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UNCOUNTABLE K-BESSEL AND K-HILBERT SYSTEMS IN NONSEPARABLE BANACH SPACES

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Abstract. We consider the uncountable K-Besselness and K-Hilbertness of systems in nonseparable Banach spaces with respect to nonseparable Banach space of systems of scalars K. The criteria for K-Besselness and K-Hilbertness of systems in case where the space K is a space of systems consisting of coefficients of some uncountable unconditional basis are found. The relationship between the K-Besselness and K-Hilbertness of the system and the existence of the uncountable unconditional basis with the space of coefficients K is established.

1. Introduction

The concept of frame has been probably introduced by R.J. Duffin and A.C. Schaeffer in 1952 [1] in the study of non-harmonic Fourier series with respect to perturbed exponential systems. In this seminal work, Duffin and Schaefer established some properties of exponential frames. In the same work, they introduced the concept of abstract frame in separable Hilbert space and extended some properties of frames consisting of perturbed exponential systems to this concept. The interest to frames has grown in the 1980s due to wide applications of wavelet methods in various fields of natural science. Standing at the crossroads of theory and practice, the wavelets are widely used in processing and encoding of signals and different kinds of images (satellite images, roentgenograms of internal organs, etc), in pattern recognition, in the study of the properties of crystal surfaces and nano-objects, and in many other fields. Today, there are a lot of monographs and review articles dedicated to this direction of approximation theory. For theoretical aspects of this direction we refer the readers to Ch. Chui [2], Y. Meyer [3], I. Daubechies [4], S. Mallat [5], R. Young [6], Ch. Heil [7], O. Christensen [8-10], etc.

In subsequent years, the concept of frame has been generalized to various mathematical structures (for example, Banach frames, p-frames, etc) and new methods for establishing frames have been elaborated. One of these methods is a perturbation method. A lot of results have been obtained in this direction in the context of

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classical Paley-Wiener theorem on perturbation of an orthonormal basis (for more details see O. Christensen [8-10] and Ch. Heil [7]).

Frames in Banach spaces were first considered by K.H. Gröchenig [11] in 1991. He introduced the concepts of atomic decomposition and Banach frame. It should be noted that, unlike Hilbert's case, the definition of Banach frame does not, in general, provide the decomposition of arbitrary element of Banach space (or of arbitrary element of the closure of the linear span of the system under consideration). In special cases, such a decomposition exists. L^p-case has been considered by A. Aldroubi, Q. Sun, W.-Sh. Tang in [12] where the concept of p-frame has been introduced and the atomic decomposition with regard to L_p-subspaces invariant with respect to the shift operator has been obtained. This idea has been extended to the general Banach case by Christensen O. and Stoeva D. T. [10]. Also, the concept of q-Riesz basis for a Banach space has been introduced in these works, which generalizes the one of Riesz basis introduced by N.K. Bari [13]. Note that similar results have been obtained in [14-20]. There are different generalizations of frames, and this research field has been continuously growing over the last years (see, e.g., [10; 12; 21-31]).

Frames draw growing interest also from a theoretical point of view. As an example, we can mention the connection between the theory of frames and the well-known problem of Kadison and Singer (1959). Modified, but equivalent forms of this problem have been studied in different branches of mathematics such as theory of frames, theory of operators, time-frequency analysis, etc. (for more details see [32-38]).

In the context of applications to some problems of mechanics and mathematical physics, since recently there arose great interest in the study of different mathematical problems in non-standard function spaces such as Lebesgue spaces of variable summability, Morrey-type spaces, grand Lebesgue spaces, etc. (for more details see Cruz-Uribe [39], Kokilashvili V., Meskhi A., Rafeiro H., Samko S. [40], Adams D.R. [41], Bardaro C., Musileak J., Vinti G. [42], etc.). It's worth noting that in most cases these spaces (like, for example, Morrey-type spaces, grand Lebesgue spaces, etc.) are not separable. That's what makes the study of frames in non-separable spaces interesting.

Note that, in general, the case of nonseparable space is not considered in the approximation theory due to objective reasons. The examples with nonseparable spaces are mostly exotic. Meanwhile, from a purely theoretical point of view, it would be interesting to develop approximation theory to the case of nonseparable space. But, to do so, of course you first have to define the corresponding concepts of the theory of Bessel-Hilbert systems and the theory of frames for the case of nonseparable space, and then to extend the basic facts of these theories to nonseparable case. Perhaps, this was first done in [43, 44], where the concept of uncountable Hilbert frame was defined and the basic provisions of the above theories were extended to nonseparable case.

We consider the uncountable K-Besselness and K-Hilbertness of systems in nonseparable Banach spaces with respect to nonseparable Banach space of systems of scalars K. The criteria for K-Besselness and K-Hilbertness of systems in case where the space K is a space of systems consisting of coefficients of some uncountable unconditional basis are found. The relationship between the K-Besselness and

K-Hilbertness of the system and the existence of the uncountable unconditional basis with the space of coefficients K is established.

2. Some notations and auxiliary facts

Let X and Y be nonseparable Banach spaces with the norm $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. By $L(X,Y)$ we denote the space of all bounded linear operators acting from X to Y. A space conjugate to X will be denoted by X^*. $L(M)$ will be a linear span of the $M \subset X$, $\overline{M}$ wll denote a closure of the set M in X, I be an uncountable index set, I^a be a set of no-more-than countable subsets of I, I_0 be a set of finite subsets of I, and $\{\varphi_\alpha\}_{\alpha\in I}$ be some system in X.

Definition 2.1. *The system $\{\varphi_\alpha\}_{\alpha\in I}$ is called an uncountable unconditional basis in X if $\forall x \in X$ $\exists!\lambda = \{\lambda_\alpha\}_{\alpha\in I}$: $\{\alpha \in I : \lambda_\alpha \neq 0\} \in I^a$ $x = \sum_{\alpha\in I} \lambda_\alpha\varphi_\alpha$ (unconditionally).*

Example 2.2. *Let $l_p(I)$, $1 \leq p < +\infty$, be a set of systems of scalars $\lambda = \{\lambda_\alpha\}_{\alpha\in I}$ such that $\omega_\lambda = \{\alpha \in I : \lambda_\alpha \neq 0\} \in I^a$ and $\sum_{\alpha\in\omega_\lambda} |\lambda_\alpha|^p < +\infty$. $l_p(I)$ is a nonseparable Banach space with the norm*

$$\|\lambda\| = \left(\sum_{\alpha\in\omega_\lambda} |\lambda_\alpha|^p\right)^{\frac{1}{p}}, \lambda = \{\lambda_\alpha\}_{\alpha\in I} \in l_p(I).$$

Let $\delta_{\alpha\beta}$ be the Kronecker symbol and $\delta_\alpha = \{\delta_{\alpha\beta}\}_{\beta\in I}$. The system $\{\delta_\alpha\}_{\alpha\in I}$ is an uncountable unconditional basis for $l_p(I)$. In fact, it is not difficult to show that for $\forall\lambda = \{\lambda_\alpha\}_{\alpha\in I} \in l_p(I)$ there exists a unique representation in the form of unconditionally convergent series $\lambda = \sum_{\alpha\in\omega_\lambda} \lambda_\alpha\delta_\alpha$.

