \in P_j$, $S_g^W(\phi_\beta)$ is $\frac{1}{2\sqrt{C_g}}$-concentrated in the ball $B_{2^{j+\frac{2}{p}}}$, since the sequence $(\phi_\beta)_{\beta\in P_j}$ satisfies the conditions of Corollary 3.6, which shows that P_j is finite and

$$Card(P_j) \leqslant \frac{\|g\|^2_{\nu_\alpha,2}}{C_g} \frac{\pi^{\frac{n}{2}} 2^{\frac{2}{p}(4\alpha+2n+6)}\Gamma(\alpha+1)\Gamma(\frac{2\alpha+n+4}{2})}{2\Gamma(2\alpha+n+4)} 2^{j(4\alpha+2n+6)}, \qquad (3.7)$$

in particular $\lim_{j\longrightarrow -\infty} Card(P_j) = 0$. □

Theorem 3.8. *(Shapiro's dispersion theorem for S_g^W) Let g be a Weinstein wavelet and $(\phi_\beta)_{\beta\in\mathbb{N}^{n+1}}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$. Then, for every positive real number p and for every non-empty subset $\Omega \subset \mathbb{N}^{n+1}$, we have*

$$\sum_{\beta\in\Omega} \varrho_p^p(S_g^W(\phi_\beta)) \geqslant (Card(\Omega))^{1+\frac{p}{4\alpha+2n+6}} \frac{1}{2^{p+1}} \left(\frac{2C_g}{C_{\alpha,p,n}\|g\|^2_{\nu_\alpha,2}}\right)^{\frac{p}{4\alpha+2n+6}}.$$

Proof. For $k \in \mathbb{Z}, k \geqslant j_0$, we denote by $Q_k = \cup_{j=j_0}^k P_j$ then according to (3.7), we have

$$\begin{aligned} Card(Q_k) &= \sum_{j=j_0}^k Card(P_j) \\ &\leqslant \frac{\|g\|^2_{\nu_\alpha,2}}{C_g} \frac{\pi^{\frac{n}{2}} 2^{\frac{2}{p}(4\alpha+2n+6)}\Gamma(\alpha+1)\Gamma(\frac{2\alpha+n+4}{2})}{(2^{4\alpha+2n+7}-2)\Gamma(2\alpha+n+4)} 2^{(k+1)(4\alpha+2n+6)} \\ &= \frac{C_{\alpha,p,n}\|g\|^2_{\nu_\alpha,2}}{2C_g} 2^{(k+1)(4\alpha+2n+6)}. \end{aligned}$$

Now, if $Card(\Omega) > \frac{C_{\alpha,p,n}\|g\|^2_{\nu_\alpha,2}}{2C_g} 2^{j_0(4\alpha+2n+6)}$ then we can choose an integer $k > j_0$ such that

$$\frac{C_{\alpha,p,n}\|g\|^2_{\nu_\alpha,2}}{2C_g} 2^{(k-1)(4\alpha+2n+6)} < Card(\Omega) \leqslant \frac{C_{\alpha,p,n}\|g\|^2_{\nu_\alpha,2}}{2C_g} 2^{k(4\alpha+2n+6)}, \qquad (3.8)$$

so that $Card(Q_{k-1}) < \dfrac{Card(\Omega)}{2}$, hence by (3.8), we obtain

$$\begin{aligned}\sum_{\beta\in\Omega}(\varrho_p(S_g^W(\phi_\beta)))^p &\geqslant \sum_{\beta\in Q_{k-1}}(\varrho_p(S_g^W(\phi_\beta)))^p\\ &\geqslant \sum_{j=k}^{Card(\Omega)}\sum_{\beta\in P_j}(\varrho_p(S_g^W(\phi_\beta)))^p\\ &\geqslant \frac{Card(\Omega)}{2}2^{(k-1)p}\\ &\geqslant (Card(\Omega))^{1+\frac{p}{4\alpha+2n+6}}\frac{1}{2^{p+1}}\left(\frac{2C_g}{C_{\alpha,p,n}\|g\|^2_{\nu_\alpha,2}}\right)^{\frac{p}{4\alpha+2n+6}}.\end{aligned}$$

Finally, if $Card(\Omega) \leqslant \frac{C_{\alpha,p,n}\|g\|^2_{\nu_\alpha,2}}{2C_g}2^{j_0(4\alpha+2n+6)}$, then

$$\begin{aligned}\sum_{\beta\in\Omega}(\varrho_p(S_g^W(\phi_\beta)))^p &\geqslant Card(\Omega)2^{j_0p}\\ &\geqslant (Card(\Omega))^{1+\frac{p}{4\alpha+2n+6}}\left(\frac{2C_g}{C_{\alpha,p,n}\|g\|^2_{\nu_\alpha,2}}\right)^{\frac{p}{4\alpha+2n+6}}.\end{aligned}$$

□

Theorem 3.9. *(Shapiro's umbrella theorem for S_g^W) Let g be a Weinstein wavelet, $(\phi_\beta)_{\beta\in K}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$ and $K\subset\mathbb{N}^{n+1}$ be a non-empty subset. If there is a function $g\in L^2(d\gamma_\alpha)$ such that*

$$S_g^W(\phi_\beta)(a,x)|\leqslant g(a,x),$$

for every $\beta\in K$, then K is finite.

