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REMARKS ON A 1-D NONLOCAL IN TIME FRACTIONAL DIFFUSION EQUATION WITH INHOMOGENEOUS SOURCE

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Abstract. In this paper, we deal with the fractional diffusion equation with Riemann-Liouville in the form

$$\mathbf{D}^{\alpha}_{0+}u - u_{xx} = F(x,t).$$

Under some various assumptions of the input data ψ, F, we study the well-posedness of our problem. To achieve our purpose, we use the techniques of Fourier series expansion in Hilbert scales. In particular, we apply Fourier analysis method and combine with some estimates of Mittag-Lefler functions to establish the existence and uniqueness of solutions to our problem on the Sobolev spaces.

1. Introduction

In the process of studying a number of physical models and problems of practical significance, it was realized that it was necessary to study and consider diffusion models with fraction derivatives rather than models with classical derivative. Fractional calculation has many important applications in many different fields of science and engineering, such as in biological population models, signal processing, fluid mechanics, electrical networks, and electromagnetism. , electrochemical, optical and viscosity [8, 9, 12, 31]. As far as we know, there are currently several definitions for fraction derivatives and fraction integrals, such as Riemann-Liouville, Caputo, Hadamard, Riesz, Grünwald-Letnikov, Marchaud, etc. Some works are attracting the attention of the community, like A. Debbouche and his group [4, 5, 6], E. Karapinar et al [13, 14, 15, 16, 17, 18, 19, 20], M. Benchohra [28, 29, 30]. Although most of them have been extensively studied, most mathematicians are interested and studied the two derivative Caputo derivative and Riemann-Liouville.

In this paper, we consider the fractional Sobolev equation

$$\begin{cases} \mathbf{D}^{\alpha}_{0+}u - u_{xx} = F(x,t), \ (x,t) \in (0,\pi)\times(0,T), \\ u = 0, \ (x,t) \in \partial\Omega \times (0,T), \\ \epsilon_1 t^{1-\alpha} u|_{t=0} + \epsilon_2 u(x,T) = \psi(x) \end{cases} \tag{1}$$

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where $\mathbf{D}^{\alpha}_{0^+} v$ denotes a RiemannLiouville fractional derivative of v with order α, $0 < \alpha \le 1$. It is defined by

$$\mathbf{D}^{\alpha}_{0^+} v(t) = \frac{d}{dt}\Big(\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-r)^{-\alpha} v(r)dr \Big) \tag{2}$$

and $\mathbf{D}^{\alpha}_{0^+} v(t) =: \dfrac{d}{dt} v(t)$ if $\alpha = 1$. Diffusion equations derive from the many diffusion phenomena that occur widely in nature. They are proposed as mathematical models of physical problems in many fields, such as phase transition, biochemistry. Equations of time fractional reactions occur in describing "memory" in physics, for example plasma turbulence [24], fractal geometry [26], and single-molecular protein dynamics [25]. When $\mathbf{D}_{0^+}$ is replaced by the Caputo derivative and $\epsilon_2 = 0$, the above equation has been studied quite in detail in the previous paper[27]. We can refer the reader to some interesting papers on fractional diffusion equation, for example [1, 2].

In [22], the authors investigated the existence of solutions of fractional differential equations with integral boundary conditions as follows

$$\begin{cases} \mathbf{D}^{\alpha}_{0^+} u = F(t, u(t)), \ (x,t) \in (0,\pi) \times (0,T), \\ t^{1-\alpha} u|_{t=0} = \lambda \displaystyle\int_0^T u(t)dt + h, \ h \in \mathbb{R} \end{cases} \tag{3}$$

where $\lambda \ge 0$. In [23], C. Zhai and R. Jiang studied the following non-local problem

$$\begin{cases} \mathbf{D}^{\alpha}_{0^+} u + F(t, v(t)) = a, t \in (0,T), \\ \mathbf{D}^{\beta}_{0^+} v + F(t, u(t)) = b, t \in (0,T), \\ u(0) = 0, \ u(T) = \displaystyle\int_0^T \phi(t) u(t) dt, \\ v(0) = 0, \ v(T) = \displaystyle\int_0^T \phi(t) v(t) dt, \end{cases} \tag{4}$$

where $1 < \alpha, \beta < 2$. In recent paper [3], T.B. Ngoc and his coauthors considered the backward problem for nonlinear model

$$\mathbf{D}^{\alpha}_{0^+} u - u_{xx} = F(x,t,u).$$

To the best of our knowledge, there are not any result concerning on Problem (1). Motivated by this work [3], we first consider a nonlocal in time problem for the equation

$$\mathbf{D}^{\alpha}_{0^+} u - u_{xx} = F(x,t).$$

Our main goal in this paper is to give the existence and uniqueness of the mild solution for Problem (1). The regularity estimates for the mild solution are established in some various spaces.

This article is arranged as follows. Section 2 gives some preliminary and mild solution. In Section 3, we present our main results including two main theorems. Finally, the proof of some theorems is completed in section 4.

2. Preliminaries

Consider the Mittag-Leffler function, which is defined by

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+\beta)}$$

($z\in\mathbb{C}$), for $\alpha>0$ and $\beta\in\mathbb{R}$. When $\beta=1$, it is abbreviated as $E_\alpha(z)=E_{\alpha,1}(z)$. We call to mind the following lemmas (see for example [7]. We have the following lemma which useful for next proof.

Lemma 2.1. *Let $0<\alpha<1$. Then the function $z\mapsto E_{\alpha,\alpha}(z)$ has no negative root. Moreover, there exists a constant $\overline{C}_\alpha$ such that*

$$0\le E_{\alpha,\alpha}(-z)\le\frac{\overline{C}_\alpha}{1+z},\;\; z>0. \tag{5}$$

For positive number $r\ge0$, we also define the Hilber scale space

$$H^r(0,\pi)=\left\{w\in L^2(0,\pi):\sum_{p=1}^{\infty}p^{2r}\langle w,\sqrt{\frac{2}{\pi}}\sin(px)\rangle^2<+\infty\right\}, \tag{6}$$

with the following norm $\|u\|_{H^r(0,\pi)}=\left(\sum_{p=1}^{\infty}p^{2r}\langle w,\sqrt{\frac{2}{\pi}}\sin(px)\rangle^2\right)^{\frac{1}{2}}$.