Let's state a criterion for uncountable unconditional basicity in nonseparable Banach spaces (see [45]).

Theorem 2.3. *In order for the system $\{\varphi_\alpha\}_{\alpha\in I}$ to be an uncountable unconditional basis in X, it is necessary and sufficient that the following conditions hold:*
1) the system $\{\varphi_\alpha\}_{\alpha\in I}$ is complete in X;
2) the system $\{\varphi_\alpha\}_{\alpha\in I}$ is minimal in X;
3) $\exists M > 0 : \forall J \in I_0, \forall x \in X$ $\left\|\sum_{\alpha\in J} \varphi^*_\alpha(x)\varphi_\alpha\right\|_X \leq M \|x\|_X$, where $\{\varphi^*_\alpha\}_{\alpha\in I}$ is a system biorthogonal to $\{\varphi_\alpha\}_{\alpha\in I}$.

If $\{\varphi_\alpha\} \subset X$ forms an uncountable unconditional basis for X, then the space K_φ of all possible systems of scalars $\lambda = \{\lambda_\alpha\}_{\alpha\in I}$, such that $\omega_\lambda \in I^\alpha$ and the series $\sum_{\alpha\in I} \lambda_\alpha\varphi_\alpha$ converges unconditionally, is a Banach space with the norm

$$\|\lambda\|_{K_\varphi} = \sup_{J\in I_0} \left\|\sum_{\alpha\in I} \lambda_\alpha\varphi_\alpha\right\|_X .$$

3. K-Bessel and K-Hilbert Systems

Let X be a nonseparable Banach space, I be some uncountable index set, and I^a be a set of no-more-than countable subsets of I. Consider the minimal system $\{x_\alpha\}_{\alpha\in I} \subset X$ with the biorthogonal system $\{x^*_\alpha\}_{\alpha\in I} \subset X^*$. Let K be a nonseparable Banach space of systems of scalars $\lambda = \{\lambda_\alpha\}_{\alpha\in I}$ such that $\{\alpha \in I : \lambda_\alpha \neq 0\} \in I^a$.

The space K is called a CB-space if the system $\{\delta_\alpha\}_{\alpha\in I} \subset K$, $\delta_\alpha = \{\delta_{\alpha\beta}\}_{\beta\in I}$ forms an uncountable unconditional basis for K, i.e. $\forall\lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K$ the relation

$$\lambda = \sum_{\alpha\in I} \lambda_\alpha \delta_\alpha = \sum_{i=1}^{\infty} \lambda_{\alpha_i} \delta_{\alpha_i}$$

holds, where $\{\lambda_{\alpha_i}\}_{i\in N}$ is a sequence of arbitrary permutations of non-zero elements $\lambda = \{\lambda_\alpha\}_{\alpha\in I}$. Let $\{\delta_\alpha^*\}_{\alpha\in I} \subset K^*$ be a system biorthogonal to $\{\delta_\alpha\}_{\alpha\in I}$.

The next definition is a generalization of the concept of Bessel sequence.

Definition 3.1. *The pair $\{x_\alpha; x_\alpha^*\}$ is called K-Bessel if the condition $\{x_\alpha^*(x)\}_{\alpha\in I} \in K$, $\forall x \in X$, holds. If the system $\{x_\alpha\}_{\alpha\in I}$ is complete in X and the pair $\{x_\alpha; x_\alpha^*\}$ is K-Bessel in X, then $\{x_\alpha\}_{\alpha\in I}$ is called K-Bessel in X.*

Example 3.2. *Let $e_\alpha(t) = e^{i\alpha t}$, $t \in R$. Assume $V = span\{e_\alpha\}_{\alpha\in R}$. It is clear that $\forall x \in V$ $\exists M_x$: $|x(t)| \le M_x$. Therefore $\forall p \in [1; +\infty)$ there exists $\overline{\lim\limits_{T\to\infty}} \frac{1}{2T}\int_{-T}^{T} |x(t)|^p\, dt$. Let's show that*

$$\|x\|_p = \overline{\lim_{T\to\infty}} \left(\frac{1}{2T} \int_{-T}^{T} |x(t)|^p\, dt \right)^{\frac{1}{p}}$$

is a norm in V. Obviously, $\|x\|_p \ge 0$. Let $\|x\|_p = 0$ and $x = \sum_k c_k e^{i\alpha_k t}$. For $\forall p \in [1; +\infty)$, from

$$\frac{1}{2T}\int_{-T}^{T} |x(t)|^2\, dt = \int_{-T}^{T} (2T)^{-\frac{1}{p}} |x(t)| (2T)^{-\frac{1}{q}} |x(t)|\, dt \le M_x \left(\frac{1}{2T}\int_{-T}^{T} |x(t)|^p\, dt \right)^{\frac{1}{p}}$$

we obtain

$$\overline{\lim_{T\to\infty}} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\, dt \le M_x \overline{\lim_{T\to\infty}} \left(\frac{1}{2T}\int_{-T}^{T} |x(t)|^p\, dt \right)^{\frac{1}{p}} = 0,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

However

$$\overline{\lim_{T\to\infty}} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\, dt = \sum_k |c_k|^2,$$

hence, $x = 0$. Further, $\forall\lambda$ we have

$$\|\lambda x\|_p = \overline{\lim_{T\to\infty}} \left(\frac{1}{2T}\int_{-T}^{T} |\lambda x(t)|^p\, dt \right)^{\frac{1}{p}} = |\lambda|\, \|x\|_p\,.$$

At last, by Minkowski inequality,

$$\left(\frac{1}{2T}\int_{-T}^{T} |x(t)+y(t)|^p\, dt \right)^{\frac{1}{p}} \le \left(\frac{1}{2T}\int_{-T}^{T} |x(t)|^p\, dt \right)^{\frac{1}{p}} + \left(\frac{1}{2T}\int_{-T}^{T} |x(t)|^p\, dt \right)^{\frac{1}{p}}.$$

Passing to the upper limit as $T \to +\infty$ we obtain $\|x+y\|_p \le \|x\|_p + \|y\|_p$. Consequently, $\|\cdot\|_p$ is a norm in the linear space V. Denote the obtained normed space by V_p. Define a scalar product in the space V_2 as follows:

$$(x, y)_V = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\overline{y(t)}dt.$$

The system $\{e_\alpha\}_{\alpha\in R}$ *is orthonormal in* V_2. *Denote by* $L_p^V(R)$, $1 \le p < +\infty$, *the completion of the space* V_p. *Using Holder inequality, it is not difficult to show that* $L_p^V(R) \subset L_2^V(R)$ *for* $p > 2$ *and* $L_2^V(R) \subset L_p^V(R)$ *for* $p < 2$. *The space* $L_p^V(R)$ *is nonseparable. To show this, it obviously suffices to show the nonseparability of the space* V_p.