Proof. Using analogous proof as for Theorem 3.3 of [17] page 11, we obtain the result. □

4. Time-frequency localization operators and their properties

Definition 4.1. Let g be a Weinstein wavelet on $\mathbb{R}_+^{n+1}$ in $L^2(d\nu_\alpha)$ and σ be a bounded nonnegative function on $]0,+\infty[\times\mathbb{R}_+^{n+1}$. The time-frequency localization operator $\mathcal{L}_g(\sigma)$ with Weinstein wavelet g and symbol σ is formally defined as

$$\begin{aligned}\mathcal{L}_g(\sigma)(f) &= \frac{1}{C_g}\int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}}\sigma(a,y)S_g^W(f)(a,y)g_{a,y}d\gamma_\alpha(a,y)\\ &= (S_g^W)^*\sigma S_g^W(f),\ f\in L^2(d\nu_\alpha).\end{aligned}\tag{4.1}$$

We note that if $\sigma=1$, then by the inversion formula (2.6), we have $\mathcal{L}_g(\sigma)(f)=f$. If σ is compactly supported on Σ, then $\mathcal{L}_g(\sigma)(f)$ is interpreted as the part of f that lies essentially in Σ.

It is usually more convenient to use the alternative weak definition of $\mathcal{L}_g(\sigma)$ given by

$$\langle \mathcal{L}_g(\sigma)(f), h\rangle_{\nu_\alpha}$$
$$= \frac{1}{C_g}\int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}} \sigma(a,y)S_g^W(f)(a,y)\overline{S_g^W(h)(a,y)}d\gamma_\alpha(a,y),\ f,h\in L^2(d\nu_\alpha). \quad (4.2)$$

In this section, we shall keep our focus on localization operators $\mathcal{L}_g(\sigma)$ with symbol $\sigma = \chi_\Sigma$, where g is a Weinstein wavelet on $\mathbb{R}_+^{n+1}$ in $L^2(d\nu_\alpha)$ and Σ is subset of $]0,+\infty[\times\mathbb{R}_+^{n+1}$ with finite measure $\gamma_\alpha(\Sigma) < \infty$.

Theorem 4.2. *Let g be a Weinstein wavelet on $\mathbb{R}_+^{n+1}$ in $L^2(d\nu_\alpha)$ such that $\|g\|_{\nu_\alpha,2} = 1$ and Σ is subset of $]0,+\infty[\times\mathbb{R}_+^{n+1}$. Then*
(i) the localization $\mathcal{L}_g(\Sigma)$ is in S_∞ and we have

$$\|\mathcal{L}_g(\Sigma)\|_{S_\infty} \leqslant \frac{1}{\sqrt{C_g}}. \quad (4.3)$$

(ii) the localization $\mathcal{L}_g(\Sigma)$ is a compact operator and even trace class with

$$Tr(\mathcal{L}_g(\Sigma)) = \Lambda_\alpha(g,\Sigma),$$

where

$$\Lambda_\alpha(g,\Sigma) = \frac{1}{C_g}\int\int_\Sigma \|g_{a,y}\|^2_{\nu_\alpha,2}d\gamma_\alpha(a,y). \quad (4.4)$$

Proof. (i) Let g be a Weinstein wavelet on $\mathbb{R}_+^{n+1}$ in $L^2(d\nu_\alpha)$. From Corollary 2.7 we get for $f\in L^2(d\nu_\alpha)$

$$\begin{aligned}\|\mathcal{L}_g(\Sigma)(f)\|_{\nu_\alpha,2} &= \|(S_g^W)^*(\chi_\Sigma S_g^W(f))\|_{\nu_\alpha,2}\\ &\leqslant \frac{1}{C_g}\|\chi_\Sigma S_g^W(f)\|_{\gamma_\alpha,2}\\ &\leqslant \frac{1}{C_g}\|S_g^W(f))\|_{\gamma_\alpha,2}\\ &\leqslant \frac{1}{\sqrt{C_g}}\|f\|_{\nu_\alpha,2}.\end{aligned}$$

so, $\mathcal{L}_g(\Sigma)\in\mathcal{B}(L^p(d\nu_\alpha))$ and $\|\mathcal{L}_g(\Sigma)\|_{S_\infty}\leqslant \frac{1}{\sqrt{C_g}}$.
(ii) We now show that $\mathcal{L}_g(\Sigma)$ is a compact operator. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $L^2(d\nu_\alpha)$ such that $f_k\longrightarrow 0$ weakly in $L^2(d\nu_\alpha)$ as $k\longrightarrow\infty$. It is enough to prove that

$$\lim_{k\longrightarrow+\infty}\|\mathcal{L}_g(\Sigma)(f_k)\|_{\nu_\alpha,2} = 0.$$

We have

$$\begin{aligned}\|\mathcal{L}_g(\Sigma)(f_k)\|^2_{\nu_\alpha,2} &\leqslant \frac{1}{C_g^2}\int_{\mathbb{R}_+^{n+1}}\int\int_\Sigma |S_g^W(f_k)(a,y)g_{a,y}(x)|^2d\gamma_\alpha(a,y)d\nu_\alpha(x)\\ &\leqslant \frac{1}{C_g^2}\int_{\mathbb{R}_+^{n+1}}\int\int_\Sigma |\langle f_k, g_{a,y}\rangle_{\nu_\alpha}|^2|g_{a,y}(x)|^2d\gamma_\alpha(a,y)d\nu_\alpha(x).\end{aligned}$$

Using the fact that $f_k\longrightarrow 0$ weakly in $L^2(d\nu_\alpha)$, we deduce that

$$\forall a>0, \forall(x,y)\in(\mathbb{R}_+^{n+1})^2,\quad \lim_{k\longrightarrow+\infty}|\langle f_k, g_{a,y}\rangle_{\nu_\alpha}|^2|g_{a,y}(x)|^2 = 0.$$