3. Main results

Theorem 3.1. *Let $\epsilon_1,\epsilon_2>0$ and $F\in L^\infty(0,T;L^2(0,\pi))$. Then Problem (1) has a mild solution in $L^q(0,T;H^d(0,\pi))$ for any $1<q<\frac{1}{1-\alpha}$ and $d<3/2$. Then we have*

$$\|u\|_{L^q(0,T;H^d(0,\pi))}\le\mathcal{C}_1\|\psi\|_{H^d(0,\pi)}+\mathcal{C}_2\|F\|_{L^\infty(0,T;L^2(0,\pi))}, \tag{7}$$

where the hidden constant depends on $\alpha,\epsilon_1,\epsilon_2,s,T$ and $\mathcal{C}_1=\overline{C}_\alpha\epsilon_1^{-1}\Gamma(\alpha)\Big(\int_0^T t^{(\alpha-1)q}dt\Big)^{1/q}$ and $\mathcal{C}_2=\mathcal{M}(\epsilon_1,\epsilon_2,\alpha)\Big(\int_0^T t^{(\alpha-1)q}dt\Big)^{1/q}+\overline{M}(\alpha,s)\Big(\int_0^T t^{(\alpha-1)q}dt\Big)^{1/q}$.

Theorem 3.2. *Let u^* be the mild solution of the following problem*

$$\begin{cases}\mathbf{D}^\alpha_{0+}u-u_{xx}=F(x,t),\;(x,t)\in(0,\pi)\times(0,T),\\ u=0,\;(x,t)\in\partial\Omega\times(0,T),\\ t^{1-\alpha}u|_{t=0}=\psi(x)\end{cases} \tag{8}$$

a) If $F\in L^\infty(0,T;L^2(0,\pi))$ and $\psi\in H^d(0,\pi)$ then $u^*\in L^q(0,T;H^d(0,\pi))$.
b) If $\psi\in H^\mu(0,\pi)$ and $F\in L^2(0,T;H^\mu(0,\pi))$ for $\mu\ge0$. Assume that $\alpha>1/2$ and $0<h<\frac{2\alpha-1}{2}$. Then we get $t^\gamma u\in L^\infty(0,T;H^\nu(0,\pi))$ and $u^*\in L^2(0,T;H^\nu(0,\pi))$ where $\nu\le\mu+2h$ and $\gamma\ge1+\alpha h-\alpha$. And we also get

$$\|t^\gamma u^*\|_{L^\infty(0,T;H^\nu(0,\pi))}\lesssim T^{\gamma+\alpha-\alpha h-1}\|\psi\|_{H^\mu(0,\pi)}+T^{\gamma+\alpha-1/2-\alpha h}\|F\|_{L^2(0,T;H^\mu(0,\pi))} \tag{9}$$

and

$$\|u^*\|_{L^2(0,T;H^\nu(0,\pi))}\lesssim\|\psi\|_{H^\mu(0,\pi)}+\|F\|_{L^2(0,T;H^\mu(0,\pi))} \tag{10}$$

where the hiden constants depend on $\alpha,h,\gamma\nu,\mu$.

Theorem 3.3. *Let $\epsilon_2 > 0$ and $F \in L^\infty(0,T;L^2(0,\pi))$ and $\psi \in H^\mu(0,\pi)$. Assume that u^{1,ϵ_2} be the solution of Problem (1) with $\epsilon_1 = 1, \epsilon_2 > 0$. Let $h \in (0,1)$. Then we have the following*

$$\left\| u_{1,\epsilon_2} - u^* \right\|_{L^q(0,T;H^d(0,\pi))} \lesssim \epsilon_2 \|\psi\|_{H^\mu(0,\pi)} + \epsilon_2 \|F\|_{L^\infty(0,T;L^2(0,\pi))}. \tag{11}$$

where $d \le \mu + 2h$. The hiden constant in the above estimation depends on $\alpha, h, \gamma\nu, \mu$.

3.1. **Proof of Theorem** (3.1)**.** Let us first to give the explicit formula of the mild solution of Problem (1). The separation of variables helps us to yield the solution of (1) which is defined by Fourier series $u(x,t) = \sqrt{\frac{2}{\pi}} \sum_{p=1}^{\infty} u_p(t) \sin(px)$, where $u_p(t) = \left\langle u(t,\cdot), \sqrt{\frac{2}{\pi}} \sin(px) \right\rangle$. It becomes to the fractional ordinary differential equation

$$\mathbf{D}^\alpha_{0^+} u_p(t) + p^2 u_p(t) = F_p(t).$$

Let $\varphi = t^{1-\alpha} u|_{t=0}$. Then we get the following identity

$$\begin{aligned} u_p(t) &= \Gamma(\alpha) t^{\alpha-1} E_{\alpha,\alpha}\left(-p^2 t^\alpha\right) \varphi_p \\ &\quad + \int_0^t (t-\xi)^{\alpha-1} E_{\alpha,\alpha}\left(-p^2 (t-\xi)^\alpha\right) F_p(\xi) d\xi. \end{aligned} \tag{12}$$

The non-local in time condition allows us to confirm that

$$\begin{aligned} &\epsilon_1 \varphi_p + \epsilon_2 \Gamma(\alpha) T^{\alpha-1} E_{\alpha,\alpha}\left(-p^2 T^\alpha\right) \varphi_p \\ &\qquad + \epsilon_2 \int_0^T (T-\xi)^{\alpha-1} E_{\alpha,\alpha}\left(-p^2 (T-\xi)^\alpha\right) F_p(\xi) d\xi = \psi_p. \end{aligned} \tag{13}$$

Due to the uniqueness property of Fourier expansion, one has

$$\varphi_p = \frac{\psi_p - \epsilon_2 \int_0^T (T-\xi)^{\alpha-1} E_{\alpha,\alpha}\left(-p^2 (T-\xi)^\alpha\right) F_p(\xi) d\xi}{\epsilon_1 + \epsilon_2 \Gamma(\alpha) T^{\alpha-1} E_{\alpha,\alpha}\left(-p^2 T^\alpha\right)}. \tag{14}$$