Let's first establish the nonseparability of the space V_1. *Let* $M = \{e_\alpha\}_{\alpha\in R}$. *Let's estimate the number* $\|e_\alpha - e_\beta\|_1$ *for different* $\forall e_\alpha, e_\beta \in M$. *As the system* $\{e_\alpha\}_{\alpha\in R}$ *is orthonormal, we have* $\|e_\alpha - e_\beta\|_2^2 = \|e_\alpha\|_2^2 + \|e_\beta\|_2^2 = 2$. *Therefore,*

$$2 = \|e_\alpha - e_\beta\|_2^2 = \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} \left|e^{i\alpha t} - e^{i\beta t}\right|^2 dt \le$$

$$\le \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} \left|e^{i\alpha t} - e^{i\beta t}\right| (\left|e^{i\alpha t}\right| + \left|e^{i\beta t}\right|) dt =$$

$$= 2\overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} \left|e^{i\alpha t} - e^{i\beta t}\right| dt = 2\,\|e_\alpha - e_\beta\|_1\,,$$

i.e. $\|e_\alpha - e_\beta\|_1 \ge 1$. *Consequently,* V_1 *is nonseparable. Now let's consider the space* V_p *for* $p > 1$. *Assume the contrary, i.e. assume that* V_p *is separable and* M *is a countable set everywhere dense in* V_p. *Then* $\forall x \in V$ $\exists x_n \in M$: $\lim\limits_{n\to\infty} \|x - x_n\|_p = 0$. *As* $\forall x \in V$

$$\|x\|_1 = \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} |x(t)|\, dt \le \overline{\lim_{T\to\infty}} \left(\frac{1}{2T} \int_{-T}^{T} |x(t)|^p\, dt \right)^{\frac{1}{p}} = \|x\|_p\,,$$

we obtain $\lim\limits_{n\to\infty} \|x - x_n\|_1 = 0$. *Consequently,* M *is a countable set everywhere dense in* V_1, *which contradicts the nonseparability of* V_1. *So the space* V_p *is nonseparable.*

Define a functional in V_p *by the following equality:*

$$e_\alpha^*(x) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t) e^{-i\alpha t} dt.$$

The linearity of e_α^* *is obvious. We have*

$$|e_\alpha^*(x)| \le \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} |x(t)|\, dt \le \overline{\lim_{T\to\infty}} \left(\frac{1}{2T} \int_{-T}^{T} |x(t)|^p\, dt \right)^{\frac{1}{p}},$$

i.e. $e_\alpha^* \in (V_p)^*$. *Extending* e_α^* *by continuity onto* $L_p^V(R)$, *we obtain* $e_\alpha^* \in (L_p^V(R))^*$. *It is clear that* $e_\alpha^*(e_\beta) = \delta_{\alpha\beta}$, *i.e. the systems* $\{e_\alpha\}_{\alpha\in R}$ *and* $\{e_\alpha^*\}_{\alpha\in R}$ *are biorthogonal.*
Let $p > 2$ *and* $\forall x \in L_p^V(R)$. *As* $x \in L_2^V(R)$, *it follows from Bessel's inequality that there are no more than a countable number of Fourier coefficients* $e_\alpha^*(x) = (x, e_\alpha)_V$ *that are different from zero and* $\{e_\alpha^*(x)\}_{\alpha\in R} \in l_2(R)$. *Further, from* $l_2(R) \subset l_p(R)$ *we obtain* $\{e_\alpha^*(x)\}_{\alpha\in R} \in l_p(R)$. *Consequently, for* $p > 2$ *the system* $\{e_\alpha\}_{\alpha\in R}$ *is* $l_p(R)$*-Besselian in* $L_p^V(R)$.

The following theorem is true:

Theorem 3.3. *Let* K *be a* CB*-space with an uncountable unconditional basis* $\{\delta_\alpha\}_{\alpha\in I}$. *Then, in order for the pair* $\{x_\alpha; x_\alpha^*\}$ *to be* K*-Bessel in* X, *it is necessary* *and, in case of completeness of* $\{x_\alpha\}_{\alpha\in I}$ *in* X*, sufficient that there exists an operator* $T \in L(X,K)$ *such that* $Tx_\alpha = \delta_\alpha$*,* $\forall\alpha \in I$*.*

Proof. Necessity. Let $\{x_\alpha; x_\alpha^*\}$ be K-Bessel in X. Then $\forall x \in X$ $\{x_\alpha^*(x)\}_{\alpha\in I} \in K$. Consider the operator $T : X \to K$ defined by the formula

$$T(x) = \sum_{\alpha\in I} x_\alpha^*(x)\delta_\alpha. \tag{3.1}$$

It is clear that the equality $Tx_\alpha = \delta_\alpha$, $\forall\alpha \in I$, holds. Define for every $\omega \in I^a$ the operator

$$T_\omega(x) = \sum_{\alpha\in\omega} x_\alpha^*(x)\delta_\alpha.$$

We have $T_\omega \in L(X,K)$ (see [6]). Is easy to show $\exists B > 0$ for $\forall x \in X$ we have

$$\|T_\omega(x)\|_K = \left\|\{x_\alpha^*(x)\}_{\alpha\in\omega}\right\|_K \le B\left\|\{x_\alpha^*(x)\}_{\alpha\in I}\right\|_K$$

we obtain that

$$\sup_{\omega\in I^a}\|T_\omega(x)\|_K \le B\left\|\{x_\alpha^*(x)\}_{\alpha\in I}\right\|_K < +\infty.$$

Then, by Banach-Steinhaus theorem, we have $\sup\limits_{\omega\in I^a}\|T_\omega\| < +\infty$. As the boundedness of T is equivalent to the condition $\sup\limits_{\omega\in I^a}\|T_\omega\| < +\infty$, the operator T is bounded.

