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ON THE CONVERGENCE OF A FIFTH-ORDER ITERATIVE METHOD IN BANACH SPACES

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Abstract. This paper is devoted to convergence study and analysis of a fifth-order iterative method to solve nonlinear equations in Banach spaces. The idea of the Lipschitz condition on the second Fréchet derivative has been used to obtain semilocal convergence balls, R-order of convergence and error bounds by following the main theorem. The local convergence follows under weak-Lipschitz-type conditions. Theoretical results are verified through numerical examples, including integral equation and boundary value problem. It is observed that better results have been obtained in terms of accuracy and number of iterations in comparison with the well-known existing algorithms using similar information. The basins of attraction of the presented method show good performance as compared to already established methods which enhance the applicability of our approach.

1. Introduction

Solving nonlinear equations or system of nonlinear equations is a challenging task in numerical analysis and various other branches of applied sciences. Many real-life problems arising in science and engineering [1], [2], [3] can be modeled to algebraic and differential equations whose solutions require solving such equations. Thus, many researchers have extensively studied these problems and different methods have been developed to find their solution. In this study, we consider the problem of approximating a solution x^* of the equation

$$F(x) = 0, \tag{1.1}$$

where $F : \Omega \subseteq X \to Y$ is a nonlinear operator on an open convex subset Ω of a Banach space X with values in a Banach space Y. The solution of this type can rarely be found in a closed form. So iterative methods are used. The solution x^* can be obtained as a fixed point of some function $\Phi : \Omega \subseteq X \to Y$ by means of fixed point iteration [4, 5]

$$x_{n+1} = \Phi(x_n), \quad n = 0, 1, 2, \ldots$$

Our aim here is to focus on using techniques of functional analysis to obtain domains that contain solutions of (1.1). Uniqueness conditions for these domains are also

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established. Starting from one initial approximation of a solution $x^*$of the equation $F(x) = 0$, a sequence $\{x_n\}$ of approximations is constructed such that $\{\|x_n - x^*\|\}$ is decreasing and a better approximation to the solution x^* is obtained at every step. There are a variety of iterative methods for solving (1.1). The quadratically convergent Newton's method is most widely used and is given as:

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad n = 0, 1, 2, \dots \tag{1.2}$$

where x_0 is the initial point and $F'(x_n)^{-1} \in \mathcal{L}(Y, X)$, where $\mathcal{L}(Y, X)$ is the set of bounded linear operators from Y into X.

Three types of studies are done to prove the convergence of iterative algorithms: local, semilocal and global. First, the local convergence is based on the information around a solution x^* to find estimates of the computed radii of the convergence balls, (see [6, 7, 9, 8, 10, 11, 12]). The convergence ball of an iterative method is important as it shows the extent of difficulty for choosing initial guess for iterative method. Second, the semilocal convergence is based on the information around an initial approximation x_0, to obtain conditions ensuring the convergence of sequence generated by the iterative method to the solution x^*, (see [13, 14, 15, 16, 17]). Third, the global study of the convergence guarantees the convergence of the sequence to the solution x^* in a domain and independent of initial approximation x_0 (see [18, 19]). The convergence of Newton's method in Banach spaces was established by Kantorovich in [1]. Rall in [20] established the convergence of Newton's method by using recurrence relations. With the same approach, various researchers established semilocal convergence of higher order methods in Banach spaces (see [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38] and references there in).

Inspired by ongoing research, the main goal and motivation of the paper is to discuss semilocal and local convergence in Banach spaces of a three-step fifth order iterative method introduced by Sharma [39]. The fifth order iterative scheme in Banach spaces is given by;

$$\begin{cases} u_n = x_n - \Gamma_n F(x_n), \\ y_n = x_n + \frac{1}{2}(u_n - x_n), \\ z_n = x_n - F'(y_n)^{-1}F(x_n), \\ x_{n+1} = z_n - [2F'(y_n)^{-1} - F'(x_n)^{-1}]F(z_n), \end{cases} \tag{1.3}$$

where $\Gamma_n = F'(x_n)^{-1}$, for $n \in \mathbb{N}$. The main advantage to study the convergence analysis of (1.3) in Banach spaces is to focus on the initial data as well as on the solution obtained. Semilocal convergence analysis for this method is developed using recurrence relations under second order derivative of Fréchet satisfying Lipschitz condition in Banach spaces. Based on these recurrence relations, an existence and uniqueness theorem is established along with error bounds for the solution. Several examples are worked out in which radii of convergence balls is computed using the established theorem. The local convergence is established under weak-Lipschitz-type conditions on first Fréchet derivative to extend its applicability. The weak-Lipschitz-type continuity condition contains particular cases of the Lipschitz and Hölder continuity conditions and is valid for the problems where the Lipschitz and Hölder continuity conditions fail. We also analyze basins of attraction of (1.3) and compare with methods in [23], [38] and [40]. A variety of examples are solved to demonstrate the applicability of proposed approach. In comparison to method in [23], the differentiability conditions of the semilocal convergence in this paper

are mild.
The structure of the paper is given as follows: In Section 2, we give some basic definitions, preliminary results and define the auxiliary functions. In Section 3, the recurrence relations are constructed in order to establish the semilocal convergence including radius of convergence, error bounds and uniqueness results, which is completed in Section 4. The local convergence of (1.3) is presented in Section 5. Various numerical examples are considered to verify theoretical results in Section 6. Section 7 depicts the results of global convergence in an example system. Finally conclusion is given in Section 8.

2. Preliminary results

Definition 2.1. Let X and Y be Banach spaces. An operator F that maps X into Y is Fréchet differentiable at x_0, if there exists a bounded linear operator A from X into Y such that

$$\lim_{\|\Delta x\|\to 0} \frac{\|F(x_0+\Delta x)-F(x_0)-A\Delta x\|}{\|\Delta x\|}=0.$$

The linear operator A is called the first Fréchet derivative of F at x_0 and is denoted by $F'(x_0)$.

Definition 2.2. A sequence $\{x_n\}$ in X is said to be convergent to a point x^*, if

$$\lim_{n\to\infty} \|x_n - x^*\| = 0.$$

Definition 2.3. A sequence $\{x_n\}$ converges with R-order at least $\tau > 1$, if there are constants $K \in (0,\infty)$ and $\gamma \in (0,1)$ such that $\|x_n\| \leq K\gamma^{\tau^n}$, $n \in \mathbb{Z}_+$.

Let $F : \Omega \subseteq X \to Y$ be a nonlinear twice Fréchet differentiable operator on an open convex domain Ω. We assume that the inverse of F' at x_0, $\quad \Gamma_0 = F'(x_0)^{-1} \in \mathcal{L}(Y,X)$ exists at some $x_0 \in \Omega$, where $\mathcal{L}(Y,X)$ is set of bounded linear operators from Y into X. In the following we will assume that $y_0, z_0 \in \Omega$ and

(C1) $\|\Gamma_0 F(x_0)\| \leq \eta_0$,
(C2) $\|\Gamma_0\| \leq \beta_0$,
(C3) $\|F''(x)\| \leq M, x \in \Omega$,
(C4) there exists a positive real number N such that

$$\|F''(x) - F''(y)\| \leq N\|x-y\|, \text{ for each } x, y \in \Omega. \tag{2.1}$$

We firstly give an approximation of the operator F in the following lemma, which will be used in next derivation.

Lemma 2.4. Assume that the nonlinear operator $F : \Omega \subseteq X \to Y$ is continuously twice Fréchet differentiable where Ω is an open convex set and X and Y are Banach spaces. Then we have

$$
\begin{aligned}
F(x_{n+1}) =& F''(x_n)(u_n - x_n)F'(y_n)^{-1}\big[F'(y_n) - F'(x_n)\big]\Gamma_n F(z_n) \\
&+ \int_0^1 \big[F''(x_n + t(u_n - x_n)) - F''(x_n)\big](u_n - x_n)(x_{n+1} - z_n)dt \\
&- \frac{1}{2}\int_0^1 \left[F''(x_n + \frac{1}{2}t(u_n - x_n)) - F''(x_n)\right](u_n - x_n)(x_{n+1} - z_n)dt \\
&+ \frac{1}{2}\int_0^1 \left[F''(x_n + \frac{1}{2}t(u_n - x_n)) - F''(x_n)\right](u_n - x_n)\Gamma_n F(z_n)dt \\
&+ \int_0^1 \big[F'(z_n + t(x_{n+1} - z_n)) - F'(u_n)\big](x_{n+1} - z_n)dt. \qquad (2.2)
\end{aligned}
$$