Combining (12) and (14), we arrive at

$$\begin{aligned} u_p(t) &= \frac{\Gamma(\alpha) t^{\alpha-1} E_{\alpha,\alpha}\left(-p^2 t^\alpha\right) \psi_p}{\epsilon_1 + \epsilon_2 \Gamma(\alpha) T^{\alpha-1} E_{\alpha,\alpha}\left(-p^2 T^\alpha\right)} \\ &\quad - \frac{\epsilon_2 \Gamma(\alpha) t^{\alpha-1} E_{\alpha,\alpha}\left(-p^2 t^\alpha\right) \left(\int_0^T (T-\xi)^{\alpha-1} E_{\alpha,\alpha}\left(-p^2 (T-\xi)^\alpha\right) F_p(\xi) d\xi \right)}{\epsilon_1 + \epsilon_2 \Gamma(\alpha) T^{\alpha-1} E_{\alpha,\alpha}\left(-p^2 T^\alpha\right)} \\ &\quad + \int_0^t (t-\xi)^{\alpha-1} E_{\alpha,\alpha}\left(-p^2 (t-\xi)^\alpha\right) F_p(\xi) d\xi. \end{aligned} \tag{15}$$

By the theory of Fourier series, the mild solution u is given by

$$\begin{aligned} u(x,t) &= \sum_{p=1}^{\infty} \frac{\Gamma(\alpha) t^{\alpha-1} E_{\alpha,\alpha}\left(-p^2 t^\alpha\right) \psi_p}{\epsilon_1 + \epsilon_2 \Gamma(\alpha) T^{\alpha-1} E_{\alpha,\alpha}\left(-p^2 T^\alpha\right)} \Big(\sqrt{\frac{2}{pi}} \sin(px) \Big) \\ &- \sum_{p=1}^{\infty} \frac{\epsilon_2 \Gamma(\alpha) t^{\alpha-1} E_{\alpha,\alpha}\left(-p^2 t^\alpha\right) \left(\int_0^T (T-\xi)^{\alpha-1} E_{\alpha,\alpha}\left(-p^2 (T-\xi)^\alpha\right) F_p(\xi) d\xi \right)}{\epsilon_1 + \epsilon_2 \Gamma(\alpha) T^{\alpha-1} E_{\alpha,\alpha}\left(-p^2 T^\alpha\right)} \Big(\sqrt{\frac{2}{pi}} \sin(px) \Big) \\ &+ \sum_{p=1}^{\infty} \left(\int_0^t (t-\xi)^{\alpha-1} E_{\alpha,\alpha}\left(-p^2 (t-\xi)^\alpha\right) F_p(\xi) d\xi \right) \Big(\sqrt{\frac{2}{pi}} \sin(px) \Big) \\ &= \mathscr{A}_1(x,t) + \mathscr{A}_2(x,t) + \mathscr{A}_3(x,t). \end{aligned} \tag{16}$$

The upper bound of the Mittag-Lefler implies that

$$\begin{aligned}\|\mathscr{A}_1(.,t)\|^2_{H^d(\Omega)} &= \sum_{p=1}^{\infty}\left(\frac{\Gamma(\alpha)t^{\alpha-1}p^{2d}E_{\alpha,\alpha}\left(-p^2t^\alpha\right)\psi_p}{\epsilon_1+\epsilon_2\Gamma(\alpha)T^{\alpha-1}E_{\alpha,\alpha}\left(-p^2T^\alpha\right)}\right)^2\\ &\le |\overline{C}_\alpha|^2\epsilon_1^{-2}|\Gamma(\alpha)|^2t^{2\alpha-2}\sum_{p=1}^{\infty}p^{2d}\psi_p^2\\ &= \overline{C}_\alpha\epsilon_1^{-2}|\Gamma(\alpha)|^2t^{2\alpha-2}\|\psi\|^2_{H^d(0,\pi)},\end{aligned}\tag{17}$$

where we note that $\epsilon_1+\epsilon_2\Gamma(\alpha)T^{\alpha-1}E_{\alpha,\alpha}\left(-p^2T^\alpha\right)>\epsilon_1$. Hence, we get

$$\|\mathscr{A}_1(.,t)\|_{H^d(\Omega)} \le \overline{C}_\alpha\epsilon_1^{-1}\Gamma(\alpha)t^{\alpha-1}\|\psi\|_{H^d(\Omega)}.\tag{18}$$

Take any $0<s<1$ then we get that

$$E_{\alpha,\alpha}\left(-p^2(t-\xi)^\alpha\right)\le\frac{\overline{C}_\alpha}{1+p^2(t-\xi)^\alpha}\le\overline{C}_\alpha p^{-2s}(t-\xi)^{-\alpha s}.\tag{19}$$

Hence, we arrive at the following bound

$$\begin{aligned}&\int_0^t(t-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(-p^2(t-\xi)^\alpha\right)F_p(\xi)d\xi\\ &\le\overline{C}_\alpha p^{-2s}\int_0^t(t-\xi)^{\alpha-1-\alpha s}|F_p(\xi)|d\xi\\ &\le\overline{C}_\alpha p^{-2s}\|F\|_{L^\infty(0,T;L^2(0,\pi))}\int_0^t(t-\xi)^{\alpha-1-\alpha s}d\xi\\ &=\frac{\overline{C}_\alpha}{\alpha-\alpha s}p^{-2s}\|F\|_{L^\infty(0,T;L^2(0,\pi))}t^{\alpha-\alpha s}.\end{aligned}\tag{20}$$

Using this inequality, we find that

$$\begin{aligned}&\|\mathscr{A}_2(.,t)\|^2_{H^d(0,\pi)}\\ &=\sum_{p=1}^{\infty}p^{2d}\left(\frac{\epsilon_2\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}\left(-p^2t^\alpha\right)\left(\int_0^T(T-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(-p^2(T-\xi)^\alpha\right)F_p(\xi)d\xi\right)}{\epsilon_1+\epsilon_2\Gamma(\alpha)T^{\alpha-1}E_{\alpha,\alpha}\left(-p^2T^\alpha\right)}\right)^2\\ &\le\epsilon_2^2\epsilon_1^{-2}|\Gamma(\alpha)|^2t^{2\alpha-2}\sum_{p=1}^{\infty}p^{2d}\Big|E_{\alpha,\alpha}\left(-p^2t^\alpha\right)\Big|^2\left(\int_0^T(T-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(-p^2(T-\xi)^\alpha\right)F_p(\xi)d\xi\right)^2\\ &\le\epsilon_2^2\epsilon_1^{-2}|\Gamma(\alpha)|^2\left(\frac{\overline{C}_\alpha T^{\alpha-\alpha s}}{\alpha-\alpha s}\right)^2t^{2\alpha-2}\left(\sum_{p=1}^{\infty}p^{2d-4s}\right).\end{aligned}\tag{21}$$