Sufficiency. Let the system $\{x_\alpha\}_{\alpha\in I}$ be complete in X and there exist $T \in L(X,K)$ such that $Tx_\alpha = \delta_\alpha$, $\forall\alpha \in I$. Consider $\forall x \in X$. From $Tx \in K$ it follows that $\{\delta_\alpha^*(Tx)\}_{\alpha\in I} \in K$. We have

$$\delta_{\alpha\beta} = \delta_\alpha^*(\delta_\beta) = \delta_\alpha^*(Tx_\beta) = T^*\delta_\alpha^*(x_\beta), \forall\alpha,\beta \in I.$$

Hence, due to the completeness of $\{x_\alpha\}_{\alpha\in I}$, we obtain

$$T^*\delta_\alpha^* = x_\alpha^*. \tag{3.2}$$

Thus, taking into account (3.2), we obtain $\{x_\alpha^*(x)\}_{\alpha\in I} = \{\delta_\alpha^*(Tx)\} \in K$. □

Corollary 3.4. *Let the system* $\{x_\alpha\}_{\alpha\in I}$ *be complete in* X *and* K *be a* CB*-space with an uncountable unconditional basis* $\{\delta_\alpha\}_{\alpha\in I}$*. The system* $\{x_\alpha\}_{\alpha\in I}$ *is* K*-Bessel in* X *only when there exists a number* $M > 0$ *such that*

$$\|\{\lambda_\alpha\}\|_K \le M\left\|\sum_\alpha \lambda_\alpha x_\alpha\right\|_X \tag{3.3}$$

for every finite set of scalars $\{\lambda_\alpha\}$*.*

Proof. Let $\{x_\alpha\}_{\alpha\in I}$ be K-Bessel in X. Then, by Theorem 3.3, there exists a bounded operator $T \in L(X,K)$ defined by the formula (3.1). Let $\{\lambda_\alpha\}$ be an arbitrary finite set of scalars. Let $x = \sum_\alpha \lambda_\alpha x_\alpha$. Then $Tx = \sum_\alpha \lambda_\alpha\delta_\alpha$ and

$$\|\{\lambda_\alpha\}\|_K = \|Tx\|_K \le \|T\|\,\|x\|_X = \|T\|\left\|\sum_\alpha \lambda_\alpha x_\alpha\right\|_X.$$

On the contrary, let there exist a number $M > 0$ such that the inequality (3.3) holds for every finite set of scalars $\{\lambda_\alpha\}$. Consider arbitrary $\lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K$.

Define the operator $T : L\left(\{x_\alpha\}_{\alpha\in I}\right) \to K$ by the formula

$$T\left(\sum_\alpha \lambda_\alpha x_\alpha\right) = \{\lambda_\alpha\}.$$

Due to the minimality of the system $\{x_\alpha\}_{\alpha\in I}$, such an operator is defined correctly. By (3.3), the operator T is bounded. Continuing the operator T continuously to the whole X, we obtain the boundedness of T. As $Tx_\alpha = \delta_\alpha$, $\forall\alpha \in I$, by Theorem 3.3, the system $\{x_\alpha\}_{\alpha\in I}$ is K-Bessel in X. □

Remark. *Note that for $p \geq 2$ $l_p(R)$- Besselianness of systems $e_\alpha(t) = e^{i\alpha t}$ in $L_p^V(R)$ immediately implies of Corollary 3.4. Indeed, for every finite set of scalars $\{\lambda_\alpha\}$ we have*

$$\|\{\lambda_\alpha\}\|_{l_p(R)} \leq \|\{\lambda_\alpha\}\|_{l_2(R)} = \left\|\sum_\alpha \lambda_\alpha e_\alpha\right\|_2 \leq \left\|\sum_\alpha \lambda_\alpha e_\alpha\right\|_p.$$

The next definition is a generalization of the concept of Hilbert sequence.

Definition 3.5. *The pair $\{x_\alpha; x_\alpha^*\}$ is called K-Hilbert in X if the following condition holds: $\forall\lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K$ $\exists x \in X$: $\lambda = \{x_\alpha^*(x)\}_{\alpha\in I}$. If the system $\{x_\alpha\}_{\alpha\in I}$ is complete in X and the pair $\{x_\alpha; x_\alpha^*\}$ is K-Hilbert in X, then $\{x_\alpha\}_{\alpha\in I}$ is called K-Hilbert in X.*

Example 3.6. *The system $\{e_\alpha\}_{\alpha\in R}$ is $l_p(R)$-Hilbertian in $L_p^V(R)$ for $p < 2$. Indeed, consider $\forall\lambda = \{\lambda_\alpha\}_{\alpha\in R} \in l_p(R)$. Then $\lambda \in l_2(R)$. As $\{e_\alpha\}_{\alpha\in R}$ is an orthonormal system in $L_2^V(R)$, there exists $x \in L_2^V(R)$ such that $e_\alpha^*(x) = (x, e_\alpha)_V = \lambda_\alpha$. From $L_2^V(R) \subset L_p^V(R)$ we obtain that $x \in L_p^V(R)$. Hence, $\{e_\alpha\}_{\alpha\in R}$ is $l_p(R)$-Hilbertian in $L_p^V(R)$ for $p < 2$.*

The following theorem is true:

Theorem 3.7. *Let K be a CB-space with an uncountable unconditional basis $\{\delta_\alpha\}_{\alpha\in I}$. Then, in order for the pair $\{x_\alpha; x_\alpha^*\}$ to be K-Hilbert in X, it is sufficient and, in case of completeness of $\{x_\alpha^*\}_{\alpha\in I}$ in X^*, necessary that there exists an operator $T \in L(K, X)$ such that $T\delta_\alpha = x_\alpha$, $\forall\alpha \in I$.*

Proof. Necessity. Let $\{x_\alpha; x_\alpha^*\}$ be K-Hilbert in X and the system $\{x_\alpha^*\}_{\alpha\in I}$ be complete in X^*. Then $\forall\lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K$ $\exists x \in X$ such that $\lambda = \{x_\alpha^*(x)\}_{\alpha\in I}$. Due to the completeness of $\{x_\alpha^*\}_{\alpha\in I}$ in X^*, such an element is unique. Consider the operator $T : K \to X$ defined by the formula $T\lambda = x$. Obviously, $T\delta_\alpha = x_\alpha$, $\forall\alpha \in I$. It remains to show the boundedness of T. To this end, let's first prove its closedness. Let the sequence $\lambda_n = \left\{\lambda_\alpha^{(n)}\right\}_{\alpha\in I} \in K$ converge in K to $\lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K$ as $n \to \infty$, the sequence $T\lambda_n = x_n$ converge in X to $y \in X$ as $n \to \infty$, and $T\lambda = x$. It is clear that $x_\alpha^*(x_n)$ converges to $x_\alpha^*(y)$ as $n \to \infty$. $\forall\alpha \in I$ we have

$$\left|\lambda_\alpha^{(n)} - \lambda_\alpha\right| = |\delta_\alpha^*(\lambda_n - \lambda)| \leq \|\delta_\alpha^*\| \, \|\lambda_n - \lambda\|_K \to 0, n \to \infty.$$

As $\lambda_\alpha^{(n)} = x_\alpha^*(x_n)$ and $\lambda_\alpha = x_\alpha^*(x)$, we have $x_\alpha^*(x) = x_\alpha^*(y)$. Hence, due to the completeness of $\{x_\alpha^*\}_{\alpha\in I}$, we obtain $x = y$. Consequently, $T\lambda_n$ converges in X to $T\lambda$ as $n \to \infty$, i.e. the operator T is closed. Then, by the closed graph theorem, T is bounded.