Proof. From last step of (1.3), we have

$$F'(y_n)(x_{n+1} - z_n) + \big[2F'(x_n) - F'(y_n)\big]\Gamma_n F(z_n) = 0.$$

By Taylor's theorem, we obtain

$$
\begin{aligned}
F(x_{n+1}) &= F(z_n) + F'(u_n)(x_{n+1} - z_n) + \int_0^1 \big[F'(z_n + t(x_{n+1} - z_n)) - F'(u_n)\big](x_{n+1} - z_n)dt \\
&= F(z_n) + (F'(u_n) - F'(y_n))(x_{n+1} - z_n) - \big[2F'(x_n) - F'(y_n)\big]\Gamma_n F(z_n) \\
&\quad + \int_0^1 \big[F'(z_n + t(x_{n+1} - z_n)) - F'(u_n)\big](x_{n+1} - z_n)dt. \qquad (2.3)
\end{aligned}
$$

Similarly, we obtain

$$F'(u_n) = F'(x_n) + F''(x_n)(u_n - x_n) + \int_0^1 \big[F''(x_n + t(u_n - x_n)) - F''(x_n)\big](u_n - x_n)dt,$$

and

$$F'(y_n) = F'(x_n) + \frac{1}{2}F''(x_n)(u_n - x_n) + \frac{1}{2}\int_0^1 \left[F''(x_n + \frac{1}{2}t(u_n - x_n)) - F''(x_n)\right](u_n - x_n)dt.$$

It follows that

$$
\begin{aligned}
F'(u_n) - F'(y_n) &= \frac{1}{2}F''(x_n)(u_n - x_n) + \int_0^1 \big[F''(x_n + t(u_n - x_n)) - F''(x_n)\big](u_n - x_n)dt \\
&\quad - \frac{1}{2}\int_0^1 \left[F''(x_n + \frac{1}{2}t(u_n - x_n)) - F''(x_n)\right](u_n - x_n)dt, \qquad (2.4)
\end{aligned}
$$

and

$$
\begin{aligned}
2F'(x_n) - F'(y_n) &= F'(x_n) - \frac{1}{2}F''(x_n)(u_n - x_n) \\
&\quad - \frac{1}{2}\int_0^1 \left[F''(x_n + \frac{1}{2}t(u_n - x_n)) - F''(x_n)\right](u_n - x_n)dt. \qquad (2.5)
\end{aligned}
$$

Substituting (2.4) and (2.5) into (2.3), we can obtain (2.2). □

We now define the following scaler functions that will be often used in the later developments. Let

$$g(t) = \frac{8 + 4t^2 - t^3}{(2-t)^3}, \tag{2.6}$$

$$h(t) = \frac{1}{1 - tg(t)}, \tag{2.7}$$

$$\phi(u,v) = \left[\frac{4u^2}{(2-u)^2} + \frac{v(3+u)}{2(2-u)} + \frac{u(2+u)^2}{2(2-u)^2}p(u,v)\right]p(u,v), \tag{2.8}$$

where

$$p(u,v) = \frac{u^2}{(2-u)^2} + \frac{v}{2(2-u)^2} + \frac{4v}{3(2-u)^3}.$$

Let $q(u) = g(u)u - 1$. Since $q(0) = -1$ and $q(2) = +\infty$, then $q(u)$ has at least a zero in $(0,2)$. Let s is the smallest positive zero of the scalar function $g(u)u - 1$. Some properties of the functions g, h, ϕ defined by (2.6)-(2.8) in the interval $(0,s)$ are given in the following lemma.

Lemma 2.5. Let the real functions g, h and ϕ be given in (2.6)-(2.8). Then

(i) $g(u)$ and $h(u)$ are increasing and $g(u) > 1, h(u) > 1$ for $u \in (0,s)$,
(ii) $\phi(u,v)$ is increasing for $u \in (0,s), v > 0$.

Proof. The proof is obvious. □

Assume that conditions in (C1)-(C4) hold. We now denote $a_0 = M\beta\eta, b_0 = N\beta\eta^2$ and $d_0 = h(a_0)\phi(a_0,b_0)$. Let $a_0 < s$ and $h(a_0)d_0 < 1$. Furthermore, we can define the following sequences for each $n = 0,1,2,\ldots$.

$$\eta_{n+1} = d_n\eta_n, \tag{2.9}$$

$$\beta_{n+1} = h(a_n)\beta_n, \tag{2.10}$$

$$a_{n+1} = M\beta_{n+1}\eta_{n+1}, \tag{2.11}$$

$$b_{n+1} = N\beta_{n+1}\eta_{n+1}^2, \tag{2.12}$$

$$d_{n+1} = h(a_{n+1})\phi(a_{n+1},b_{n+1}). \tag{2.13}$$

From the definitions of a_{n+1}, b_{n+1}, (2.9) and (2.10), we also have

$$a_{n+1} = h(a_n)d_na_n, \tag{2.14}$$

$$b_{n+1} = h(a_n)d_n^2b_n. \tag{2.15}$$

Next, we shall study some properties of the previous scaler sequences. Later developments will require the following lemma.

Lemma 2.6. Let the real functions g, h and ϕ be given in (2.6)-(2.8). If

$$a_0 < s \text{ and } h(a_0)d_0 < 1, \tag{2.16}$$

then we have

(i) $h(a_n) > 1$ and $d_n < 1$ for each $n = 0,1,2,\ldots$
(ii) the sequences $\{\eta_n\}, \{a_n\}, \{b_n\}$ and $\{d_n\}$ are decreasing ,

(iii) $g(a_n)a_n < 1$ and $h(a_n)d_n < 1$ for each $n = 0, 1, 2, \ldots$.

Proof. By Lemma (2.5) and (2.16), $h(a_0) > 1$ and $d_0 < 1$ hold. It follows from (2.9), (2.14) and (2.15) that $\eta_1 < \eta_0, a_1 < a_0, b_1 < b_0$. Moreover, by Lemma (2.5), we have $h(a_1) < h(a_0)$ and $\phi(a_1, b_1) < \phi(a_0, b_0)$. This yields $d_1 < d_0$ and (ii) holds. Based on these results we obtain $g(a_1)a_1 < g(a_0)a_0 < 1$ and $h(a_1)d_1 < h(a_0)d_0 < 1$ and (iii) holds. By induction we can derive that items (i), (ii) and (iii) hold. □

Lemma 2.7. Let the real functions g, h and ϕ be given in (2.6)-(2.8). Let $\theta \in (0, 1)$, then $g(\theta u) < g(u), h(\theta u) < h(u)$ and $\phi(\theta u, \theta^2 v) < \theta^4 \phi(u, v)$ for $u \in (0, s)$.

Proof. For $\theta \in (0, 1)$ and $u \in (0, s)$, by using (2.6 - 2.8), the lemma can be easily proved. □

Lemma 2.8. Under the assumptions of Lemma (2.6). Let $\gamma = h(a_0)d_0$ and $\lambda = 1/h(a_0)$, then

$$d_n \leq \lambda\gamma^{5^n}, \text{for each } n = 0, 1, 2, \ldots, \tag{2.17}$$

and

$$\prod_{i=0}^{n} d_i \leq \lambda^{n+1}\gamma^{\frac{5^{n+1}-1}{4}}, n \geq 0. \tag{2.18}$$

Proof. Since $a_1 = \gamma a_0, b_1 = h(a_0)d_0^2 b_0 < \gamma^2 b_0$, by Lemma (2.7) we have

$$d_1 < h(\gamma a_0)\phi(\gamma a_0, \gamma^2 b_0) < \gamma^4 d_0 = \gamma^{5^1-1}d_0 = \lambda\gamma^{5^1}.$$

Suppose $d_k \leq \lambda\gamma^{5^k}, k \geq 1$, then by Lemma (2.6), we have $a_{k+1} < a_k$ and $h(a_k)d_k < 1$. Thus

$$\begin{aligned} d_{k+1} &< h(a_k)\phi(h(a_k)d_k a_k, h(a_k)d_k^2 b_k) \\ &< h(a_k)\phi(h(a_k)d_k a_k, h(a_k)^2 d_k^2 b_k) \\ &< h(a_k)^4 d_k^5 < \lambda\gamma^{5^{k+1}}. \end{aligned}$$

Therefore it holds that $d_n \leq \lambda\gamma^{5^n}, n \geq 0$.
By (2.17), we get

$$\prod_{i=0}^{n} d_i \leq \prod_{i=0}^{n} \lambda\gamma^{5^i} = \lambda^{n+1}\gamma^{\sum_{i=0}^{n} 5^i} = \lambda^{n+1}\gamma^{\frac{5^{n+1}-1}{4}}, n \geq 0.$$

This shows that (2.18) holds. □

Lemma 2.9. Under the assumptions of Lemma (2.6). Let $\gamma = h(a_0)d_0$ and $\lambda = 1/h(a_0)$. Then the sequence $\{\eta_n\}$ satisfies

$$\eta_n \leq \eta\lambda^n\gamma^{\frac{5^n-1}{4}}, n \geq 0, \tag{2.19}$$

and the sequence $\{\eta_n\}$ converges to 0.