On the other hand as $f_k \longrightarrow 0$ weakly in $L^2(d\nu_\alpha)$ as $k \longrightarrow +\infty$, then there exists a positive constant A such that $\|f_k\|_{\nu_\alpha,2} \leqslant A$. So, $\forall a > 0, \forall (x, y) \in (\mathbb{R}^{n+1}_+)^2$

$$|\langle f_k, g_{a,y}\rangle_{\nu_\alpha}|^2 \leqslant \|f_k\|^2_{\nu_\alpha,2}\|g_{a,y}\|^2_{\nu_\alpha,2}.$$

Moreover, by Fubini's theorem and (2.4), we have

$$\|\mathcal{L}_g(\Sigma)(f_k)\|^2_{\nu_\alpha,2} \leqslant \frac{A^2\gamma(\Sigma)}{C_g^2}\|g\|^4_{\nu_\alpha,2} = \frac{A^2\gamma(\Sigma)}{C_g^2}.$$

By the dominated Convergence Theorem, $\lim_{k\longrightarrow+\infty} \|\mathcal{L}_g(\Sigma)(f_k)\|_{\nu_\alpha,2} = 0$ and therefore $\mathcal{L}_g(\Sigma)$ is compact.
To show that $\mathcal{L}_g(\Sigma)$ is trace class, we let $\{u_k\}_{k=1}^{+\infty}$ be an arbitrary orthonormal basis of $L^2(d\nu_\alpha)$, and we calculate

$$\begin{aligned}\sum_{k=1}^{+\infty}\Big|\langle \mathcal{L}_g(\Sigma)(u_k), u_k\rangle_{\nu_\alpha}\Big| &= \sum_{k=1}^{+\infty}\Big|\langle (S_g^W)^*(\chi_\Sigma S_g^W(u_k)), u_k\rangle_{\nu_\alpha}\Big| \\ &= \frac{1}{C_g}\sum_{k=1}^{+\infty}\Big|\langle \chi_\Sigma S_g^W(u_k)), S_g^W(u_k)\rangle_{\gamma_\alpha}\Big| \\ &= \frac{1}{C_g}\sum_{k=1}^{+\infty}\int\int_\Sigma \Big|S_g^W(u_k)(a, y)\Big|^2 d\gamma_\alpha(a, y).\end{aligned}$$

Using Fubini's theorem and (2.3), we get

$$\begin{aligned}\sum_{k=1}^{+\infty}\Big|\langle \mathcal{L}_g(\Sigma)(u_k), u_k\rangle_{\nu_\alpha}\Big| &= \frac{1}{C_g}\int\int_\Sigma \sum_{k=1}^{+\infty}\Big|\langle u_k, g_{a,y}\rangle_{\nu_\alpha}\Big|^2 d\gamma_\alpha(a, y) \\ &= \frac{1}{C_g}\int\int_\Sigma \|g_{a,y}\|^2_{\nu_\alpha,2} d\gamma_\alpha(a, y) \\ &= \Lambda_\alpha(g, \Sigma).\end{aligned}$$

Therefore $\mathcal{L}_g(\Sigma)$ is trace class with

$$\|\mathcal{L}_g(\Sigma)\|_{S_1} = Tr(\mathcal{L}_g(\Sigma)) = \Lambda_\alpha(g, \Sigma).$$

□

Proposition 4.3. *Let g and g' be a Weinstein wavelet on $\mathbb{R}^{n+1}_+$ in $L^2(d\nu_\alpha)$ and Σ is subset of $]0, +\infty[\times\mathbb{R}^{n+1}_+$. Then for every $f \in L^2(d\nu_\alpha)$*

$$|\langle(\mathcal{L}_g(\Sigma) - \mathcal{L}_{g'}(\Sigma))(f), f\rangle_{\nu_\alpha}| \leqslant (\frac{1}{\sqrt{C_{g-g'}}} + \frac{\sqrt{C_{g'}}}{C_{g-g'}} + \frac{\sqrt{C_{g-g'}}}{C_{g'}})\|f\|^2_{\nu_\alpha,2}.$$

Proof. From the boundedness of the time-frequency localization operators in $L^2(d\nu_\alpha)$, we get

$$\begin{aligned}
&|\langle(\mathcal{L}_g(\Sigma)-\mathcal{L}_{g'}(\Sigma))(f),f\rangle_{\nu_\alpha}|\\
&=\left|\langle(((S_g^W)^*(\chi_\Sigma S_g^W)-(S_{g'}^W)^*(\chi_\Sigma S_{g'}^W))(f),f\rangle_{\nu_\alpha}\right|\\
&=\left|\langle(((S_{g-g'}^W)^*(\chi_\Sigma S_{g-g'}^W)+(S_{g-g'}^W)^*(\chi_\Sigma S_{g'}^W))+(S_{g'}^W)^*(\chi_\Sigma S_{g-g'}^W))(f),f\rangle_{\nu_\alpha}\right|\\
&\leqslant\left|\langle(((S_{g-g'}^W)^*(\chi_\Sigma S_{g-g'}^W)(f),f\rangle_{\nu_\alpha}\right|+\left|\langle((S_{g-g'}^W)^*(\chi_\Sigma S_{g'}^W))(f),f\rangle_{\nu_\alpha}\right|\\
&+\left|\langle((S_{g'}^W)^*(\chi_\Sigma S_{g-g'}^W))(f),f\rangle_{\nu_\alpha}\right|\\
&\leqslant\|\mathcal{L}_{g-g'}(\Sigma)(f)\|_{\nu_\alpha,2}\|f\|_{\nu_\alpha,2}+\|((S_{g-g'}^W)^*(\chi_\Sigma S_{g'}^W))(f)\|_{\nu_\alpha,2}\|f\|_{\nu_\alpha,2}\\
&+\|((S_{g'}^W)^*(\chi_\Sigma S_{g-g'}^W))(f)\|_{\nu_\alpha,2}\|f\|_{\nu_\alpha,2}\\
&\leqslant\Big(\frac{1}{\sqrt{C_{g-g'}}}+\frac{\sqrt{C_{g'}}}{C_{g-g'}}+\frac{\sqrt{C_{g-g'}}}{C_{g'}}\Big)\|f\|_{\nu_\alpha,2}^2.
\end{aligned}$$