Sufficiency. Let there exist an operator $T \in L(K,X)$ such that $T\delta_\alpha = x_\alpha$, $\forall\alpha \in I$. Consider arbitrary $\lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K$. Let $T\lambda = x$. From $\lambda = \sum_{\alpha\in I} \lambda_\alpha \delta_\alpha$ we obtain

$$x = T\lambda = \sum_{\alpha\in I} \lambda_\alpha T\delta_\alpha = \sum_{\alpha\in I} \lambda_\alpha x_\alpha.$$

Therefore, $x^*_\alpha(x) = \lambda_\alpha$, i.e. $\lambda = \{x^*_\alpha(x)\}_{\alpha\in I}$. □

Corollary 3.8. *Let the system $\{x_\alpha\}_{\alpha\in I}$ be complete in X, the system $\{x^*_\alpha\}_{\alpha\in I}$ be complete in X^*, and K be a CB-space with an uncountable unconditional basis $\{\delta_\alpha\}_{\alpha\in I}$. The system $\{x_\alpha\}_{\alpha\in I}$ is K-Hilbert in X only when these exists a number $M > 0$ such that*

$$\left\| \sum_\alpha \lambda_\alpha x_\alpha \right\|_X \le M \left\| \{\lambda_\alpha\} \right\|_K \tag{3.4}$$

for every finite set of scalars $\{\lambda_\alpha\}$.

Proof. Let $\{x_\alpha\}_{\alpha\in I}$ be K-Hilbert in X. By Theorem 3.7, there exists a bounded operator $T \in L(K,X)$ such that $T\delta_\alpha = x_\alpha$, $\forall\alpha \in I$. Then, for every finite set $\{\lambda_\alpha\}$ we have

$$T\{\lambda_\alpha\} = T(\sum_\alpha \lambda_\alpha \delta_\alpha) = \sum_\alpha \lambda_\alpha x_\alpha.$$

Hence

$$\left\| \sum_\alpha \lambda_\alpha x_\alpha \right\|_X = \left\| T\{\lambda_\alpha\} \right\| \le \|T\| \left\| \{\lambda_\alpha\} \right\|_K.$$

On the contrary, let the relation (3.4) hold for every finite set $\{\lambda_\alpha\}$. Define the operator T for finite systems $\{\lambda_\alpha\}$ by the formula $T\{\lambda_\alpha\} = \sum_\alpha \lambda_\alpha x_\alpha$. From the inequality (3.4) it follows that this operator is bounded. Continuing it continuously to the whole K, we obtain the operator $T \in L(K,X)$. As $T\delta_\alpha = x_\alpha$,$\forall\alpha \in I$, from Theorem 3.7 it follows that the system $\{x_\alpha\}_{\alpha\in I}$ is K-Hilbert in X. □

Remark. *Note that for $p \le 2$ $l_p(R)$- Hilbertianness of systems $e_\alpha(t) = e^{i\alpha t}$ in $L^V_p(R)$ immediately implies of Corollary 3.8. Indeed, for every finite set of scalars $\{\lambda_\alpha\}$ we have*

$$\left\| \sum_\alpha \lambda_\alpha e_\alpha \right\|_p \le \left\| \sum_\alpha \lambda_\alpha e_\alpha \right\|_2 = \left\| \{\lambda_\alpha\} \right\|_{l_2(R)} \le \left\| \{\lambda_\alpha\} \right\|_{l_p(R)}.$$

Now let's consider the Besselianness and the Hilbertianness of systems with respect to the space generated by an uncountable unconditional basis.

Theorem 3.9. *Let Y be a Banach space with an uncountable unconditional basis $\{\varphi_\alpha\}_{\alpha\in I}$ and K_φ be a space generated by the system $\{\varphi_\alpha\}_{\alpha\in I}$. Then, in order for the pair $\{x_\alpha; x^*_\alpha\}$ to be K_φ-Bessel in X, it is necessary and, in case of completeness of $\{x_\alpha\}_{\alpha\in I}$ in X, sufficient that there exists an operator $T \in L(X,Y)$ such that $Tx_\alpha = \varphi_\alpha$, $\forall\alpha \in I$.*

Proof. Necessity. Let the pair $\{x_\alpha; x^*_\alpha\}$ be K_φ-Bessel in X. Then, $\forall x \in X$ we have $\{x^*_\alpha(x)\}_{\alpha\in I} \in K_\varphi$. Define the operator $T : X \to Y$ by setting $T(x) = \sum_{\alpha\in I} x^*_\alpha(x)\varphi_\alpha$. It is clear that T is linear and $Tx_\alpha = \varphi_\alpha$, $\forall\alpha \in I$. Let's prove its boundedness. For

$\forall \omega \in I^a$, consider the operator

$$T_\omega(x) = \sum_{\alpha\in\omega} x_\alpha^*(x)\varphi_\alpha.$$

Then $T_\omega \in L(X,Y)$, and $\forall x \in X$ we obtain

$$\|T_\omega(x)\|_{K_\varphi} = \left\|\{x_\alpha^*(x)\}_{\alpha\in\omega}\right\|_{K_\varphi} \le \left\|\{x_\alpha^*(x)\}_{\alpha\in I}\right\|_{K_\varphi}.$$

Thus

$$\sup_{\omega\in I^a} \|T_\omega(x)\|_{K_\varphi} \le \left\|\{x_\alpha^*(x)\}_{\alpha\in I}\right\|_{K_\varphi} < +\infty.$$

Then, by Banach-Steinhaus theorem we obtain $\sup\limits_{\omega\in I^a} \|T_\omega\| < +\infty$. As the boundedness of T is equivalent to the condition $\sup\limits_{\omega\in I^a} \|T_\omega\| < +\infty$, the operator T is bounded.