Proof. From(2.9) and (2.17), we have

$$\eta_n = d_{n-1}\eta_{n-1} = d_{n-1}d_{n-2}\eta_{n-2} = \cdots = \eta\left(\prod_{i=0}^{n-1} d_i\right) \leq \eta\lambda^n\gamma^{\frac{5^n-1}{4}}, n \geq 0.$$

Because $\lambda < 1$ and $\gamma < 1$, it follows that $\eta_n \to 0$ as $n \to \infty$. □

3. Recurrence relations for the method

We firstly give an approximation of the operator F in the following lemma.

Lemma 3.1. Assume that the nonlinear operator $F : \Omega \subseteq X \to Y$ is continuously twice Fréchet differentiable operator where Ω is an open convex set and X and Y are Banach spaces. Then we have

$$\begin{aligned} F(z_n) = -\frac{1}{2}\int_0^1 & F''(x_n + t(y_n - x_n))\Gamma_n(F'(y_n) - F'(x_n))(z_n - x_n)^2 dt \\ & +\frac{1}{2}\int_0^1 \big[F''(x_n) - F''(x_n + t(y_n - x_n))\big](z_n - x_n)^2 dt \\ & + \int_0^1 \big[F''(x_n + t(z_n - x_n)) - F''(x_n)\big](1 - t)dt(z_n - x_n)^2. \end{aligned} \tag{3.1}$$

Proof. Since

$$F(x_n) + F'(y_n)(z_n - x_n) = 0, \tag{3.2}$$

and

$$y_n - x_n = \frac{1}{2}\Gamma_n(F'(y_n) - F'(x_n))(z_n - x_n) + \frac{1}{2}(z_n - x_n), \tag{3.3}$$

by Taylor's theorem and by (3.3), we have

$$\begin{aligned} &(F'(x_n) - F'(y_n))(z_n - x_n) = \\ &\quad -\frac{1}{2}\int_0^1 F''(x_n + t(y_n - x_n)) \left[\Gamma_n(F'(y_n) - F'(x_n))(z_n - x_n)^2 + (z_n - x_n)^2\right] dt. \end{aligned} \tag{3.4}$$

Again by Taylor's theorem, we have

$$\begin{aligned} F(z_n) &= F(x_n) + F'(x_n)(z_n - x_n) + \frac{1}{2}F''(x_n)(z_n - x_n)^2 \\ &\quad + \int_0^1 \big[F''(x_n + t(z_n - x_n)) - F''(x_n)\big](1 - t)dt(z_n - x_n)^2 \\ &= F(x_n) + F'(y_n)(z_n - x_n) + (F'(x_n) - F'(y_n))(z_n - x_n) + \frac{1}{2}F''(x_n)(z_n - x_n)^2 \\ &\quad + \int_0^1 \big[F''(x_n + t(z_n - x_n)) - F''(x_n)\big](1 - t)dt(z_n - x_n)^2. \end{aligned} \tag{3.5}$$

Substituting (3.2) and (3.4) in (3.5), we get (3.1). □

We denote $B(x, r) = \{y \in X : \|y - x\| < r\}$ and $\overline{B(x, r)} = \{y \in X : ||y - x|| \leq r\}$ in this paper. In the following, the recurrence relations are derived for the method given by (1.3) under the assumptions mentioned in previous section.
For $n = 0$, the existence of Γ_0 implies the existence of u_0 and y_0. This gives us

$$\|u_0 - x_0\| = \|\Gamma_0 F(x_0)\| \leq \eta_0. \tag{3.6}$$

This means that $u_0, y_0 \in B(x_0, R\eta)$, where $R = \mathrm{g}(a_0)/(1-d_0)$. Furthermore, we have

$$\begin{aligned}\|I-\Gamma_0 F'(y_0)\| &\le \|\Gamma_0\|\|F'(x_0)-F'(y_0)\|\\ &\le M\|\Gamma_0\|\|y_0-x_0\|\\ &\le \frac{1}{2}a_0 < 1.\end{aligned}$$

Since the assumption $a_0 < s < 1$, by the Banach lemma it follows that $F'(y_0)^{-1}$ exists and

$$\|F'(y_0)^{-1}\| \le \frac{\|\Gamma_0\|}{1-\|\Gamma_0\|\|F'(x_0)-F'(y_0)\|} \le \frac{1}{1-\frac{1}{2}a_0}\|\Gamma_0\| = \frac{2}{2-a_0}\|\Gamma_0\|. \quad (3.7)$$

we obtain

$$\|z_0-x_0\| = \|F'(y_0)^{-1}F(x_0)\| \le \frac{2}{2-a_0}\|\Gamma_0 F(x_0)\|. \quad (3.8)$$

Consequently,

$$\begin{aligned}\|x_1-z_0\| &\le \left(\|F'(y_0)^{-1}\|\|F'(y_0)-F'(x_0)\| + \|F'(y_0)^{-1}F'(x_0)\|\right)\|\Gamma_0 F(z_0)\|\\ &\le \frac{2+a_0}{2-a_0}\beta\|F(z_0)\|. \quad (3.9)\end{aligned}$$

Since

$$\begin{aligned}F(z_0) &= F(x_0)+F'(x_0)(z_0-x_0)+\int_0^1 [F'(x_0+t(z_0-x_0))-F'(x_0)]\,dt(z_0-x_0)\\ &= F(x_0)+F'(y_0)(z_0-x_0)+(F'(x_0)-F'(y_0))(z_0-x_0)\\ &\quad + \int_0^1 [F'(x_0+t(z_0-x_0))-F'(x_0)]\,dt(z_0-x_0). \quad (3.10)\end{aligned}$$

then

$$\|F(z_0)\| \le \frac{4-a_0}{(2-a_0)^2}M\eta_0^2. \quad (3.11)$$

From(3.8), (3.9) and (3.11), we have

$$\|x_1-x_0\| \le \|x_1-z_0\|+\|z_0-x_0\| \le \mathrm{g}(a_0)\eta_0. \quad (3.12)$$

From the assumption $d_0 < 1/h_0 < 1$, it follows that $x_1 \in B(x_0, R\eta)$.
By $a_0 < s$ and $\mathrm{g}(a_0) < \mathrm{g}(s)$, we have

$$\begin{aligned}\|I-\Gamma_0 F'(x_1)\| &\le \|\Gamma_0\|\|F'(x_0)-F'(x_1)\|\\ &\le M\|\Gamma_0\|\|x_1-x_0\| \le a_0\mathrm{g}(a_0) < 1.\end{aligned}$$

It follows by Banach lemma that $\Gamma_1 = [F'(x_1)]^{-1}$ exists and

$$\begin{aligned}\|\Gamma_1\| &\le \frac{\|\Gamma_0\|}{1-\|\Gamma_0\|\|F'(x_0)-F'(x_1)\|}\\ &\le \frac{\beta_0}{1-a_0\mathrm{g}(a_0)} = h(a_0)\beta_0 = \beta_1. \quad (3.13)\end{aligned}$$

By Lemma (2.4) and Lemma (3.1), we get

$$\|F(z_0)\| \le \frac{1}{4}Ma_0\|z_0-x_0\|^2+\frac{1}{4}N\|y_0-x_0\|\|z_0-x_0\|^2+\frac{1}{6}N\|z_0-x_0\|^3. \quad (3.14)$$

and

$$\|F(x_1)\| \le \left[\frac{a_0}{2-a_0}M\|u_0-x_0\| + \frac{1}{8}N\|u_0-x_0\|^2\right]\beta\|F(z_0)\| \\ + \left[\frac{5}{8}N\|u_0-x_0\|^2 + M\|u_0-z_0\|\right]\|x_1-z_0\| + \frac{1}{2}M\|x_1-z_0\|^2. \tag{3.15}$$

From (3.13), (3.14) and (3.15), we have

$$\|u_1-x_1\| = \|\Gamma_1 F(x_1)\| \le \|\Gamma_1\|\|F(x_1)\| \\ \le h(a_0)\phi(a_0,b_0)\eta_0 = d_0\eta_0 = \eta_1. \tag{3.16}$$

Because of $g(a_0) > 1$, we obtain

$$\|u_1-x_0\| \le \|u_1-x_1\| + \|x_1-x_0\| \\ \le (g(a_0)+d_0)\eta_0 < g(a_0)(1+d_0)\eta < R\eta. \tag{3.17}$$

which shows that u_1 and hence $y_1 \in B(x_0, R\eta)$.
In addition, we have

$$M\|\Gamma_1\|\|\Gamma_1 F(x_1)\| \le h(a_0)d_0a_0 = a_1, \tag{3.18}$$

$$N\|\Gamma_1\|\|\Gamma_1 F(x_1)\|^2 \le h(a_0)d_0^2b_0 = b_1. \tag{3.19}$$

Repeating the above derivation, we can obtain the system of recurrence relations given in next lemma.