□

5. Weinstein wavelet Scalograms

The main aim of this section is to generalize the results proved by Ghobber in [13], in the context of Weinstein wavelet.

5.1. Calderón-Toeplitz Operator.

Definition 5.1. Let g be a Weinstein wavelet on $\mathbb{R}_+^{n+1}$ in $L^2(d\nu_\alpha)$. We define the Weinstein wavelet scalogram of as

$$\phi_g^W(f)(a,s)=|S_g^W(f)(a,s)|^2,\quad (a,s)\in]0,+\infty[\times\mathbb{R}_+^{n+1}.$$

From the Plancherel formula of S_g^W, we have

$$\int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}}\phi_g^W(f)(a,s)d\gamma_\alpha(a,s)=C_g\|f\|_{\nu_\alpha,2}^2. \tag{5.1}$$

Definition 5.2. We define the Calderón-Toeplitz operator

$$T_{g,\Sigma}:S_g^W(L^2(d\nu_\alpha))\longrightarrow S_g^W(L^2(d\nu_\alpha))$$

by

$$T_{g,\Sigma}F=P_gP_\Sigma F. \tag{5.2}$$

Proposition 5.3. *The operator $T_{g,\Sigma}$ is trace-class and satisfies*

$$0\leqslant T_{g,\Sigma}\leqslant P_\Sigma\leqslant I, \tag{5.3}$$

and

$$T_{g,\Sigma}=S_g^W\mathcal{L}_g(\Sigma)(S_g^W)^*. \tag{5.4}$$

Proof. For every $F\in S_g^W(L^2(d\nu_\alpha))$, we have

$$\begin{aligned}
\langle T_{g,\Sigma}F,F\rangle_{\gamma_\alpha}&=\langle P_gP_\Sigma F,F\rangle_{\gamma_\alpha}=\langle P_\Sigma F,F\rangle_{\gamma_\alpha}\\
&=\int\int_\Sigma|F(a,y)|^2d\gamma_\alpha(a,y).
\end{aligned}$$

This gives (5.3), and in particular shows that $T_{g,\Sigma}$ is bounded and positive.
Now, we want to prove (5.4). By (4.1), we have

$$\mathcal{L}_g(\Sigma)(f)=(S_g^W)^*\chi_\Sigma S_g^W(f)=(S_g^W)^*P_\Sigma S_g^W(f),\ f\in L^2(d\nu_\alpha).$$

Therefore

$$(S_g^W \mathcal{L}_g(\Sigma)(S_g^W)^*)F = P_g P_\Sigma F = T_{g,\Sigma}F, \; F \in S_g^W(L^2(d\nu_\alpha)).$$

Then, the time-frequency operator $\mathcal{L}_g(\Sigma)$ and the Calderón-Toeplitz operator $T_{g,\Sigma}$ are related by

$$S_g^W \mathcal{L}_g(\Sigma)(S_g^W)^* = T_{g,\Sigma}.$$

□

Let $\mathcal{M}_{g,\Sigma} : L^2(d\gamma_\alpha) \longrightarrow L^2(d\gamma_\alpha)$ be the inflated operator defined by

$$\mathcal{M}_{g,\Sigma} = P_g P_\Sigma P_g.$$

The advantage of $\mathcal{M}_{g,\Sigma}$ compared to $T_{g,\Sigma}$ is that it is defined on $L^2(d\gamma_\alpha)$ and consequently its spectral properties can be easily related to its integral kernel.

Lemma 5.4. *The trace of $\mathcal{M}_{g,\Sigma}$ is given by*

$$Tr(\mathcal{M}_{g,\Sigma}) = Tr(T_{g,\Sigma}) = \Lambda_\alpha(g, \Sigma),$$

where $\Lambda_\alpha(g, \Sigma)$ is given by relation (4.4).

Proof. Since $T_{g,\Sigma}$ is positive and trace class, then using the decomposition $L^2(d\gamma_\alpha) = S_g^W(L^2(d\nu_\alpha)) + (S_g^W(L^2(d\nu_\alpha)))^\perp$, we deduce that $\mathcal{M}_{g,\Sigma}$ is also positive and trace class with

$$Tr(\mathcal{M}_{g,\Sigma}) = Tr(T_{g,\Sigma}).$$

Now, let $\{\phi_k\}_{k=1}^{+\infty}$ be an arbitrary orthonormal basis for $S_g^W(L^2(d\nu_\alpha))$. Then if we denote by $\psi_k = (S_g^W(\phi_k))^*$, then $\{\psi_k\}_{k=1}^{+\infty}$ is an orthonormal basis for $L^2(d\nu_\alpha)$. Thus by (4.2) and Fubini's theorem