Sufficiency. Let the system $\{x_\alpha\}_{\alpha\in I}$ be complete in X and there exist $T \in L(X,Y)$ such that $Tx_\alpha = \varphi_\alpha$, $\forall\alpha \in I$. Let $\{\varphi_\alpha^*\}_{\alpha\in I}$ be a system biorthogonal to $\{\varphi_\alpha\}_{\alpha\in I}$. Consider $\forall x \in X$. We have

$$\delta_{\alpha\beta} = \varphi_\alpha^*(\varphi_\beta) = \varphi_\alpha^*(Tx_\beta) = T^*\varphi_\alpha^*(x_\beta).$$

Due to the completeness of $\{x_\alpha\}_{\alpha\in I}$, we obtain $T^*\varphi_\alpha^* = x_\alpha^*$. Then

$$x_\alpha^*(x) = T^*\varphi_\alpha^*(x) = \varphi_\alpha^*(Tx),$$

and therefore,

$$\{x_\alpha^*(x)\}_{\alpha\in I} = \{\varphi_\alpha^*(Tx)\}_{\alpha\in I} \in K_\varphi.$$

□

Theorem 3.10. *Let Y be a Banach space with an uncountable unconditional basis $\{\varphi_\alpha\}_{\alpha\in I}$ and K_φ be a space generated by the system $\{\varphi_\alpha\}_{\alpha\in I}$. Then, in order for the pair $\{x_\alpha; x_\alpha^*\}$ to be K_φ-Hilbert in X, it is sufficient and, in case of completeness of $\{x_\alpha^*\}_{\alpha\in I}$ in X^*, necessary that there exists an operator $T \in L(Y,X)$ such that $T\varphi_\alpha = x_\alpha$, $\forall\alpha \in I$.*

Proof. Necessity. Let $\{x_\alpha; x_\alpha^*\}$ be K_φ-Hilbert in X and $\{x_\alpha^*\}_{\alpha\in I}$ be complete in X^*. Suppose every $\lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K_\varphi$ is corresponded to $x \in X$ such that $\lambda = \{x_\alpha^*(x)\}_{\alpha\in I}$. Due to the completeness of the system $\{x_\alpha^*\}_{\alpha\in I}$ in X^*, this element is unique. Let $y = \sum_{\alpha\in I} x_\alpha^*(x)\varphi_\alpha$. Define the operator $T : Y \to X$ by the formula $T(y) = x$. It is clear that T is linear and $T\varphi_\alpha = x_\alpha$, $\forall\alpha \in I$. Let's show the closedness of the operator T. Let $y_n \in Y$, $y_n \to y$ and $T(y_n) = x_n \to x$ as $n \to \infty$. Also let $y = \sum_{\alpha\in I} \lambda_\alpha\varphi_\alpha$. As $y_n = \sum_{\alpha\in I} x_\alpha^*(x_n)\varphi_\alpha$, $\forall\alpha \in I$ we obtain

$$|\lambda_\alpha - x_\alpha^*(x_n)| = \left|\varphi_\alpha^*\left(\sum_{\beta\in I}\lambda_\beta\varphi_\beta - \sum_{\alpha\in I} x_\beta^*(x_n)\varphi_\beta\right)\right| \le \|\varphi_\alpha^*\|\,\|y - y_n\|_Y \to 0$$

as $n \to \infty$. On the other hand, $x_\alpha^*(x_n) \to x_\alpha^*(x)$ as $n \to \infty$. Thus, $y = \sum_{\alpha\in I} x_\alpha^*(x)\varphi_\alpha$, and therefore, $T(y) = x$. Then, by the closed graph theorem, the operator T is bounded.

Sufficiency. Let there exist $T \in L(Y,X)$ such that $T\varphi_\alpha = x_\alpha$, $\forall\alpha \in I$. Consider $\forall\lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K_\varphi$ and let $y = \sum_{\alpha\in I} \lambda_\alpha\varphi_\alpha$. Let $T(y) = x$. We have

$$\delta_{\alpha\beta} = x_\alpha^*(x_\beta) = x_\alpha^*(T\varphi_\beta) = T^*x_\alpha^*(\varphi_\beta), \forall\alpha,\beta \in I.$$

Hence, $T^*x_\alpha^* = \varphi_\alpha^*$, and, consequently, $\forall\alpha \in I$ we obtain

$$\lambda_\alpha = \varphi_\alpha^*(y) = T^*x_\alpha^*(y) = x_\alpha^*(Ty) = x_\alpha^*(x),$$

i.e. the pair $\{x_\alpha; x_\alpha^*\}$ is K_φ-Hilbert in X. □

Theorem 3.11. *Let K be a reflexive CB-space with an uncountable unconditional basis $\{\delta_\alpha\}_{\alpha\in I}$. Then*

1) if the pair $\{x_\alpha; x_\alpha^*\}$ is K-Hilbert in X and $\{x_\alpha^*\}_{\alpha\in I} \subset X^*$ is complete, then $\{x_\alpha^*\}_{\alpha\in I}$ is K^*-Bessel in X^*;

2) if the pair $\{x_\alpha; x_\alpha^*\}$ is K-Bessel in X and $\{x_\alpha\}_{\alpha\in I}$ is complete in X, then the pair $\{x_\alpha; x_\alpha^*\}$ is K^*-Hilbert in X^*.

Proof. 1) Let $\{x_\alpha; x_\alpha^*\}$ be K-Hilbert in X. Then, by Theorem 3.7, there exists an operator $T \in L(K, X)$ such that $T\delta_\alpha = x_\alpha$, $\forall\alpha \in I$. It is not difficult to show that $T^*x_\alpha^* = \delta_\alpha^*$, $\forall\alpha \in I$. As the system $\{\delta_\alpha^*\}_{\alpha\in I}$ forms an uncountable unconditional basis for K^* and the system $\{x_\alpha^*\}_{\alpha\in I}$ is complete in X^*, it follows from Theorem 3.3 that $\{x_\alpha^*\}_{\alpha\in I}$ is K^*-Bessel inX^*.

2) Now let the pair $\{x_\alpha; x_\alpha^*\}$ be K-Bessel in X. By Theorem 3.3, there exists an operator $T \in L(X, K)$ such that $Tx_\alpha = \delta_\alpha$, $\forall\alpha \in I$. As it was shown in Theorem 3.3, $T^*\delta_\alpha^* = x_\alpha^*$, $\forall\alpha \in I$. Consequently, by Theorem 3.2, the pair $\{x_\alpha; x_\alpha^*\}$ is K^*-Hilbert in X^*. □

Theorem 3.12. *Let X be a Banach space with an uncountable unconditional basis $\{\varphi_\alpha\}_{\alpha\in I}$, K_φ be a space generated by the system $\{\varphi_\alpha\}_{\alpha\in I}$, and the system $\{x_\alpha\}_{\alpha\in I}$ be complete in X. Then, in order for the system $\{x_\alpha\}_{\alpha\in I}$ to be K_φ-Bessel in X, it is necessary and sufficient that the operator $A : K_\varphi \to K_\varphi$ defined by the expression $A(\lambda) = \left\{\sum_{\alpha\in I} x_\beta^*(\varphi_\alpha)\lambda_\alpha\right\}_{\beta\in I}$, $\lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K_\varphi$, is bounded in K_φ.*