Lemma 3.2. Let the assumptions of Lemma (2.6) and conditions (C1)-(C4) hold. Then the following items are true for each $n = 0, 1, 2, \dots$

(i) There exists $\Gamma_n = [F'(x_n)]^{-1}$ and $\|\Gamma_n\| \le \beta_n$,
(ii) $\|\Gamma_n F(x_n)\| \le \eta_n$,
(iii) $M\|\Gamma_n\|\|\Gamma_n F(x_n)\| \le a_n$,
(iv) $N\|\Gamma_n\|\|\Gamma_n F(x_n)\|^2 \le b_n$,
(v) $\|x_{n+1}-x_n\| \le g(a_n)\eta_n$,
(vi) $\|x_{n+1}-x_0\| \le R\eta$, where $R = \frac{g(a_0)}{1-d_0}$,
(vii) $R < 1/a_0$.

Proof. The proof of (i)-(v) follows by using the above mentioned way and invoking the induction hypothesis. We only consider (vi). By (v) and Lemma (2.9) we obtain

$$\|x_{n+1}-x_0\| \le \|x_{n+1}-x_n\| + \|x_n-x_{n-1}\| + \cdots + \|x_1-x_0\| \\ \le g(a_n)\eta_n + g(a_{n-1})\eta_{n-1} + \cdots + g(a_0)\eta_0 \\ \le g(a_0)[\eta_n + \eta_{n-1} + \cdots + \eta_0] \\ \le g(a_0)[\lambda^n\gamma^{\frac{5^n-1}{4}} + \lambda^{n-1}\gamma^{\frac{5^{n-1}-1}{4}} + \cdots + 1]\eta.$$

By Bernoulli's inequality, for every real number $x > -1$ and every integer $k \ge 0$, we have $(1+x)^k - 1 \ge kx$. Thus

$$\|x_{n+1}-x_0\| \le g(a_0)\frac{1-(\lambda\gamma)^{n+1}}{1-\lambda\gamma}\eta = g(a_0)\frac{1-(d_0)^{n+1}}{1-d_0}\eta < R\eta,$$

since $\lambda\gamma = d_0$ and $d_0 < 1$.
Finally, From the definition of R and d_0 it can be obtained that

$$R = \frac{g(a_0)}{1-d_0} = \frac{g(a_0)}{1-h(a_0)\phi(a_0,b_0)} < 1/a_0.$$

The lemma is proved. □

4. Semilocal convergence

Now we give a theorem to establish the semilocal convergence of (1.3), the existence and uniqueness of the solution and the domain in which it is located, along with a priori error bounds, which lead to the R-order of convergence at least five of iteration (1.3).

Theorem 4.1. Let X and Y be two Banach spaces and $F : \Omega \subseteq X \to Y$ be a nonlinear twice Fréchet differentiable operator on a non-empty open convex subset Ω. g, h and ϕ are defined by (2.6)-(2.8). $a_0 = M\beta\eta, b_0 = N\beta\eta^2$ and $d_0 = h(a_0)\phi(a_0, b_0)$ satisfy $a_0 < s$ and $h(a_0)d_0 < 1$, $\overline{B(x_0, R\eta)} \in \Omega$ where $R = \frac{g(a_0)}{1-d_0}$. Assume that $x_0 \in \Omega$ and all conditions (C1)-(C4) hold. Then
(i) Starting from x_0, the sequence $\{x_n\}$ generated by method (1.3) converges to a solution x^* of $F(x)$ with x_n, x^* belong to $\overline{B(x_0, R\eta)}$,
(ii) x^* is the unique solution of $F(x)$ in $B\left(x_0, \dfrac{2}{M\beta} - R\eta\right) \cap \Omega$,
(iii) A priori error estimate is given by

$$\|x_n - x^*\| \le g(a_0)\eta\lambda^n\gamma^{\frac{5^n-1}{4}}\frac{1}{1-\lambda\gamma^{5^n}}, \tag{4.1}$$

where $\gamma = h(a_0)d_0$ and $\lambda = 1/h(a_0)$.

Proof. (i) By Lemma (3.2), the sequence $\{x_n\}$ is well defined in $\overline{B(x_0, R\eta)}$. Next we prove that $\{x_n\}$ is a Cauchy sequence. Since

$$\begin{aligned}
\|x_{m+n} - x_n\| &\le \|x_{m+n} - x_{m+n-1}\| + \cdots + \|x_{n+1} - x_n\| \\
&\le g(a_{m+n-1})\eta_{m+n-1} + \cdots + g(a_n)\eta_n \\
&\le g(a_{m+n-1})\gamma^{\frac{5^{m+n-1}-1}{4}}\lambda^{m+n-1}\eta + \cdots + g(a_n)\gamma^{\frac{5^n-1}{4}}\lambda^n\eta \\
&\le g(a_n)\lambda^n\left[\gamma^{\frac{5^{m+n-1}-1}{4}}\lambda^{m-1} + \cdots + \gamma^{\frac{5^n-1}{4}}\right]\eta \\
&= g(a_n)\gamma^{\frac{5^n-1}{4}}\lambda^n\left[\gamma^{\frac{5^n[5^{m-1}-1]}{4}}\lambda^{m-1} + \cdots + \gamma^{\frac{5^n[5-1]}{4}}\lambda + 1\right]\eta,
\end{aligned}$$

by Bernoulli's inequality, for every real number $x > -1$ and every integer $k \ge 0$, we have

$$(1+x)^k - 1 \ge kx.$$

Thus,

$$\|x_{m+n} - x_n\| \le g(a_0)\gamma^{\frac{5^n-1}{4}}\lambda^n\frac{1-\gamma^{m.5^n}\lambda^m}{1-\gamma^{5^n}\lambda}\eta. \tag{4.2}$$

It follows that $\{x_n\}$ is a cauchy sequence. So there exists a x^* such that $\lim\limits_{n\to\infty} x_n = x^*$.

By letting $n = 0, m \to \infty$ in (4.2), we obtain

$$\|x^* - x_0\| \leq R\eta, \tag{4.3}$$

This shows $x^* \in \overline{B(x_0, R\eta)}$.

(ii) Firstly we prove that x^* is a solution of $F(x) = 0$. Since

$$\begin{aligned}\|F'(x_n)\| &\leq \|F'(x_0)\| + \|F'(x_n) - F'(x_0)\| \\ &\leq \|F'(x_0)\| + M\|x_n - x_0\| \\ &\leq \|F'(x_0)\| + MR\eta,\end{aligned}$$

we can obtain

$$\begin{aligned}\|F(x_n)\| &\leq \|F'(x_n)\|\|z_n - x_n\| + \|F'(x_n) - F'(y_n)\|\|z_n - x_n\| \\ &\leq (\|F'(x_0)\| + MR\eta)\frac{2}{2 - a_n}\eta_n + \frac{1}{2 - a_n}M\eta_n^2 \\ &\leq (\|F'(x_0)\| + MR\eta)\frac{2}{2 - a_0}\eta_n + \frac{1}{2 - a_0}M\eta_n^2.\end{aligned} \tag{4.4}$$

By letting $n \to \infty$ in (4.4), we find that $\|F(x_n)\| \to 0$ since $\eta_n \to 0$. Hence, by continuity of F in Ω, we obtain $F(x^*) = 0$.

Now we prove the uniqueness of x^* in $B\left(x_0, \dfrac{2}{M\beta} - R\eta\right) \cap \Omega$. Firstly we see that $x^* \in B\left(x_0, \dfrac{2}{M\beta} - R\eta\right) \cap \Omega$. By using the fact $R < 1/a_0$, it follows that

$$\frac{2}{M\beta} - R\eta = \left(\frac{2}{a_0} - R\right)\eta > \frac{1}{a_0}\eta > R\eta,$$

and then $\overline{B(x_0, R\eta)} \subseteq B\left(x_0, \dfrac{2}{M\beta} - R\eta\right) \cap \Omega$.