$$\begin{aligned}
\sum_{k=1}^{+\infty} \langle T_{g,\Sigma}(\phi_k), \phi_k \rangle_{\gamma_\alpha} &= \sum_{k=1}^{+\infty} \langle \mathcal{L}_g(\Sigma)(S_g^W)^*(\phi_k), (S_g^W)^*(\phi_k) \rangle_{\nu_\alpha} \\
&= \sum_{k=1}^{+\infty} \int\int_\Sigma \left| S_g^W(\psi_k)(a, y) \right|^2 d\gamma_\alpha(a, y) \\
&= \int\int_\Sigma \sum_{k=1}^{+\infty} \left| \langle \psi_k, g_{a,y} \rangle_{\nu_\alpha} \right|^2 d\gamma_\alpha(a, y) \\
&= \int\int_\Sigma \|g_{a,y}\|_{\nu_\alpha,2}^2 d\gamma_\alpha(a, y) \\
&= \Lambda_\alpha(g, \Sigma).
\end{aligned}$$

Therefore, by (2.8) and (2.9), the operator $T_{g,\Sigma}$ is trace class with

$$Tr(T_{g,\Sigma}) = \Lambda_\alpha(g, \Sigma).$$

□

Proposition 5.5. *The trace of $T_{g,\Sigma}^2$ is given by*

$$Tr(T_{g,\Sigma}^2) = \int\int_\Sigma \int\int_\Sigma \left| \mathcal{K}_g(a, y; a_1, y_1) \right|^2 d\gamma_\alpha(a, y) d\gamma_\alpha(a_1, y_1).$$

Proof. Since $\mathcal{M}_{g,\Sigma}$ is positive, then

$$Tr(T_{g,\Sigma}^2) = Tr(\mathcal{M}_{g,\Sigma}^2).$$

Using (3.1) and (5.2), we get

$$\mathcal{M}_{g,\Sigma}F(a,y) = T_{g,\Sigma}(P_gF)(a,y) = \langle F, \mathcal{H}_g((a,y);(.,.))\rangle_{\gamma_\alpha},$$

with

$$\begin{aligned}&\mathcal{H}_g((a,y);(u,v))\\&= \int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}} \chi_\Sigma(a',y')\mathcal{K}_g((a',y');(u,v))\mathcal{K}_g((a,y);(a',y'))d\gamma_\alpha(a',y').\end{aligned}$$

This means that $\mathcal{M}_{g,\Sigma}$ has integral kernel $\mathcal{H}_g((a,y);(u,v))$, and therefore

$$\mathcal{M}_{g,\Sigma}^2F(a,y) = \int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}} F(a',y')\mathscr{H}_g((a,y);(a',y'))d\gamma_\alpha(a',y'),$$

where

$$\mathscr{H}_g((a,y);(a',y')) = \int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}} \mathcal{H}_g((a,y);(u,v))\mathcal{H}_g((u,v);(a',y'))d\gamma_\alpha(u,v)$$

Then

$$\begin{aligned}&Tr(\mathcal{M}_{g,\Sigma}^2)\\&= \int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}} \mathscr{H}_g((a,y);(a,y))d\gamma_\alpha(a,y)\\&= \int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}}\int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}} \mathcal{H}_g((a,y);(u,v))\mathcal{H}_g((u,v);(a,y))d\gamma_\alpha(a,y)d\gamma_\alpha(u,v)\\&= \int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}}\int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}} \chi_\Sigma(x_1,y_1)\chi_\Sigma(x_2,y_2)\mathcal{N}((x_1,y_1);(x_2,y_2))d\gamma_\alpha(x_1,y_1)d\gamma_\alpha(x_2,y_2),\end{aligned}$$

where

$$\begin{aligned}\mathcal{N}((x_1,y_1);(x_2,y_2)) = &\int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}}\int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}} \mathcal{K}_g((x_2,y_2);(a,y))\mathcal{K}_g((a,y);(x_1,y_1))\\&\times \mathcal{K}_g((x_1,y_1);(u,v))\mathcal{K}_g((u,v);(x_2,y_2))d\gamma_\alpha(a,y)d\gamma_\alpha(u,v).\end{aligned}$$

On the other hand by (2.5), (2.7) and (3.2), we obtain

$$\begin{aligned}&\int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}} \mathcal{K}_g((x_2,y_2);(a,y))\mathcal{K}_g((a,y);(x_1,y_1))d\gamma_\alpha(a,y)\\&\qquad= \frac{1}{C_g^2}\int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}} S_g^W(g_{x_2,y_2})(a,y)\overline{S_g^W(g_{x_1,y_1})(a,y)}d\gamma_\alpha(a,y)\\&\qquad= \frac{1}{C_g}\int_{\mathbb{R}_+^{n+1}} g_{x_2,y_2}(t)\overline{g_{x_1,y_1}(t)}d\nu_\alpha(t)\\&\qquad= \mathcal{K}_g((x_2,y_2);(x_1,y_1)).\end{aligned}$$

By the same way, we have

$$\int_0^{+\infty}\int_{\mathbb{R}_+^{n+1}} \mathcal{K}_g((x_1,y_1);(u,v))\mathcal{K}_g((u,v);(x_2,y_2))d\gamma_\alpha(u,v) = \mathcal{K}_g((x_1,y_1);(x_2,y_2)).$$