Notice that the infinite series $\sum_{p=1}^\infty p^{4s-2d}$ is convergent for $4s>1+2d$. It is obvious to see that if $d<3/2$, we can choose s such that $1+2d<4s$ and $0<s<1$. Using the fact that $E_{\alpha,\alpha}\left(-p^2t^\alpha\right)\le\overline{C}_\alpha$ and together with (20), we know that

$$\|\mathscr{A}_2(.,t)\|_{H^d(0,\pi)}\le\mathcal{M}(\epsilon_1,\epsilon_2,\alpha)t^{\alpha-1}\|F\|_{L^\infty(0,T;L^2(0,\pi))},\tag{22}$$

where

$$\mathcal{M}(\epsilon_1,\epsilon_2,\alpha)=\frac{|\overline{C}_\alpha|^2}{\alpha-\alpha s}\epsilon_2\epsilon_1^{-1}\Gamma(\alpha)T^{\alpha-\alpha s}\sqrt{\sum_{p=1}^{\infty}p^{2d-4s}}$$

Thanks to the inequality (20), we confirm that

$$\begin{aligned}\|\mathscr{A}_3(.,t)\|_{H^d(0,\pi)} &= \sqrt{\sum_{p=1}^{\infty}\Big(\int_0^t (t-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(-p^2(t-\xi)^{\alpha}\right)F_p(\xi)d\xi\Big)^2} \\ &\le \frac{\overline{C}_\alpha}{\alpha-\alpha s}\|F\|_{L^\infty(0,T;L^2(0,\pi))}t^{\alpha-\alpha s}\sqrt{\sum_{p=1}^{\infty}p^{2d-4s}} \\ &\le \overline{M}(\alpha,s)t^{\alpha-1}\|F\|_{L^\infty(0,T;L^2(0,\pi))} \qquad (23)\end{aligned}$$

where

$$\overline{M}(\alpha,s)=\frac{\overline{C}_\alpha}{\alpha-\alpha s}T^{1-\alpha s}\sqrt{\sum_{p=1}^{\infty}p^{2d-4s}}.$$

Combining (18), (22) and (23), we find that

$$\begin{aligned}\|u(.,t)\|_{H^d(0,\pi)} &\le \|\mathscr{A}_1(.,t)\|_{H^d(0,\pi)}+\|\mathscr{A}_2(.,t)\|_{H^d(0,\pi)}+\|\mathscr{A}_3(.,t)\|_{H^d(0,\pi)} \\ &\le \overline{C}_\alpha\epsilon_1^{-1}\Gamma(\alpha)t^{\alpha-1}\|\psi\|_{H^d(\Omega)}+\mathcal{M}(\epsilon_1,\epsilon_2,\alpha)t^{\alpha-1}\|F\|_{L^\infty(0,T;L^2(0,\pi))} \\ &+\overline{M}(\alpha,s)t^{\alpha-1}\|F\|_{L^\infty(0,T;L^2(0,\pi))}. \qquad (24)\end{aligned}$$

Therefore, since the condition $1<q<\frac{1}{1-\alpha}$, we obtain that

$$\begin{aligned}\|u\|_{L^q(0,T;H^d(0,\pi))} &= \Big(\int_0^T\|u(.,t)\|^q_{H^d(0,\pi)}dt\Big)^{1/q} \le \overline{C}_\alpha\epsilon_1^{-1}\Gamma(\alpha)\|\psi\|_{H^d(0,\pi)}\Big(\int_0^T t^{(\alpha-1)q}dt\Big)^{1/q} \\ &+\mathcal{M}(\epsilon_1,\epsilon_2,\alpha)\Big(\int_0^T t^{(\alpha-1)q}dt\Big)^{1/q}\|F\|_{L^\infty(0,T;L^2(0,\pi))} \\ &+\overline{M}(\alpha,s)\Big(\int_0^T t^{(\alpha-1)q}dt\Big)^{1/q}\|F\|_{L^\infty(0,T;L^2(0,\pi))} \\ &\lesssim \|\psi\|_{H^d(\Omega)}+\|F\|_{L^\infty(0,T;L^2(0,\pi))} \qquad (25)\end{aligned}$$

where the hidden constant depends on $\alpha,\epsilon_1,\epsilon_2,s,T$.

3.2. **Proof of Theorem** (3.2)**.** *Proof a.*
Let us recall the mild solution

$$\begin{aligned}u^*(x,t) &= \sum_{p=1}^{\infty}\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}\left(-p^2t^\alpha\right)\psi_p\Big(\sqrt{\frac{2}{\pi}}\sin(px)\Big) \\ &+\sum_{p=1}^{\infty}\int_0^t(t-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(-p^2(t-\xi)^\alpha\right)F_p(\xi)d\xi\Big(\sqrt{\frac{2}{\pi}}\sin(px)\Big) \\ &= \mathscr{B}_1(x,t)+\mathscr{B}_2(x,t). \qquad (26)\end{aligned}$$

The upper bound of the Mittag-Lefler implies that

$$\begin{aligned}\|\mathscr{B}_1(.,t)\|^2_{H^d(\Omega)} &= \sum_{p=1}^{\infty}\left(\Gamma(\alpha)t^{\alpha-1}p^{2d}E_{\alpha,\alpha}\left(-p^2t^\alpha\right)\psi_p\right)^2 \\ &\le |\overline{C}_\alpha|^2|\Gamma(\alpha)|^2t^{2\alpha-2}\sum_{p=1}^{\infty}p^{2d}\psi_p^2 = |\overline{C}_\alpha|^2|\Gamma(\alpha)|^2t^{2\alpha-2}\|\psi\|^2_{H^d(0,\pi)}, \qquad (27)\end{aligned}$$