Let $y^* \in B\left(x_0, \dfrac{2}{M\beta} - R\eta\right) \cap \Omega$ is another zero of $F(x)$. By Taylor's theorem, we have

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))dt(y^* - x^*). \tag{4.5}$$

Since

$$\begin{aligned}&\|\Gamma_0\|\left\|\int_0^1 \left[F'(x^* + t(y^* - x^*)) - F'(x_0)\right]dt\right\| \\ &\leq M\beta\int_0^1 \|x^* + t(y^* - x^*) - x_0\|dt \\ &\leq M\beta\int_0^1 \big((1 - t)\|x^* - x_0\| + t\|y^* - x_0\|\big)dt \\ &< \frac{M\beta}{2}\left(R\eta + \frac{2}{M\beta} - R\eta\right) = 1.\end{aligned} \tag{4.6}$$

It follows by Banach lemma that $\displaystyle\int_0^1 (F'(x^* + t(y^* - x^*))dt$ is invertible and hence $y^* = x^*$.

By letting $m \to \infty$ in (4.2), we obtain (4.1) and furthermore

$$\|x_n - x^*\| \leq \frac{g(a_0)\eta}{\gamma^{1/4}(1-d_0)} \left(\gamma^{1/4}\right)^{5^n}. \tag{4.7}$$

This means that the method given by (1.3) is of R-order convergence at least five. □

5. Local Convergence

In this section we present a local convergence analysis using the hypotheses only on the first derivative that appears in method (1.3).
Let $w_0 : [0, +\infty) \to [0, +\infty)$ be a continuous and nondecreasing function satisfying $w_0(0) = 0$.
Define the parameter s_0 by

$$s_0 = \sup\{t \geq 0 : w_0(t) < 1\}. \tag{5.1}$$

Define functions g_1, h_1, g_2 and h_2 on the interval, $[0, s_0)$ by

$$g_1(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{1 - w_0(t)},$$
$$h_1(t) = g_1(t) - 1,$$
$$g_2(t) = \frac{1}{2}(1 + g_1(t)),$$

and

$$h_2(t) = g_2(t) - 1,$$

where $w : [0, s_0) \to [0, +\infty)$ is a continuous and nondecreasing function satisfying $w(0) = 0$.
We have $h_1(0) = -1, h_2(0) = \frac{-1}{2}, h_1(t) \to +\infty$ as $t \to s_0^-$ and $h_2(t) \to +\infty$ as $t \to s_0^-$.
It follows by intermediate value theorem that equations $h_1(t) = 0, h_2(t) = 0$ have solutions in the interval $(0, s_0)$. Denote by r_1 and r_2, respectively the smallest such solutions. Define parameter s by

$$s = \max\{t \in [0, s_0] : w_0(g_2(t)t) < 1\}, \tag{5.2}$$

and the functions g_3 and h_3 on the interval $[0, s)$ by

$$g_3(t) = g_1(t) + \frac{(w_0(t) + w_0(g_2(t)t) \int_0^1 v(\theta t))d\theta}{(1 - w_0(g_2(t)t))(1 - w_0(t))}$$

and

$$h_3(t) = g_3(t) - 1,$$

where $v : [0, s) \to [0, +\infty)$ is a continuous and nondecreasing function. We get $h_3(0) = -1$ and $h_3(t) \to +\infty$ as $t \to s^-$.
Denote by r_3 the smallest solution of equation $h_3(t) = 0$ in $(0, s)$.
Moreover, define parameter s_1 by

$$s_1 = \max\{t \in [0, s) : w_0(g_3(t)t) < 1\} \tag{5.3}$$

and function s, g_4, h_4 on $[0, s_1)$ by

$$g_4(t) = [1 + \frac{(w_0(g_2(t)t) + w_0(g_3(t)t)) \int_0^1 v(\theta g_3(t)t) d\theta g_3(t)}{(1 - w_0(g_3(t)t))(1 - w_0(g_2(t)t))} + \frac{(w_0(t) + w_0(g_2(t)t)) \int_0^1 v(\theta g_3(t)t) d\theta g_3(t)}{(1 - w_0(g_2(t)t))(1 - w_0(t))}] g_3(t) \tag{5.4}$$

and $h_4(t) = g_4(t) - 1$. we get $h_4(0) = -1$ and $h_4(t) \to +\infty$as $t \to s_1^-$. Denote by r_4 the smallest solution of equation $h_4(t) = 0$ in $(0, s_1)$.
Define the radius of convergence r by

$$r = \min\{r_i\}, i = 1, 2, 3, 4. \tag{5.5}$$

Then, we have that for each $t \in [0, r)$,

$$0 \leq g_i(t) < 1. \tag{5.6}$$

The local convergence analysis that follows uses the preceding notation and conditions (A):

: $(a_1) F : \Omega \subseteq X \to Y$ is a continuously Fréchet differentiable operator and there exists $x^* \in \Omega$ such that $F(x^*) = 0$ with $F'(x^*)^{-1} \in \mathcal{L}(Y, X)$

: (a_2)There exists $w_0 : [0, +\infty) \to [0, +\infty)$ continuous and nondecreasing function satisfying $w_0(0) = 0$ and for each $x \in \Omega$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq w_0(\|x - x^*\|).$$

Let $\Omega_0 = \Omega \cap U(x^*, s_0), \Omega_1 = \Omega \cap U(x^*, s)$.

: (a_3) There exists functions $w : [0, +\infty) \to [0, +\infty)$ with $w(0) = 0$ and $v : [0, s) \to [0, +\infty)$ is a continuous and nondecreasing such that

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq w(\|x - y\|) \quad \text{for each} \quad x, y \in \Omega_0 \quad \text{and}$$

$$\|F'(x^*)^{-1}F'(x)\| \leq v(\|x - x^*\|) \quad \text{for each} \quad x, y \in \Omega_1.$$

: $(a_4)\overline{U}(x^*, r) \subseteq \Omega$.

: (a_5) There exists $R \geq r$ such that $\int_0^1 v(\theta R) d\theta < 1$.

Let $\Omega_2 = \Omega \cap \overline{U}(x^*, R)$

Theorem 5.1. Suppose that the conditions (A) hold. Then, sequence $\{x_n\}$ starting from $x_0 \in U(x^*, r) - \{x^*\}$ and generated by method (1.3) is well defined in $U(x^*, r)$, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following estimates hold

$$\|u_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r, \tag{5.7}$$

$$\|y_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{5.8}$$

$$\|z_n - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{5.9}$$

and

$$\|x_{n+1} - x^*\| \leq g_4(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{5.10}$$

where the functions g_i are defined previously. Furthermore, the point x^* is the only solution of equation $F(x) = 0$ in Ω_2.

Proof. We shall show estimates (5.7)- (5.10) using mathematical induction. Let $x \in U(x^*, r) - \{x^*\}$. Using (5.5) and (a_2), we have in turn that

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq w_0(\|x - x^*\|) \leq w_0(r) < 1. \tag{5.11}$$

It follows from (5.11) and Banach lemma on invertible operators that $F'(x)^{-1} \in \mathcal{L}(Y, X)$ and

$$\|F'(x)^{-1}F'(x^*)\| \leq \frac{1}{1 - w_0(\|x - x^*\|)}. \tag{5.12}$$

In particular for $x = x_0, u_0$ and y_0 are well defined by the first and second substep of method (1.3), respectively. We can write

$$\begin{aligned} u_0 - x^* =& x_0 - x^* - F'(x_0)^{-1}F(x_0) \\ =& (F'(x_0)^{-1}F'(x^*)) \left(\int_0^1 F'(x^*)^{-1}\big(F'(x^* + \theta(x_0 - x^*)) - F'(x_0)\big) \right) (x_0 - x^*) d\theta. \end{aligned} \tag{5.13}$$

So by (5.5), (a_3), (5.12) and (5.13), we get in turn that

$$\begin{aligned} \|u_0 - x^*\| &\leq \|(F'(x_0)^{-1}F'(x^*))\| \left\| \left(\int_0^1 F'(x^*)^{-1}\big(F'(x^* + \theta(x_0 - x^*)) - F'(x_0)\big) \right) (x_0 - x^*) d\theta \right\| \\ &\leq \frac{\int_0^1 w((1-\theta))\|x_0 - x^*\| d\theta \|x_0 - x^*\|}{1 - w_0(\|x_0 - x^*\|)} = g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \\ &\leq \|x_0 - x^*\| < r, \end{aligned} \tag{5.14}$$

so (5.7) holds for $n = 0$ and $u_0 \in U(x^*, r)$.
In view of the second substep of method (1.3), (5.5) and (5.14), we obtain in turn that

$$\begin{aligned} \|y_0 - x^*\| &= \frac{1}{2}\|(x_0 - x^*) + (u_0 - x^*)\| \\ &\leq \frac{1}{2}\left(\|x_0 - x^*\| + \|(u_0 - x^*)\|\right) \\ &\leq \frac{1}{2}\left(1 + g_1(\|x_0 - x^*\|)\right)\|x_0 - x^*\| \\ &\leq g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \end{aligned} \tag{5.15}$$

so (5.8) holds for $n = 0$ and $y_0 \in U(x^*, r)$.
By (a_3), we can write for $x \in U(x^*, r)$

$$F(x) = F(x) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*)) d\theta (x_0 - x^*), \tag{5.16}$$

so

$$\|F'(x^*)^{-1}F(x)\| \leq \int_0^1 v(\theta\|x_0 - x^*\|) d\theta \|x_0 - x^*\|. \tag{5.17}$$