Therefore

$$\begin{aligned}&Tr(\mathcal{M}^2_{g,\Sigma})\\&=\int\int_\Sigma\int\int_\Sigma \mathcal{K}_g((x_2,y_2);(x_1,y_1))\mathcal{K}_g((x_1,y_1);(x_2,y_2))d\gamma_\alpha(x_1,y_1)d\gamma_\alpha(x_2,y_2)\\&=\int\int_\Sigma\int\int_\Sigma |\mathcal{K}_g((x_2,y_2);(x_1,y_1))|^2 d\gamma_\alpha(x_1,y_1)d\gamma_\alpha(x_2,y_2).\end{aligned}$$

□

5.2. **Eigenvalues and Eigenfunctions.** Since the localization operator $\mathcal{L}_g(\Sigma) = (S^W_g)^*\chi_\Sigma S^W_g$ that we consider is a compact and self-adjoint operator, the spectral theorem gives the following spectral representation

$$\mathcal{L}_g(\Sigma)(f) = \sum_{k=1}^{+\infty} u_k\langle f, v_k\rangle_{\nu_\alpha} v_k, \quad f \in L^2(d\nu_\alpha), \tag{5.5}$$

where $\{u_k\}_{k=1}^\infty$ are the positive eigenvalues arranged in a nonincreasing manner and $\{v_k\}_{k=1}^\infty$ is the corresponding orthonormal set of eigenfunctions. Note that $u_k \searrow 0$ and by (4.3), we have for all $k \geqslant 1$

$$0 \leqslant u_k \leqslant u_1 \leqslant 1.$$

We denote by $\mathcal{C}(\Sigma,\varepsilon,g)$ the set of functions in $L^2(d\nu_\alpha)$ that are (ε,g)-concentrated in a subset Σ

$$\mathcal{C}(\Sigma,\varepsilon,g) = \{f \in L^2(d\nu_\alpha), \quad \langle \mathcal{L}_g(\Sigma)(f), f\rangle_{\nu_\alpha} \geqslant (1-\varepsilon)\|f\|^2_{\nu_\alpha,2}\}.$$

Moreover, since

$$\langle \mathcal{L}_g(\Sigma)(f), f\rangle_{\nu_\alpha} = \sum_{k=1}^{+\infty} u_k|\langle f, v_k\rangle_{\nu_\alpha}|^2 = \frac{1}{C_g}\int\int_\Sigma |S^W_g(f)(a,y)|^2 d\gamma_\alpha(a,y),$$

then the operator $\mathcal{L}_g(\Sigma)$ is useful in studying the following optimization problem:

$$\text{Maximize} \quad \frac{1}{C_g}\int\int_\Sigma |S^W_g(f)(a,y)|^2 d\gamma_\alpha(a,y), \quad \|f\|_{\nu_\alpha,2} = 1,$$

which aims to look for functions in $\mathcal{C}(\Sigma,\varepsilon,g)$ that are well concentrated in Σ. It follows that the first eigenfunction v_1 satisfies

$$u_1 = \langle \mathcal{L}_g(\Sigma)(v_1), v_1\rangle_{\nu_\alpha} = \max \quad \{\langle \mathcal{L}_g(\Sigma)(f), f\rangle_{\nu_\alpha}, \quad \|f\|_{\nu_\alpha,2} = 1.\}$$

Now, if v_k is an eigenfunction of $\mathcal{L}_g(\Sigma)$ with eigenvalue $u_k \geqslant 1-\varepsilon$, then from the spectral representation $\langle \mathcal{L}_g(\Sigma)(v_k), v_k\rangle_{\nu_\alpha} = u_k \geqslant 1-\varepsilon$. Hence, by (5.5) the eigenfunction v_k is in $\mathcal{C}(\Sigma,\varepsilon,g)$. Moreover, the min- max lemma for self-adjoint operators states that (see e.g.[22, Section 95]),

$$u_k = \max \quad \{\langle \mathcal{L}_g(\Sigma)(f), f\rangle_{\nu_\alpha}, \quad \|f\|_{\nu_\alpha,2} = 1, f \perp v_1,...,v_{k-1}\}.$$

So that the eigenvalues of $\mathcal{L}_g(\Sigma)$ determine the number of orthogonal functions that are in $\mathcal{C}(\Sigma,\varepsilon,g)$. Let V_N be the span of the first N eigenfunctions of $\mathcal{L}_g(\Sigma)$ corresponding to the N largest eigenvalues $\{u_k\}_{k=1}^N$, then for all $f \in V_n = \text{span}\{v_k\}_{k=1}^N$

$$\langle \mathcal{L}_g(\Sigma)(f), f\rangle_{\nu_\alpha} = \sum_{k=1}^{+\infty} u_k|\langle f, v_k\rangle_{\nu_\alpha}|^2 \geqslant u_N\sum_{k=1}^{+\infty}|\langle f, v_k\rangle_{\nu_\alpha}|^2 = u_N\|f\|^2_{\nu_\alpha,2}.$$

This implies that a function f in V_N is in $\mathcal{C}(\Sigma, 1-u_N, g)$.

5.3. **Scalogram of a subspace.** Given an N-dimensional subspace V of $L^2(d\nu_\alpha)$, P_V the orthogonal projection onto V with projection kernel κ_V , i.e.

$$P_V f(.) = \int_{\mathbb{R}^{n+1}_+} \kappa_V(.;y) f(y) d\nu_\alpha(y).$$

Recall that if $\{e_k\}_{k=1}^N$ is an orthonormal basis of V , then

$$\kappa_V(s;y) = \sum_{k=1}^N e_k(s)\overline{e_k(y)}.$$

The kernel κ_V is independent of the choice of orthonormal basis for V.