By a similar way as above, we also obtain that

$$\|\mathscr{B}_2(.,t)\|_{H^d(0,\pi)} \le \overline{M}_1(\alpha,d)t^{\alpha-1}\|F\|_{L^\infty(0,T;L^2(0,\pi))} \tag{28}$$

where

$$\overline{M}_1(\alpha,d) = \frac{\overline{C}_\alpha}{\alpha-\alpha s}T^{1-\alpha s}\sqrt{\sum_{p=1}^\infty p^{2d-4s}}.$$

Combining (27) and (28), we find that

$$\begin{aligned}\|u^*(.,t)\|_{H^d(0,\pi)} &\le \|\mathscr{B}_1(.,t)\|_{H^d(0,\pi)} + \|\mathscr{B}_2(.,t)\|_{H^d(0,\pi)} \\ &\le \overline{C}_\alpha|\Gamma(\alpha)|t^{\alpha-1}\|\psi\|_{H^d(0,\pi)} + \overline{M}_1(\alpha,d)t^{\alpha-1}\|F\|_{L^\infty(0,T;L^2(0,\pi))}.\end{aligned} \tag{29}$$

Proof b.
From (26), we get

$$\begin{aligned}t^\gamma u^*(x,t) &= \sum_{p=1}^\infty \Gamma(\alpha)t^{\gamma+\alpha-1}E_{\alpha,\alpha}\left(-p^2t^\alpha\right)\psi_p\big(\sqrt{\frac{2}{\pi}}\sin(px)\big) \\ &+ \sum_{p=1}^\infty\left(t^\gamma\int_0^t(t-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(-p^2(t-\xi)^\alpha\right)F_p(\xi)d\xi\right)\big(\sqrt{\frac{2}{\pi}}\sin(px)\big) \\ &= \mathscr{D}_1(x,t)+\mathscr{D}_2(x,t).\end{aligned} \tag{30}$$

Therefore, we obtain that

$$\left\|t^\gamma u^*(.,t)\right\|_{H^\nu(0,\pi)} \le \left\|\mathscr{D}_1(.,t)\right\|_{H^\nu(0,\pi)} + \left\|\mathscr{D}_2(.,t)\right\|_{H^\nu(0,\pi)} \tag{31}$$

The condition of h as above implies that $0<h<1$, so we get immediately that

$$E_{\alpha,\alpha}\left(-p^2t^\alpha\right) \le \frac{\overline{C}_\alpha}{1+p^2t^\alpha} \le \frac{\overline{C}_\alpha}{(1+p^2t^\alpha)^h} \le \overline{C}_\alpha p^{-2h}t^{-\alpha h}. \tag{32}$$

Hence

$$\Gamma(\alpha)t^{\gamma+\alpha-1}E_{\alpha,\alpha}\left(-p^2t^\alpha\right) \le \Gamma(\alpha)\overline{C}_\alpha p^{-2h}t^{\gamma+\alpha-1-\alpha h}. \tag{33}$$

which leads to

$$\begin{aligned}\left\|\mathscr{D}_1(.,t)\right\|_{H^\nu(0,\pi)} &= \sqrt{\sum_{p=1}^\infty p^{2\nu}\Big(\Gamma(\alpha)t^{\gamma+\alpha-1}E_{\alpha,\alpha}\left(-p^2t^\alpha\right)\Big)^2|\psi_p|^2} \\ &\le \overline{C}_\alpha\Gamma(\alpha)t^{\gamma+\alpha-\alpha h-1}\sqrt{\sum_{p=1}^\infty p^{2\nu-4h}|\psi_p|^2} \\ &= \overline{C}_\alpha\Gamma(\alpha)t^{\gamma+\alpha-\alpha h-1}\|\psi\|_{H^{\nu-2h}(0,\pi)}.\end{aligned} \tag{34}$$

For the term $\mathscr{D}_1(x,t)$, we notice that

$$\Big|\int_0^t(t-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(-p^2(t-\xi)^\alpha\right)F_p(\xi)d\xi\Big| \le \overline{C}_\alpha p^{-2h}\int_0^t(t-\xi)^{\alpha-1-\alpha h}|F_p(\xi)|d\xi. \tag{35}$$

which allows us to deduce that

$$\begin{aligned}\left\|\mathscr{D}_2(.,t)\right\|^2_{H^\nu(0,\pi)} &= \sum_{p=1}^{\infty} p^{2\nu}\left(t^\gamma \int_0^t (t-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(-p^2(t-\xi)^\alpha\right)F_p(\xi)d\xi\right)^2 \\ &\le |C_\alpha|^2 \sum_{p=1}^{\infty} p^{2\nu-4h}t^{2\gamma}\left(\int_0^t (t-\xi)^{2\alpha-2-2\alpha h}d\xi\right)\left(\int_0^t |F_p(\xi)|^2 d\xi\right) \\ &\le \frac{|C_\alpha|^2}{2\alpha-1-2\alpha h} t^{2\gamma+2\alpha-1-2\alpha h}\|F\|^2_{L^2(0,T;H^{\nu-2h}(0,\pi))}. \qquad (36)\end{aligned}$$

where we have used Hölder inequality. Combining (31), (34) and (36), we find that

$$\begin{aligned}\left\|t^\gamma u^*(.,t)\right\|_{H^\nu(0,\pi)} &\le \overline{C}_\alpha \Gamma(\alpha)t^{\gamma+\alpha-\alpha h-1}\|\psi\|_{H^{\nu-2h}(0,\pi)} \\ &+ \frac{C_\alpha}{\sqrt{2\alpha-1-2\alpha h}} t^{\gamma+\alpha-1/2-\alpha h}\|F\|_{L^2(0,T;H^{\nu-2h}(0,\pi))}. \qquad (37)\end{aligned}$$

Since $\nu - 2h \le \mu$, we find that

$$\|\psi\|_{H^{\nu-2h}(0,\pi)} \le C_{1,\mu,\nu,h}\|\psi\|_{H^\mu(0,\pi)} \qquad (38)$$

and

$$\|F\|_{L^2(0,T;H^{\nu-2h}(0,\pi))} \le C_{2,\mu,\nu,h}\|F\|_{L^2(0,T;H^\mu(0,\pi))}. \qquad (39)$$