Then, by the third substep of method (1.3) we can write

$$z_0 - x^* = (x_0 - x^* - F'(x_0)^{-1}F(x_0)) + F'(y_0)^{-1}(F'(x_0) - F'(y_0))F'(x_0)^{-1}F(x_0). \tag{5.18}$$

Then, by (5.5), (5.14-5.18)

$$\begin{aligned}\|z_0 - x^*\| \leq &\left[g_1(\|x_0 - x^*\|) + \frac{w_0(\|x_0 - x^*\|) + w_0(\|y_0 - x^*\|)\int_0^1 v(\theta\|x_0 - x^*\|)d\theta}{(1 - w_0(\|y_0 - x^*\|))(1 - w_0(\|x_0 - x^*\|))}\right]\|x_0 - x^*\| \\ &\leq g_3(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \end{aligned} \tag{5.19}$$

which implies that (5.9) holds for $n = 0$ and $z_0 \in U(x^*, r)$.
Next, from the last substep of method (1.3) we can write

$$\begin{aligned} x_1 - x^* =&(z_0 - x^* - F'(z_0)^{-1}F(z_0)) + F'(z_0)^{-1}(F'(y_0) - F'(z_0))F'(y_0)^{-1}F(z_0) \\ &+ F'(y_0)^{-1}(F'(x_0) - F'(y_0))F'(x_0)^{-1}F(z_0), \end{aligned} \tag{5.20}$$

so

$$\begin{aligned} \|x_1 - x^*\| \leq &\frac{\int_0^1 w((1-\theta)\|z_0 - x^*\|)d\theta\|z_0 - x^*\|}{1 - w_0(\|z_0 - x^*\|)} \\ &+ \frac{(w_0(\|y_0 - x^*\|) + w_0(\|z_0 - x^*\|))\int_0^1 v(\theta\|z_0 - x^*\|)d\theta\|z_0 - x^*\|}{(1 - w_0(\|z_0 - x^*\|))(1 - w_0(\|y_0 - x^*\|))} \\ &+ \frac{(w_0(\|x_0 - x^*\|) + w_0(\|y_0 - x^*\|))\int_0^1 v(\theta\|z_0 - x^*\|)d\theta\|z_0 - x^*\|}{(1 - w_0(\|y_0 - x^*\|))(1 - w_0(\|x_0 - x^*\|))} \\ &\leq g_4(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \end{aligned} \tag{5.21}$$

which shows (5.10) for $n = 0$ and $x_1 \in U(x^*, r)$.
Notice that we also used (5.12) for $x = y_0, z_0$ and (5.17) for $x = x_0, z_0$.
The induction can clearly be completed if, we replace x_0, u_0, y_0, z_0, x_1 by $x_k, u_k, y_k, z_k, x_{k+1}$ in the preceding estimates. Then, from the estimate

$$\|x_{k+1} - x^*\| \leq c\|x_k - x^*\| < r, \text{ where } c = g_4(\|x_0 - x^*\|) \in [0, 1), \tag{5.22}$$

we obtain $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$. Let

$T = \int_0^1 F'(x^* + \theta(y^* - x^*))d\theta$ for some $y^* \in \Omega$ with $F(y^*) = 0$.
By (a_2) and (a_5), we get

$$\begin{aligned}\|F'(x^*)^{-1}(T - F'(x^*))\| &\leq \int_0^1 w_0(\theta\|x^* - y^*\|)d\theta \\ &\leq \int_0^1 w_0(\theta R) < 1, \end{aligned}$$

so $T^{-1} \in \mathcal{L}(Y, X)$. Finally, from the identity

$$0 = F(y^*) - F(x^*) = T(y^* - x^*),$$

we conclude that $x^* = y^*$. □

6. Numerical Testing

In this section, a number of numerical examples are worked out in order to check the applicability of (1.3) that now we denote by M_1^5. The values of the sequences $\{\eta_n\}$, $\{\beta_n\}$, $\{a_n\}$, $\{b_n\}$ and $\{d_n\}$ are computed for all the examples and summarized in the tables. We compare the presented method with fourth-order method by Argyros et al. [23] denoted by M_1^4, fifth-order method by Cordero et al. [38] denoted by M_2^5 and fifth-order method by Singh et al. [40] denoted by M_3^5.

The above mentioned methods are given as follows:
Fourth order method given by Argyros et.al (M_1^4):

$$\begin{aligned} y_k &= x_k - \Gamma_k F(x_k), \\ z_k &= x_k + \frac{2}{3}(y_k - x_k), \\ x_{k+1} &= y_k - \frac{3}{4}H(x_k)\left[I - \frac{3}{2}H(x_k)\right](y_k - x_k), \end{aligned} \tag{6.1}$$

where

$$H(x_k) = \Gamma_k[F'(z_k) - F'(x_k)].$$

Fifth order method given by Cordero et.al (M_2^5):

$$\begin{aligned} y_k &= x_k - \Gamma_k F(x_k), \\ z_k &= y_k - 5\Gamma_k F(y_k), \\ x_{k+1} &= z_k - \frac{1}{5}\Gamma_k(-16F(y_k) + F(z_k)). \end{aligned} \tag{6.2}$$

Fifth order method given by Singh et al. (M_3^5):

$$\begin{aligned} y_k &= x_k - \Gamma_k F(x_k), \\ z_k &= y_k - \Gamma_k F(y_k), \\ x_{k+1} &= z_k - [F'(y_k)]^{-1}F(z_k). \end{aligned} \tag{6.3}$$

Example 6.1. *Consider the equation $F(x) = 0$, where*

$$F(x) = \begin{cases} x^3 - 2x - 5 & , x \geq 0 \\ -x^3 - 2x - 13 & , x < 0 \end{cases} \tag{6.4}$$

$$F(x) = \begin{cases} x^3 - 2x - 5 & , x \geq 0 \\ -x^3 - 2x - 13 & , x < 0 \end{cases}$$

on $[-1, 3]$.
It is easy to find first derivative of F as

$$F'(x) = \begin{cases} 3x^2 - 2 & , x \geq 0 \\ -3x^2 - 2 & , x < 0 \end{cases}$$

and the second derivative as

$$F''(x) = \begin{cases} 6x & , x \geq 0 \\ -6x & , x < 0 \end{cases}$$

The second derivative F'' satisfies Lipschitz condition as,

$$\|F''(x) - F''(y)\| = 6\||x| - |y|\| \leq 6\|x - y\|.$$

Now, for the initial point $x_0 = 2$, we can obtain

$$\beta = \|F'(x_0)^{-1}\| = 0.1, \quad \eta = \|F'(x_0)^{-1}F(x_0)\| = 0.1, \quad M = 18, \quad N = 6.$$

Therefore, $a_0 = M\beta\eta = 0.18, \quad b_0 = N\beta\eta^2 = 0.006$, *which satisfy*

$$q(a_0) = a_0 g(a_0) - 1 = -0.757442 < 0,$$

and

$$d_0 h(a_0) \simeq 0.000961577 < 1.$$

This means that the hypotheses of Theorem (4.1) is satisfied. Hence the recurrence relations for the method given by (1.3) is demonstrated in Table 1. Besides

TABLE 1. Results of recurrence relations

n	η_n	β_n	a_n	b_n	d_n
0	$1.00000e-001$	$1.00000e-001$	$1.80000e-001$	$6.00000e-003$	$7.28339e-004$
1	$7.28339e-005$	$1.32023e-001$	$1.73084e-004$	$4.20213e-009$	$2.88713e-016$
2	$2.10281e-020$	$1.32046e-001$	$4.99802e-020$	$3.50329e-040$	$2.00620e-078$
3	$4.21865e-098$	$1.32046e-001$	$1.00270e-097$	$1.41002e-195$	$3.24992e-389$

TABLE 2. Results of problem (6.4)

k	*Method* M_1^4	*Method* M_1^5	*Method* M_2^5	*Method* M_3^5
1	$1.034925e-003$	$3.686228e-005$	$1.989572e-004$	$4.704464e-005$
2	$6.941069e-016$	$1.464104e-027$	$3.053532e-022$	$5.966369e-027$
3	$1.405131e-064$	$1.447150e-139$	$2.599923e-117$	$1.957486e-136$

the solution x^* belongs to $\overline{B(x_0, R\eta)} = \overline{B(2, 0.134853\dots)} \subseteq \Omega$ and is unique in $B(2, 0.976258\dots) \cap \Omega$.
Now we apply the presented method to compute (6.4) and compare it with methods M_1^4, M_2^5 and M_3^5. Displayed in Table 2 is the norm of vector functions at each iterative step. It can be observed that accuracy of the method M_1^5 is higher than the respective competitors in terms of number of significant digits gained by each method.