Definition 5.6. The scalogram of the space V with Weinstein wavelet g is defined

$$SCAL_g V(a,y) = \int_{\mathbb{R}^{n+1}_+} \int_{\mathbb{R}^{n+1}_+} \kappa_V(r;x) g_{a,y}(x) \overline{g_{a,y}(r)} d\nu_\alpha(r) d\nu_\alpha(x).$$

Lemma 5.7. *The scalogram $SCAL_g V$ is given by*

$$SCAL_g V = \sum_{k=1}^N \phi_g^W(e_k).$$

Proof. We have

$$\begin{aligned} SCAL_g V(a,y) &= \int_{\mathbb{R}^{n+1}_+} \int_{\mathbb{R}^{n+1}_+} \sum_{k=1}^N e_k(r) g_{a,y}(x) \overline{g_{a,y}(r) e_k(x)} d\nu_\alpha(r) d\nu_\alpha(x) \\ &= \sum_{k=1}^N \langle e_k, g_{a,y}\rangle_{\nu_\alpha} \overline{\langle e_k, g_{a,y}\rangle_{\nu_\alpha}} \\ &= \sum_{k=1}^N S_g^W(e_k)(a,y) \overline{S_g^W(e_k)(a,y)} \\ &= \sum_{k=1}^N |S_g^W(e_k)(a,y)|^2. \end{aligned}$$

And relation (5.1) complete the proof. □

Definition 5.8. We define the time-frequency concentration of a subspace V_N in Σ as

$$\xi_{\Sigma,g}(V_N) = \frac{1}{N} \sum_{k=1}^N \int\int_\Sigma \phi_g^W(v_k)(a,y) d\gamma_\alpha(a,y).$$

If the $\{v_k\}_{k=1}^{+\infty}$ are eigenfunctions of the localization operator $\mathcal{L}_g(\Sigma)$ then

$$\xi_{\Sigma,g}(V_N) = \frac{1}{N} \sum_{k=1}^N u_k.$$

We can see that $u_N \leqslant \xi_{\Sigma,g}(V_N) \leqslant u_1 \leqslant 1$. The min-max characterization of the eigenvalues of compact operators implies that any N-dimensional subset cannot be better concentrated in Σ, i.e. if V'_N is any N-dimensional subspace of $L^2(d\nu_\alpha)$, then

$$\xi_{\Sigma,g}(V'_N) \leqslant \xi_{\Sigma,g}(V_N).$$

5.4. **Accumulated Scalogram.** Denote by $\lceil\Lambda_\alpha(g,\Sigma)\rceil$ the smallest integer greater than or equal to $\Lambda_\alpha(g,\Sigma)$. In [2] the authors showed that the corresponding spectrograms of the first $\lceil\Lambda_\alpha(g,\Sigma)\rceil$ eigenfunctions of $\mathcal{L}_g(\Sigma)$ approximately form a partition of unity on Σ.
Define the accumulated scalogram of Σ with respect to g as the scalogram of the subspace $V_{\lceil\Lambda_\alpha(g,\Sigma)\rceil} = \mathrm{span}\{v_k\}_{k=1}^{\lceil\Lambda_\alpha(g,\Sigma)\rceil}$, where $v_k, k = 1, ..., \lceil\Lambda_\alpha(g,\Sigma)\rceil$ are the eigenfunctions of $\mathcal{L}_g(\Sigma)$, i.e. $SCAL_g V_{\lceil\Lambda_\alpha(g,\Sigma)\rceil}$.
Then

$$SCAL_g V_{\lceil\Lambda_\alpha(g,\Sigma)\rceil}(a,y) = \sum_{k=1}^{\lceil\Lambda_\alpha(g,\Sigma)\rceil} |S_g^W(v_k)(a,y)|^2.$$

Note that

$$\|SCAL_g V_{\lceil\Lambda_\alpha(g,\Sigma)\rceil}\|_{\gamma_\alpha,1} = C_g\lceil\Lambda_\alpha(g,\Sigma)\rceil = C_g\Lambda_\alpha(g,\Sigma) + O(1).$$

Moreover, since

$$\sum_{k=1}^{\lceil\Lambda_\alpha(g,\Sigma)\rceil} u_k \leqslant Tr(\mathcal{L}_g(\Sigma)) = \Lambda_\alpha(g,\Sigma),$$

then we can define the inequality

$$E(g,\Sigma) = 1 - \frac{\sum_{k=1}^{\lceil\Lambda_\alpha(g,\Sigma)\rceil} u_k}{\Lambda_\alpha(g,\Sigma)},$$

which satisfies

$$0 \leqslant E(g,\Sigma) \leqslant 1.$$

More precisely, we have the following result

Lemma 5.9. *Let $0 < \epsilon < 1$. We have*

$$0 \leqslant E(g,\Sigma) \leqslant 1 - (1-\epsilon)\min(1, \frac{k(\epsilon,\Sigma)}{\Lambda_\alpha(g,\Sigma)}),$$

where $k(\epsilon,\Sigma) = card\{k,\ u_k \geqslant 1-\epsilon\}$.