From some previous observations, we can deduce that

$$\left\|t^\gamma u^*(.,t)\right\|_{H^\nu(0,\pi)} \lesssim T^{\gamma+\alpha-\alpha h-1}\|\psi\|_{H^\mu(0,\pi)} + T^{\gamma+\alpha-1/2-\alpha h}\|F\|_{L^2(0,T;H^\mu(0,\pi))}. \qquad (40)$$

Proof c.
In the following, we need to give the estimation of $\int_0^T \left\|u(.,t)\right\|^2_{H^\nu(0,\pi)}dt$. Let us denote

$$\overline{D}_1 = \Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}\left(-p^2t^\alpha\right)\psi_p$$

and

$$\overline{D}_2 = \int_0^t (t-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(-p^2(t-\xi)^\alpha\right)F_p(\xi)d\xi.$$

Using again the result (34), we arrive at

$$\begin{aligned}\int_0^T \left\|\overline{D}_1(.,t)\right\|^2_{H^\nu(0,\pi)}dt &= \int_0^T \left(\sum_{p=1}^{\infty} p^{2\nu}\Big(\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}\left(-p^2t^\alpha\right)\Big)^2|\psi_p|^2\right)dt \\ &\le |\overline{C}_\alpha|^2|\Gamma(\alpha)|^2\|\psi\|^2_{H^{\nu-2h}(0,\pi)}\int_0^T t^{2\alpha-2\alpha h-2}dt \\ &= \frac{T^{2\alpha-2\alpha h-1}}{2\alpha-2\alpha h-1}|\overline{C}_\alpha|^2|\Gamma(\alpha)|^2\|\psi\|^2_{H^{\nu-2h}(0,\pi)}. \qquad (41)\end{aligned}$$

By using the result (36), we obtain

$$\begin{aligned}\int_0^T \left\|\overline{D}_2(.,t)\right\|^2_{H^\nu(0,\pi)}dt &\le \frac{|C_\alpha|^2}{2\alpha-1-2\alpha h}\|F\|^2_{L^2(0,T;H^{\nu-2h}(0,\pi))}\left(\int_0^T t^{2\alpha-1-2\alpha h}dt\right) \\ &= \frac{T^{2\alpha-2\alpha h}|C_\alpha|^2}{(2\alpha-1-2\alpha h)(2\alpha-2\alpha h)}\|F\|^2_{L^2(0,T;H^{\nu-2h}(0,\pi))}. \qquad (42)\end{aligned}$$

From two above observations, we deduce that

$$\|u^*\|^2_{L^2(0,T;H^\nu(0,\pi))} = \int_0^T \|u(.,t)\|^2_{H^\nu(0,\pi)} dt$$
$$\le 2\int_0^T \left\|\overline{D}_1(.,t)\right\|^2_{H^\nu(0,\pi)} dt + \int_0^T \left\|\overline{D}_2(.,t)\right\|^2_{H^\nu(0,\pi)} dt$$
$$\le 2\frac{T^{2\alpha-2\alpha h-1}}{2\alpha-2\alpha h-1}|\overline{C}_\alpha|^2|\Gamma(\alpha)|^2\|\psi\|^2_{H^{\nu-2h}(0,\pi)}$$
$$+ 2\frac{T^{2\alpha-2\alpha h}|C_\alpha|^2}{(2\alpha-1-2\alpha h)(2\alpha-2\alpha h)}\|F\|^2_{L^2(0,T;H^{\nu-2h}(0,\pi))}$$
$$\le 2\left(\frac{T^{\alpha-\alpha h-1/2}}{\sqrt{2\alpha-2\alpha h-1}}\overline{C}_\alpha\Gamma(\alpha)\|\psi\|_{H^{\nu-2h}(0,\pi)} + \frac{T^{\alpha-\alpha h}C_\alpha}{\sqrt{(2\alpha-1-2\alpha h)(2\alpha-2\alpha h)}}\|F\|_{L^2(0,T;H^{\nu-2h}(0,\pi))}\right)^2. \tag{43}$$

Using (38) and (39) and (43), we get the desired result (10).

3.3. **Proof of Theorem** (3.3)**.** Let us recall that

$$u_{\epsilon_2}(x,t) = \sum_{p=1}^\infty \frac{\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}\left(-p^2t^\alpha\right)\psi_p}{1+\epsilon_2\Gamma(\alpha)T^{\alpha-1}E_{\alpha,\alpha}\left(-p^2T^\alpha\right)}\Big(\sqrt{\frac{2}{\pi}}\sin(px)\Big)$$
$$-\sum_{p=1}^\infty \frac{\epsilon_2\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}\left(-p^2t^\alpha\right)\left(\int_0^T(T-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(-p^2(T-\xi)^\alpha\right)F_p(\xi)d\xi\right)}{1+\epsilon_2\Gamma(\alpha)T^{\alpha-1}E_{\alpha,\alpha}\left(-p^2T^\alpha\right)}\Big(\sqrt{\frac{2}{\pi}}\sin(px)\Big)$$
$$+\sum_{p=1}^\infty\left(\int_0^t(t-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(-p^2(t-\xi)^\alpha\right)F_p(\xi)d\xi\right)\Big(\sqrt{\frac{2}{\pi}}\sin(px)\Big). \tag{44}$$

and

$$u^*(x,t) = \sum_{p=1}^\infty \Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}\left(-p^2t^\alpha\right)\psi_p\Big(\sqrt{\frac{2}{\pi}}\sin(px)\Big)$$
$$+\sum_{p=1}^\infty\left(\int_0^t(t-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(-p^2(t-\xi)^\alpha\right)F_p(\xi)d\xi\right)\Big(\sqrt{\frac{2}{\pi}}\sin(px)\Big). \tag{45}$$