Example 6.2. *Consider the nonlinear integral equation $F(x) = 0$, where*

$$F(x)(s) = x(s) - \frac{4}{3} + \frac{1}{2}\int_0^1 s\cos(x(t))dt, \tag{6.5}$$

where $s \in [0,1]$, $\quad x \in \Omega = B(0,2) \subset X$. Here, $X = C[0,1]$ is the space of continuous functions on [0,1] with the max-norm

$$\|x\| = \max_{s\in[0,1]} |x(s)|.$$

We can obtain the derivatives of F given by

$$F'(x)y(s) = y(s) - \frac{1}{2}\int_0^1 s\sin(x(t))y(t)dt, \quad y \in \Omega,$$

$$F''(x)yz(s) = -\frac{1}{2}\int_0^1 s\cos(x(t))y(t)z(t)dt, \quad y, z \in \Omega.$$

Furthermore, we have

$$\|F''(x)\| \le \frac{1}{2} \equiv M, x \in \Omega$$

and the Lipschitz condition with $N = \frac{1}{2}$.

$$\|F''(x) - F''(y)\| \le \frac{1}{2}\|x - y\|, x, y \in \Omega.$$

A constant function, i.e. $x_0(t) = 4/3$, is chosen as the initial approximate solution. It follows that

$$\|F(x_0)\| \le \frac{1}{2}\cos\frac{4}{3}.$$

TABLE 3. Results recurrence relations

n	η_n	β_n	a_n	b_n	d_n
0	$2.88165e-001$	$1.95408e+000$	$2.225707e-001$	$5.092785e-002$	$6.106925e-003$
1	$1.397374e-003$	$2.879977e+000$	$2.012190e-003$	$2.811764e-006$	$1.135642e-011$
2	$1.586907e-014$	$2.885801e+000$	$2.289749e-014$	$3.633619e-028$	$1.888888e-055$
3	$2.997489e-069$	$2.885801e+000$	$4.325079e-069$	$1.296438e-137$	$2.404534e-274$

TABLE 4. Results of the system (6.6) for $m = 35$

k	*Method* M_1^4	*Method* M_1^5	*Method* M_2^5	*Method* M_3^5
1	$3.61303e-007$	$1.34443e-008$	$7.09395e-009$	$2.08428e-006$
2	$3.46796e-029$	$2.0685e-043$	$9.11033e-045$	$8.72599e-021$
3	$2.94364e-117$	$1.78337e-217$	$3.18247e-224$	$6.39529e-064$

In this case, we have

$$\|I - F'(x_0)\| \leq \frac{1}{2}\sin\frac{4}{3},$$

and then by the Banach lemma we include that Γ_0 *exists and satisfies*

$$\|\Gamma_0\| \leq \frac{2}{2-\sin\frac{4}{3}} \equiv \beta.$$

It follows that

$$\|\Gamma_0 F(x_0)\| \leq \frac{\cos\frac{4}{3}}{2-\sin\frac{4}{3}} \equiv \eta.$$

Therefore, we obtain

$$a_0 = M\beta\eta = \frac{\cos\frac{4}{3}}{(2-\sin\frac{4}{3})^2}, \quad b_0 = N\beta\eta^2 = \frac{\cos^2\frac{4}{3}}{(2-\sin\frac{4}{3})^2}.$$

As a result, we compute

$$q(a_0) = a_0 g(a_0) - 1 \simeq -0.6755 < 0,$$

and

$$d_0 h(a_0) \simeq 0.009041 < 1.$$

This means that the hypotheses of Theorem (4.1) is satisfied. Hence the recurrence relations for the method given by (1.3) is demonstrated in Table 3. Besides, the solution x^* *belongs to* $\overline{B(x_0, R\eta)} = \overline{B(4/3, 0.3357\dots)} \subseteq \Omega$ *and is unique in* $B(4/3, 1.7205\dots) \cap \Omega$.

Using Trapezoidal rule of integration with step $h = 1/m$ *to discretize (6.5), we obtain the following system of nonlinear equations*

$$0 = x_i - \frac{4}{3} + \frac{s_i}{2m}\left(\frac{1}{2}\cos(x_0) + \sum_{j=1}^{m-1}\cos(x_j) + \frac{1}{2}\cos(x_m)\right), i = 0, 1, \dots m, \quad (6.6)$$

where $s_i = t_i = i/m$ *and* $x_i = x(t_i)$. *Now we apply the presented method given by (1.3) to compute (6.6) and compare it with methods* M_1^4, M_2^5 *and* M_3^5. *We give initial guess* $x_i = 4/3, i = 0, 1, \dots m$. *In the tests, we take* $m = 35$ *in (6.6), respectively. Displayed in Table 4 is the norm of vector functions at each iterative step. From* *the numerical results, we can see that there is no clear winner between the methods M_1^5 and M_3^5 but the performance of M_1^5 is better than M_1^4 and M_2^5.*

Example 6.3. *Consider the boundary value problem*

$$y'' + y' - y^3 = 0, \quad y(0) = y(1) = 0 \tag{6.7}$$

We divide the interval $[0,1]$ into n subintervals and we set $h = \frac{1}{n}$. Let $\{z_k\}$ be the points of the subdivision with

$$0 = z_0 < z_1 < z_2 < \cdots < z_n = 1$$

and the corresponding values of the function

$$y_0 = y(z_0) = 0, y_1 = y(z_1), \ldots, y_n = y(z_n) = 0$$

Standard approximations for the first and second derivatives are given respectively by

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h}, y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}, \quad i = 1, 2, \ldots, n-1. \tag{6.8}$$

Define the operator $F : \mathbf{R^{n-1}} \to \mathbf{R^{n-1}}$ by

$$F(y) = Gy + hJy - 2h^2 g(y),$$

where

$$G = \begin{pmatrix} -4 & 2 & 0 & \cdots & 0 \\ 2 & -4 & 2 & \cdots & 0 \\ 0 & 2 & -4 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & \cdots & \cdots & -4 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

$$g(y) = \begin{pmatrix} y_1^3 \\ y_2^3 \\ \vdots \\ y_{n-1}^3 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix},$$

then, we get

$$F'(y) = G + hJ - 6h^2 \begin{pmatrix} y_1^2 & 0 & 0 & \cdots & 0 \\ 0 & y_2^2 & 0 & \cdots & 0 \\ 0 & 0 & y_3^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & \cdots & \cdots & y_{n-1}^2 \end{pmatrix},$$

$$F''(y) = -12h^2 \begin{pmatrix} y_1 & 0 & 0 & \cdots & 0 \\ 0 & y_2 & 0 & \cdots & 0 \\ 0 & 0 & y_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & \cdots & \cdots & y_{n-1} \end{pmatrix},$$

Let $x \in \mathbf{R^{n-1}}$, $A \in \mathbf{R^{n-1}} \times \mathbf{R^{n-1}}$, and define the norms of x and A by

$$\|x\| = \max_{1 \le i \le n-1} |x_i|, \quad \|A\| = \max_{1 \le i \le n-1} \sum_{k=1}^{n-1} |a_{ik}|$$

TABLE 5. Results of recurrence relations

n	η_n	β_n	a_n	b_n	d_n
0	$1.68893e-001$	$6.11998e+000$	$2.09643e-002$	$2.09486e-002$	$1.07818e-004$
1	$1.82097e-005$	$6.25536e+000$	$2.31034e-006$	$2.48908e-010$	$1.41965e-020$
2	$2.58514e-025$	$6.25538e+000$	$3.27988e-026$	$5.01653e-050$	$5.76644e-100$
3	$1.4907e-124$	$6.25538e+000$	$1.89132e-125$	$1.66809e-248$	$6.37584e-497$

TABLE 6. Results of the system (6.9) for $n = 10$

k	*Method* M_1^4	*Method* M_1^5	*Method* M_2^5	*Method* M_3^5
1	$9.353822e-008$	$7.753839e-008$	$4.752371e-008$	$2.497539e-006$
2	$8.095574e-038$	$3.517723e-042$	$1.050730e-042$	$4.521260e-034$
3	$8.304444e-187$	$2.412028e-209$	$1.098868e-209$	$2.865670e-170$