Proof. Let $0 < \epsilon < 1$, it follows that

$$u_k \geqslant 1-\epsilon,\ 1 \leqslant k \leqslant \min(k(\epsilon,\Sigma), \lceil\Lambda_\alpha(g,\Sigma)\rceil).$$

As $\lceil\Lambda_\alpha(g,\Sigma)\rceil \geqslant \min(k(\epsilon,\Sigma), \lceil\Lambda_\alpha(g,\Sigma)\rceil)$, we get

$$\sum_{k=1}^{\lceil\Lambda_\alpha(g,\Sigma)\rceil} u_k \geqslant \sum_{k=1}^{\min(k(\epsilon,\Sigma),\lceil\Lambda_\alpha(g,\Sigma)\rceil)} u_k \geqslant (1-\epsilon)\min(k(\epsilon,\Sigma), \lceil\Lambda_\alpha(g,\Sigma)\rceil).$$

Therefore

$$0 \leqslant E(g,\Sigma) \leqslant 1 - (1-\epsilon)\frac{\min(k(\epsilon,\Sigma), \lceil\Lambda_\alpha(g,\Sigma)\rceil)}{\Lambda_\alpha(g,\Sigma)}.$$

As $\lceil\Lambda_\alpha(g,\Sigma)\rceil \geqslant \Lambda_\alpha(g,\Sigma)$, we obtain the desired result. □

Acknowledgments. The authors gratefully acknowledge the approval and the support of this research study by the grant number SCI-2018-3-9-F-7841 from the Deanship of Scientific Research at the Northern Border University, Arar, KSA.

References

[1] L. D Abreu and M. Dörfler, *An inverse problem for localization operators*, Inverse Problems, **28** 115001 (2012) 1–16.

[2] L.D. Abreu, K. Gröchenig and J.L. Romero, *On accumulated spectrograms*, Trans. Am. Math. Soc. **368** (2016) 3629–3649.

[3] Z. Ben Nahia and N. Ben Salem, *Spherical harmonics and applications associated with the Weinstein operator*, In: Potential Theory- Proceedings of the ICPT f **94** (1994) 223–241.

[4] Z. Ben Nahia and N. Ben Salem, *On a mean value property associated with the Weinstein operator*, In: Potential Theory - Proceedings of the ICPT **94** (1994) 243–253.

[5] M. Brelot, *Equation de Weinstein et potentiels de Marcel Riesz*, In: Hirsch F, Mokobodzki G, editors. Seminaire de Thorie du Potentiel, No. 3 (Paris, 1976/1977). Lecture Notes in Mathematics, Vol. **681**. Berlin: Springer; (1978) 18–38.

[6] E. Cordero and K. Gröchenig, *Time-frequency analysis of localization operators*, J. Funct. Anal., **205** (1)(2003) 107–131.

[7] I. Daubechies, *Time-frequency localization operators a geometric phase space approach*, IEEE Trans. Inform. Theory, **34** (4) (1988) 605–612.

[8] F. DeMari, H. G. Feichtinger and K. Nowak, *Uniform eigenvalue estimates for time-frequency localization operators*, J. London Math. Soc., **65** (3) (2002) 720–732.

[9] H. G. Feichtinger and K. Nowak, *A Szegö-type theorem for Gabor-Toeplitz localization operators*, Michigan Math. J., **49** (1) (2001) 13–21.

[10] A. Gasmi, H. Ben Mohamed and N. Bettaibi, *Inversion of Weinstein intertwining operator and its dual using Weinstein wavelets*, An Stiint Univ Ovidius Constanta SerMat. ;**24** (1) (2016) 289–307.

[11] S. Ghobber, *Shapiro's uncertainty principle in the Dunkl setting*, Bull. Aust. Math. Soc. **92**(01)(2015) 98–110.

[12] S. Ghobber, *Time-frequency concentration and localization operators in the Dunkl setting*, J. Pseudo Differ. Oper. Appl., (2016) 1–19.

[13] S. Ghobber, *Some results on wavelet scalograms*, Int. J. Wavel Multiresolution (15) 1750019 (2017) 21 pages.

[14] S. Ghobber and S. Omri, *Time-frequnecy concentration of the windowed Hankel transform*, Integr. Transf. Spec. F. **25** (2014) 481–496.

[15] I. Gohberg, S. Goldberg and N. Krupnik, *Traces and Determinants of Linear Operators*, Birkhauser, Basel (2012).

[16] K. Gröchenig and J. Toft, *The range of localization operators and lifting theorems for modulation and Bargmann-Fock spaces*, Trans. Amer. Math. Soc.,**365** (2013) 4475–4496.

[17] NB. Hamadi and S. Omri, *Uncertainty principles for the continuous wavelet transform in the Hankel setting*, Appl. Anal; (2017) p. 1–15.

[18] P. Jaming and A.M. Powell, *Uncertainty principles for orthonormal sequences*, J. Funct. Anal. **243** (2007) 611–630.

[19] E. Malinnikova, *Orthonormal sequences in $L^2(\mathbb{R}^d)$ and time frequency localisation*, J. Fourier Anal. Appl. **16** (2010) 983–1006.

[20] H. Mejjaoli and A. Ould Ahmed Salem, *New results on the continuous Weinstein wavelet transform*, Journal of Inequalities and Applications, (1) (2017) 1–25.

[21] L.Z. Peng and J.M. Zhao, *Wavelet and Weyl transforms associated with spherical mean operator*, Integr. Equ. Oper. Theory **50** (2004) 279–290.

[22] F. Riesz and B. Sz.-Nagy, *Functional Analysis.* Frederick Ungar Publishing Co., New York (1955).

[23] H. Shapiro, *Uncertainty principles for bases in $L^2(\mathbb{R})$*, In: Proceedings of the Conference on Harmonic Analysis and Number Theory (Marseille-Luminy) (2005).

[24] A. Weinstein, *Singular partial differential equations and their applications*, Fluid dynamics and applied mathematics. New York: Gordon and Breach, (1962) 29–49.

Khaled Hleili

Faculty of Science, Department of Mathematics, Northern Border university, Arar, Saudi Arabia

E-mail address: khaled.hleili@gmail.com