Proof. Necessity. Let the pair $\{x_\alpha; x_\alpha^*\}$ be K_φ-Bessel in X. By Theorem 3.3, there exists $T \in L(X)$ such that $Tx_\alpha = \varphi_\alpha$, $\forall\alpha \in I$. Consider the operator $F^{-1}TF$, where F is an isomorphism of the spaces K_φ and X, i.e. $\forall x \in X$ $F^{-1}x = \{\varphi_\alpha^*(x)\}_{\alpha\in I}$, and $\{\varphi_\alpha^*\}_{\alpha\in I}$ is a system biorthogonal to $\{\varphi_\alpha\}_{\alpha\in I}$. For $\forall\lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K_\varphi$ we have $F\lambda = \sum_{\alpha\in I}\lambda_\alpha\varphi_\alpha$. It is not difficult to show that $T^*\varphi_\alpha^* = x_\alpha^*$, $\forall\alpha \in I$. Then

$$T\varphi_\alpha = \sum_{\beta\in I}\varphi_\beta^*(T\varphi_\alpha)\varphi_\beta = \sum_{\beta\in I}T^*\varphi_\beta^*(\varphi_\alpha)\varphi_\beta = \sum_{\beta\in I}x_\beta^*(\varphi_\alpha)\varphi_\beta.$$

Hence

$$TF(\lambda) = \sum_{\alpha\in I}\lambda_\alpha T\varphi_\alpha = \sum_{\alpha\in I}\lambda_\alpha\sum_{\beta\in I}x_\beta^*(\varphi_\alpha)\varphi_\beta = \sum_{\beta\in I}\left(\sum_{\alpha\in I}x_\beta^*(\varphi_\alpha)\lambda_\alpha\right)\varphi_\beta.$$

Thus

$$F^{-1}TF(\lambda) = \left\{\sum_{\alpha\in I}x_\beta^*(\varphi_\alpha)\lambda_\alpha\right\}_{\beta\in I},$$

i.e. $A = F^{-1}TF \in L(K_\varphi)$.

Sufficiency. Let the operator A be bounded in K_φ. Let $T = FAF^{-1}$. Obviously, $T \in L(X)$. $\forall\gamma \in I$ we obtain

$$Tx_\gamma = FAF^{-1}x_\gamma = FA(\{\varphi_\alpha^*(x_\gamma)\}_{\alpha\in I}) = F\left(\left\{\sum_{\alpha\in I}x_\beta^*(\varphi_\alpha)\varphi_\alpha^*(x_\gamma)\right\}_{\beta\in I}\right) =$$

$$= F(\{x^*_\beta(x_\gamma)\}_{\beta\in I}) = F(\{\delta_{\beta\gamma}\}_{\beta\in I}) = \varphi_\gamma.$$

By Theorem 3.9, the system $\{x_\alpha\}_{\alpha\in I}$ is K_φ-Bessel in X. □

Theorem 3.13. *Let X be a reflexive Banach space, the system $\{x_\alpha\}_{\alpha\in I}$ be complete in X and $\{x^*_\alpha\}$ be complete in X^*, the system $\{\varphi_\alpha\}_{\alpha\in I}$ form an uncountable unconditional basis for X, and K_φ be a space generated by the system $\{\varphi_\alpha\}_{\alpha\in I}$. Then, in order for the system $\{x_\alpha\}_{\alpha\in I}$ to be K_φ-Hilbert in X, it is necessary and sufficient that the operator $A : K^*_\varphi \to K^*_\varphi$ defined by the expression $A(\lambda) = \left\{\sum_{\alpha\in I} \varphi^*_\alpha(x_\beta)\lambda_\alpha\right\}_{\beta\in I}$, $\lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K^*_\varphi$, is bounded in K^*_φ.*

Proof. Let $\{x_\alpha\}_{\alpha\in I}$ be K_φ-Hilbert in X. By Theorem 3.11, the system $\{x^*_\alpha\}_{\alpha\in I}$ is K^*_φ-Bessel in X^*. Then, by Theorem 3.12, the operator $A : K^*_\varphi \to K^*_\varphi$: $A(\lambda) = \left\{\sum_{\alpha\in I} \varphi^*_\alpha(x_\beta)\lambda_\alpha\right\}_{\beta\in I}$ is bounded in K^*_φ.

On the contrary, let $A : K^*_\varphi \to K^*_\varphi$: $A(\lambda) = \left\{\sum_{\alpha\in I} \varphi^*_\alpha(x_\beta)\lambda_\alpha\right\}_{\beta\in I}$ be a linear bounded operator in K^*_φ. Then, by Theorem 3.12, the system $\{x^*_\alpha\}_{\alpha\in I}$ is K^*_φ-Bessel in X^* and, by Theorem 3.11, the system $\{x_\alpha\}_{\alpha\in I}$ is K_φ-Hilbert in X. □

Theorem 3.14. *Let the system $\{x_\alpha\}_{\alpha\in I}$ be complete in X. The, in order for Y to have an uncountable unconditional basis $\{\varphi_\alpha\}_{\alpha\in I}$ such that $\{x_\alpha\}_{\alpha\in I}$ is K_φ-Bessel in X, it is necessary and sufficient that there exist the operators $T \in L(X, Y)$ and $A : Y \to Y^*$ with $D_A = L(\{Tx_\alpha\}_{\alpha\in I})$ satisfying the following conditions:*

1) $KerT^* = \{0\}$, $T^*ATx_\alpha = x^*_\alpha, \forall\alpha \in I$;

2) $\exists M > 0 : \left\|\sum_{\alpha\in J}(ATx_\alpha)(\varphi)Tx_\alpha\right\|_Y \le M \|\varphi\|_Y$, $\forall J \in I_0$.

Proof. Necessity. Let the system $\{\varphi_\alpha\}_{\alpha\in I}$ form an uncountable unconditional basis for Y with the space of sequences of coefficients K_φ and $\{x_\alpha\}_{\alpha\in I}$ be K_φ-Bessel in X. By Theorem 3.9, there exists the operator $T \in L(X, Y)$ such that $Tx_\alpha = \varphi_\alpha$, $\forall\alpha \in I$. Obviously, $T^*\varphi^*_\alpha = x^*_\alpha$, $\forall\alpha \in I$ and $\overline{ImT} = Y$. In fact, if $\forall\varphi^* \in Y^*$ $\varphi^*T(x) = 0$, then $\varphi^*(\varphi_\alpha) = 0$, and therefore, $\varphi^* = 0$. Define on $L(\{Tx_\alpha\}_{\alpha\in I})$ the linear operator $A : Y \to Y^*$ by the formula $ATx_\alpha = \varphi^*_\alpha$. Then, $A\varphi_\alpha = \varphi^*_\alpha$ and $T^*ATx_\alpha = T^*\varphi^*_\alpha = x^*_\alpha$. Further, taking into account the condition 3) of Theorem 2.3, we obtain

$$\left\|\sum_{\alpha\in J} AT(x_\alpha)(\varphi)Tx_\alpha\right\|_Y = \left\|\sum_{\alpha\in J} \varphi^*_\alpha(\varphi)\varphi_\alpha\right\| \le M \|\varphi\|_Y .$$