For $n = 10$, we now get

$$\|F''(x) - F''(y)\| \leq 0.12\|x - y\|$$

As the solution should vanish at the end points and be positive in the interior, a reasonable choice of initial approximation seems to be $\frac{\exp(\pi x)}{100}$. This gives the following vector:
$x_0 = \{0.0136910777062484688 4\ldots, 0.01874456087585338350 6\ldots, 0.02566332395208135320 9\ldots,$
$0.0351358562428573363 77\ldots, 0.0481047738096535165 55\ldots, 0.0658606196269472485 84\ldots,$
$0.0901702861094207824 23\ldots, 0.1234528393918736858 0\ldots, 0.1690202417171154602 0\}^T$
Now we get the following results for our method: $\|\Gamma_0\| \leq \beta = 6.11998786340534387956 50$,
$\|\Gamma_0 F(x_0)\| \leq \eta = 0.168892624074745025$, $\quad \|F''(x)\| \leq M = 0.02028242900605385522 4$,
$N = 0.12, a_0 = M\beta\eta = 0.0209643406890019314, b_0 = N\beta\eta^2 = 0.0209485$. *which satisfy*

$$q(a_0) = a_0 g(a_0) - 1 = -0.9783576077847256610 < 0,$$

and

$$d_0 h(a_0) \simeq 0.000110204 < 1.$$

This means that the hypotheses of Theorem (4.1) is satisfied. Hence the recurrence relations for the method given by (1.3) is demonstrated in Table 5. This implies that the solution of the equation (6.7) exists in the ball $\overline{B(1, 0.174374\ldots)} \subseteq \Omega$ and is unique in $B(1, 15.8846\ldots) \cap \Omega$.

If we discretize the problem (6.7) by using (6.8), we obtain the following system of equations

$$2(y_{i+1} - 2y_i + y_{i-1}) + h(y_{i+1} - y_{i-1}) - 2h^2 y_i^3 = 0, \quad i = 1, 2, 3, \ldots, 9. \tag{6.9}$$

Now we apply the presented method given by (1.3) to compute (6.9) and compare it with methods M_1^4, M_2^5 and M_3^5. Displayed in Table 6 is the norm of vector functions at each iterative step. Numerical computations clearly show that the method M_1^5 is competitive with M_3^5 and behaves better than M_1^4 and M_2^5, so our method given by (1.3) can be of practical interest.

7. Global convergence

In this section, we present complex geometries of the methods in previous section based on basins of attraction when methods are applied to the complex polynomials. The basin of attraction is a useful geometrical tool for comparing convergence domains of the iterative methods (see [41, 42]). To start with, let us recall some basic dynamical concepts. Consider a rational function $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, where $\widehat{\mathbb{C}}$ is the Riemann sphere. The orbit of a point $x \in \widehat{\mathbb{C}}$ is defined as the set of successive images of x denoted by $\{x, R(x), R^2(x), \ldots, R^p(x), \ldots.\}$ In this way, a point x_0 is a fixed point of R if $R(x_0) = x_0$. A fixed point x_0 is called attracting if $\|R'(x_0)\| < 1$, repelling if $\|R'(x_0)\| > 1$ and neutral if $\|R'(x_0)\| = 1$. If $\|R'(x_0)\| = 0$, the point x_0 is superattracting. Let α be an attracting fixed point of the function R, its basins of attraction $\mathcal{A}(\alpha)$ is defined as the set of pre-images of any order such that

$$\mathcal{A}(\alpha) = \{x \in \widehat{\mathbb{C}} : R^p(x) \to \alpha \text{ for } p \to \infty\}$$

The Fatou set of the rational function R is the set of points $x \in \widehat{\mathbb{C}}$ whose orbits tend to an attractor (fixed point, periodic orbit or infinity). Its complement in $\widehat{\mathbb{C}}$ is the Julia set. That means that the basin of attraction of any fixed point belongs to the Fatou set and the boundaries of these basins of attraction belong to the Julia set.

To study dynamical behavior, we consider a system of quadratic equations, representing the intersection of two conics in $\mathbb{R}^2$ given as

$$\left.\begin{aligned} x^2 + 2y &= 3 \\ 2xy &= 1 \end{aligned}\right\}$$

presents three simple real roots that are superattractive fixed points for the methods in previous section. For generating basins of attraction associated with roots of nonlinear system of equations, we take a square $[-5, 5] \times [-5, 5]$ of 1024×1024 points, which contains all roots of concerned nonlinear system of equations and we apply the iterative method starting in every point in the square. We assign a color to each point according to the root to which the corresponding orbit of the iterative method, starting from the point, converges. If the corresponding orbit does not reach any root of the polynomial, with tolerance 10^{-3} in a maximum of 25 iterations, we mark those points with black color. For the given test problem, it can be observed in Fig.1 that all the roots of the polynomial system have their respective basins of attraction with different colors. Also the Julia set can be seen as black lines of unstable behavior. It can be easily observed that the method M_1^5 (Fig.1b) takes the lead followed by M_3^5 (Fig.1d) and M_1^4 (Fig.1a) whereas there are many divergent points in the considered region for method M_2^5 (Fig.1c).

8. Conclusions

In this paper, we have analyzed the semilocal convergence for a fifth-order iterative method in Banach spaces by using recurrence relations, giving the existence and uniqueness theorem that establishes the R-order of the method and the priori error bounds. In addition, local convergence analysis is based on Lipschitz type conditions, thereby extending the usage of the method. Theoretical results are applied on standard numerical examples like integral equation and boundary value problem to demonstrate the efficiency of our convergence analysis. Lastly, the basins of attraction of the proposed method are analyzed and compared with existing methods which shows that the performance of our method is better.

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Compliance with ethical standards

Conflict of interest The authors declare that they do not have conflict of interests.

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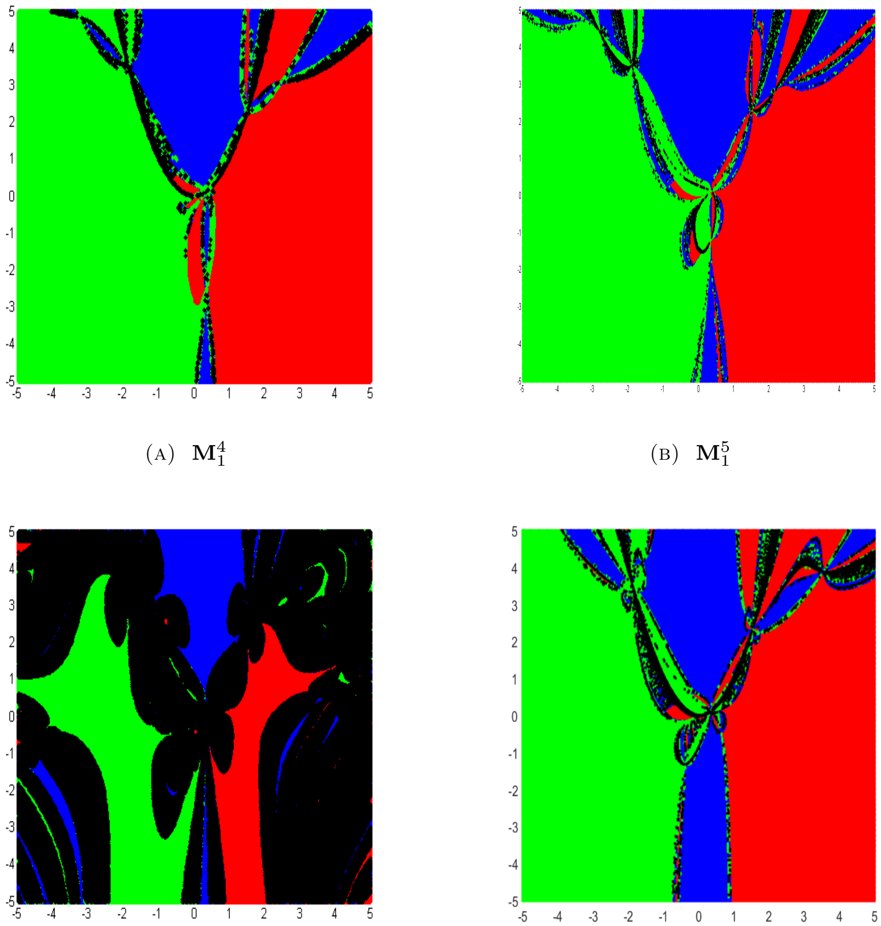

(A) $\mathbf{M}_1^4$ (B) $\mathbf{M}_1^5$

(C) $\mathbf{M}_2^5$ (D) $\mathbf{M}_3^5$

FIGURE 1. Basins of attraction for system of equations $x^2 + 2y = 3, 2xy = 1$ for various methods.

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