By subtracting (44) by (45), the following result is immediately obtained

$$u_{1,\epsilon_2}(x,t) - u^*(x,t)$$
$$= \sum_{p=1}^\infty \Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}\left(-p^2t^\alpha\right)\frac{\epsilon_2\Gamma(\alpha)T^{\alpha-1}E_{\alpha,\alpha}\left(-p^2T^\alpha\right)}{1+\epsilon_2\Gamma(\alpha)T^{\alpha-1}E_{\alpha,\alpha}\left(-p^2T^\alpha\right)}\psi_p\Big(\sqrt{\frac{2}{\pi}}\sin(px)\Big)$$
$$-\sum_{p=1}^\infty \frac{\epsilon_2\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}\left(-p^2t^\alpha\right)\left(\int_0^T(T-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(-p^2(T-\xi)^\alpha\right)F_p(\xi)d\xi\right)}{1+\epsilon_2\Gamma(\alpha)T^{\alpha-1}E_{\alpha,\alpha}\left(-p^2T^\alpha\right)}\Big(\sqrt{\frac{2}{\pi}}\sin(px)\Big)$$
$$= \mathscr{G}_1(x,t) + \mathscr{G}_2(x,t). \tag{46}$$

For estimating the term $\mathscr{G}_1$, using the upper bound of the Mittag-Lefler, we have the following inequality

$$\begin{aligned}&\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}\left(-p^2t^\alpha\right)\frac{\epsilon_2\Gamma(\alpha)T^{\alpha-1}E_{\alpha,\alpha}\left(-p^2T^\alpha\right)}{1+\epsilon_2\Gamma(\alpha)T^{\alpha-1}E_{\alpha,\alpha}\left(-p^2T^\alpha\right)}\\&\leq \epsilon_2|\Gamma(\alpha)|^2\overline{C}_\alpha p^{-2h}T^{\alpha-1-\alpha h}t^{\alpha-1}.\end{aligned}\tag{47}$$

This implies that

$$\begin{aligned}\left\|\mathscr{G}_1(.,t)\right\|^2_{H^d(0,\pi)}&=\sum_{p=1}^\infty p^{2d}\left(\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}\left(-p^2t^\alpha\right)\frac{\epsilon_2\Gamma(\alpha)T^{\alpha-1}E_{\alpha,\alpha}\left(-p^2T^\alpha\right)}{1+\epsilon_2\Gamma(\alpha)T^{\alpha-1}E_{\alpha,\alpha}\left(-p^2T^\alpha\right)}\right)^2|\psi_p|^2\\&\leq \epsilon_2^2|\Gamma(\alpha)|^4T^{2\alpha-2-2\alpha h}t^{2\alpha-2}|\overline{C}_\alpha|^2\sum_{p=1}^\infty p^{2d-4h}|\psi_p|^2\\&=\epsilon_2^2|\Gamma(\alpha)|^4T^{2\alpha-2-2\alpha h}|\overline{C}_\alpha|^2t^{2\alpha-2}\|\psi\|^2_{H^{d-2h}(0,\pi)}.\end{aligned}\tag{48}$$

So, we get immediately that

$$\left\|\mathscr{G}_1(.,t)\right\|_{H^d(0,\pi)}\leq \epsilon_2|\Gamma(\alpha)|^2T^{\alpha-1-\alpha h}\overline{C}_\alpha t^{\alpha-1}\|\psi\|_{H^{d-2h}(0,\pi)}.\tag{49}$$

By the same justification as in (22), we find that

$$\|\mathscr{G}_2(.,t)\|_{H^d(0,\pi)}\leq \epsilon_2\overline{M}_\alpha t^{\alpha-1}\|F\|_{L^\infty(0,T;L^2(0,\pi))},\tag{50}$$

where we denote

$$\overline{M}_\alpha=\frac{\Gamma(\alpha)T^{\alpha-\alpha s}|\overline{C}_\alpha|^2}{\alpha-\alpha s}\sqrt{\sum_{p=1}^\infty p^{2d-4s}}.$$

Combining (49) and (50), we arrive at

$$\begin{aligned}\|u_{1,\epsilon_2}(.,t)-u^*(.,t)\|_{H^d(0,\pi)}&\leq \|\mathscr{G}_1(.,t)\|_{H^d(0,\pi)}+\|\mathscr{G}_2(.,t)\|_{H^d(0,\pi)}\\&\leq \epsilon_2|\Gamma(\alpha)|^2T^{\alpha-1-\alpha h}\overline{C}_\alpha t^{\alpha-1}\|\psi\|_{H^{d-2h}(0,\pi)}\\&+\epsilon_2\overline{M}_\alpha t^{\alpha-1}\|F\|_{L^\infty(0,T;L^2(0,\pi))}\end{aligned}\tag{51}$$

Therefore, notice that the integral $\int_0^T t^{(\alpha-1)q}dt$ is convergent, we obtain that

$$\begin{aligned}&\left\|u_{1,\epsilon_2}-u^*\right\|_{L^q(0,T;H^d(0,\pi))}\\&\leq \epsilon_2|\Gamma(\alpha)|^2T^{\alpha-1-\alpha h}\overline{C}_\alpha\|\psi\|_{H^{d-2h}(0,\pi)}\left(\int_0^T t^{(\alpha-1)q}dt\right)^{1/q}\\&+\epsilon_2\overline{M}_\alpha\left(\int_0^T t^{(\alpha-1)q}dt\right)^{1/q}\|F\|_{L^\infty(0,T;L^2(0,\pi))}\end{aligned}\tag{52}$$

It follows from $d-2h\leq\mu$ that

$$\left\|u_{1,\epsilon_2}-u^*\right\|_{L^q(0,T;H^d(0,\pi))}\lesssim \epsilon_2\|\psi\|_{H^\mu(0,\pi)}+\epsilon_2\|F\|_{L^\infty(0,T;L^2(0,\pi))}.\tag{53}$$

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References

[1] N.H. Tuan, Y. Zhou, T.N. Thach, N.H. Can, *Initial inverse problem for the nonlinear fractional Rayleigh-Stokes equation with random discrete data* Commun. Nonlinear Sci. Numer. Simul. 78 (2019), 104873, 18 pp.

[2] N.H. Tuan, L.N. Huynh, T.B. Ngoc, Y. Zhou, *On a backward problem for nonlinear fractional diffusion equations* Appl. Math. Lett. 92 (2019), 76–84.