Sufficiency. Let there exist the operators $T \in L(X, Y)$ and $A : Y \to Y^*$ with $D_A = L(\{Tx_\alpha\}_{\alpha\in I})$ satisfying the conditions 1) and 2). Let $Tx_\alpha = \varphi_\alpha$ and $A\varphi_\alpha = \varphi^*_\alpha$. Then

$$\varphi^*_\alpha(\varphi_\beta) = A\varphi_\alpha(Tx_\beta) = T^*A\varphi_\alpha(x_\beta) = x^*_\alpha(x_\beta) = \delta_{\alpha\beta},$$

i.e. the systems $\{\varphi_\alpha\}_{\alpha\in I}$ and $\{\varphi^*_\alpha\}_{\alpha\in I}$ are biorthogonal. Further, for $\forall J \in I_0$ and $\forall\varphi \in Y$ we have

$$\left\|\sum_{\alpha\in J} \varphi^*_\alpha(\varphi)\varphi_\alpha\right\|_Y = \left\|\sum_{\alpha\in J} AT(x_\alpha)(\varphi)Tx_\alpha\right\|_Y \le M \|\varphi\|_Y .$$

Thus, the conditions 1)-3) of Theorem 2.1 are satisfied, i.e. the system $\{\varphi_\alpha\}_{\alpha\in I}$ forms an uncountable unconditional basis for Y. Let K_φ be a space generated by the system $\{\varphi_\alpha\}_{\alpha\in I}$. As $Tx_\alpha = \varphi_\alpha$, $\forall\alpha \in I$, by Theorem 3.9, the system $\{x_\alpha\}_{\alpha\in I}$ is K_φ-Bessel in X. □

Theorem 3.15. *Let X, Y be reflexive Banach spaces, the system $\{x_\alpha\}_{\alpha\in I}$ be complete in X, and the system $\{x^*_\alpha\}_{\alpha\in I}$ be complete in X^*. Then, in order for Y to have an uncountable unconditional basis $\{\varphi_\alpha\}_{\alpha\in I}$ such that $\{x_\alpha\}_{\alpha\in I}$ is K_φ-Hilbert in X, it is necessary and sufficient that there exists the operators $T \in L(X^*, Y^*)$ and $A : Y^* \to Y$ with $D_A = L(\{Tx^*_\alpha\}_{\alpha\in I})$ satisfying the following conditions:*

1) $KerT^* = \{0\}, T^*ATx^*_\alpha = x_\alpha, \forall\alpha \in I$;

2) $\exists M > 0 : \left\|\sum_{\alpha\in J} \varphi^*(ATx^*_\alpha)Tx^*_\alpha\right\|_Y \le M \|\varphi^*\|, \forall J \in I_0, \forall\varphi^* \in Y^*$.

Proof. Let $\{\varphi_\alpha\}_{\alpha\in I}$ be an uncountable unconditional basis in Y and $\{x_\alpha\}_{\alpha\in I}$ be K_φ-Hilbert in X. By Theorem 3.11, the system $\{x^*_\alpha\}_{\alpha\in I}$ is K^*_φ-Bessel in X^*. Consequently, by Theorem 3.16, there exist the operators which satisfy the given conditions.

On the contrary, let there exist the operators which satisfy the conditions 1) and 2). By Theorem 3.16, there exists an uncountable unconditional basis $\{\varphi^*_\alpha\}_{\alpha\in I}$ and $\{x^*_\alpha\}_{\alpha\in I}$ is K^*_φ-Bessel in X^*. Let $\{\varphi_\alpha\}_{\alpha\in I} \subset X$ is a system biorthogonal to $\{\varphi^*_\alpha\}_{\alpha\in I}$. Then (see [45]), $\{\varphi_\alpha\}_{\alpha\in I}$ forms an uncountable unconditional basis for X and by Theorem 3.11, the system $\{x_\alpha\}_{\alpha\in I}$ is K_φ-Hilbert in X. □

Theorem 3.16. *Let K be a CB-space with an uncountable unconditional basis $\{\delta_\alpha\}_{\alpha\in I}$, the system $\{x_\alpha\}_{\alpha\in I}$ be complete in X, and the system $\{x^*_\alpha\}_{\alpha\in I}$ be complete in X^*. Then, in order for $\{x_\alpha\}_{\alpha\in I}$ to be simultaneously K-Bessel and K-Hilbert in X, it is necessary and sufficient that $\{x_\alpha\}_{\alpha\in I}$ be an uncountable unconditional basis with the space of coefficients K.*

Proof. Necessity. Let the system $\{x_\alpha\}_{\alpha\in I}$ be simultaneously K-Bessel and K-Hilbert in X. By Theorems 3.3 and 3.7, there exist the operators $T \in L(X, K)$ and $S \in L(K, X)$ such that $\forall\alpha \in I$ $Tx_\alpha = \delta_\alpha$ and $S\delta_\alpha = x_\alpha$. Then $STx_\alpha = x_\alpha$ and $TS\delta_\alpha = \delta_\alpha$ for $\forall\alpha \in I$. Due to the completeness of $\{x_\alpha\}_{\alpha\in I}$ and the uncountable unconditional basicity of $\{\delta_\alpha\}_{\alpha\in I}$, we have $ST = I_X$, $TS = I_K$. Consequently, the operator T is boundedly invertible and $T^{-1} = S$. Consider arbitrary $x \in X$ and let $Tx = \lambda$, $\lambda = \{\lambda_\alpha\}_{\alpha\in I}$. Applying the operator S to both sides of the equality $\lambda = \sum_{\alpha\in I} \lambda_\alpha\delta_\alpha$, we obtain

$$x = \sum_{\alpha\in I} \lambda_\alpha S\delta_\alpha = \sum_{\alpha\in I} \lambda_\alpha x_\alpha,$$

i.e. the system $\{x_\alpha\}_{\alpha\in I}$ forms an uncountable unconditional basis with the space of coefficients K.

Sufficiency. Let the system $\{x_\alpha\}_{\alpha\in I}$ form an uncountable unconditional basis with the space of coefficients K. Denote by F the isomorphism between X and K, defined by the formula $F\lambda = x$, $\lambda = \{\lambda_\alpha\}_{\alpha\in I}$ and $x = \sum_{\alpha\in I} \lambda_\alpha x_\alpha$. Then $F^{-1}x_\alpha = \delta_\alpha$ and $F\delta_\alpha = x_\alpha$, $\forall\alpha \in I$. Thus, by Theorems 3.3 and 3.7, the system $\{x_\alpha\}_{\alpha\in I}$ is simultaneously K-Bessel and K-Hilbert in X. □

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