[3] T.B. Ngoc, Y. Zhou, D. O'Regan, N.H. Tuan, *On a terminal value problem for pseudoparabolic equations involving Riemann-Liouville fractional derivatives,* Appl. Math. Lett. 106 (2020), 106373, 9 pp.

[4] J. Manimaran, L. Shangerganesh, A. Debbouche, *Finite element error analysis of a time-fractional nonlocal diffusion equation with the Dirichlet energy* J. Comput. Appl. Math. 382 (2021), 113066, 11 pp

[5] J. Manimaran, L. Shangerganesh, A. Debbouche, *A time-fractional competition ecological model with cross-diffusion* Math. Methods Appl. Sci. 43 (2020), no. 8, 5197–5211

[6] N.H. Tuan, A. Debbouche, T.B. Ngoc, *Existence and regularity of final value problems for time fractional wave equations* Comput. Math. Appl. 78 (2019), no. 5, 1396–1414.

[7] I. Podlubny, *Fractional differential equations,* Academic Press, London, 1999.

[8] B. D. Coleman, W. Noll, Foundations of linear viscoelasticity, *Rev. Mod. Phys.*, 33(2) 239 (1961).

[9] P. Clément, J. A. Nohel, Asymptotic behavior of solutions of nonlinear volterra equations with completely positive kernels, *SIAM J. Math. Anal.*, 12(4) (1981), pp. 514–535.

[10] X.L. Ding, J.J. Nieto, Analytical solutions for multi-term time-space fractional partial differential equations with nonlocal damping terms, *Frac. Calc. Appl. Anal.* 21 (2018), pp. 312–335.

[11] L.C.F. Ferreira, E.J. Villamizar-Roa, Self-similar solutions, uniqueness and long-time asymptotic behavior for semilinear heat equations, *Differ. Integral Equ.*, 19(12) (2006), pp. 1349–1370.

[12] B. de Andrade, A. Viana, Abstract Volterra integrodifferential equations with applications to parabolic models with memory, *Math. Ann.*, 369 (2017), pp. 1131–1175.

[13] R. S. Adiguzel, U. Aksoy, E. Karapinar, .M. Erhan, *On the solution of a boundary value problem associated with a fractional differential equation* Mathematical Methods in the Applied Sciences https://doi.org/10.1002/mma.665

[14] H. Afshari, E, Karapınar, *A discussion on the existence of positive solutions of the boundary value problems via-Hilfer fractional derivative on b-metric spaces,* Advances in Difference Equations volume 2020, Article number: 616 (2020)

[15] H.Afshari, S. Kalantari, E. Karapinar; *Solution of fractional differential equations via coupled fixed point,* Electronic Journal of Differential Equations,Vol. 2015 (2015), No. 286, pp. 1-12

[16] B.Alqahtani, H. Aydi, E. Karapınar, V. Rakocevic, *A Solution for Volterra Fractional Integral Equations by Hybrid Contractions* Mathematics 2019, 7, 694.

[17] E. Karapinar, A.Fulga,M. Rashid, L.Shahid, H. Aydi, *Large Contractions on Quasi-Metric Spaces with an Application to Nonlinear Fractional Differential-Equations* Mathematics 2019, 7, 444.

[18] A.Salim, B. Benchohra, E. Karapinar, J. E. Lazreg, *Existence and Ulam stability for impulsive generalized Hilfer-type fractional differential equations* Adv Differ Equ 2020, 601 (2020)

[19] E. Karapinar; T.Abdeljawad; F. Jarad, *Applying new fixed point theorems on fractional and ordinary differential equations,* Advances in Difference Equations, 2019, 2019:421

[20] A.Abdeljawad, R.P. Agarwal, E. Karapinar, P.S.Kumari, *Solutions of he Nonlinear Integral Equation and Fractional Differential Equation Using the Technique of a Fixed Point with a Numerical Experiment in Extended b-Metric Space* Symmetry 2019, 11, 686.

[21] T. Jankowski, *Fractional equations of Volterra type involving a Riemann-Liouville derivative* Appl. Math. Lett. 26 (2013), no. 3, 344–350.

[22] X. Wanga, L. Wanga, Q. Zeng, *Fractional differential equations with integral boundary conditions*, J. Nonlinear Sci. Appl. 8 (2015), 309-314

[23] C. Zhai, R. Jiang, *Unique solutions for a new coupled system of fractional differential equations* Adv. Difference Equ. 2018, Paper No. 1, 12 pp.

[24] D. del-Castillo-Negrete, B. A. Carreras, V. E. Lynch; *Nondiffusive transport in plasma turbulene: A fractional diffusion approach*, Phys. Rev. Lett., 94 (2005), 065003.

[25] S. Kou; *Stochastic modeling in nanoscale biophysics: Subdiffusion within proteins*, Ann. Appl. Stat., 2 (2008), 501–535.

[26] R.R. Nigmatullin; *The realization of the generalized transfer equation in a medium with fractal geometry,* Phys. Star. Sol. B, 133 (1986), 425–430.

[27] K. Sakamoto, M. Yamamoto; *Initial value/boudary value problems for fractional diffusion-wave equations and applications to some inverse problems*, J. Math. Anal. Appl., 382 (2011), 426–447.

[28] F.S. Bachir, S. Abbas, M. Benbachir, M. Benchohra, *Hilfer-Hadamard Fractional Differential Equations, Existence and Attractivity*, Advances in the Theory of Nonlinear Analysis and its Application, 2021, Vol 5 , Issue 1, Pages 49–57.

[29] A. Salim, M. Benchohra, J. Lazreg, J. Henderson, *Nonlinear Implicit Generalized Hilfer-Type Fractional Differential Equations with Non-Instantaneous Impulses in Banach Spaces* , Advances in the Theory of Nonlinear Analysis and its Application, Vol 4 , Issue 4, Pages 332–348, 2020.

[30] Z. Baitichea, C. Derbazia, M. Benchohrab, ψ*Caputo Fractional Differential Equations with Multi-point Boundary Conditions by Topological Degree Theory*, Results in Nonlinear Analysis 3 (2020) No. 4, 167-178

[31] A. Ardjouni, A. Djoudi, *Existence and uniqueness of solutions for nonlinear hybrid implicit Caputo-Hadamard fractional differential equations* Results in Nonlinear Analysis, 2 (3), 136–142.

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