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ON APPROXIMATION BY BIVARIATE SZÁSZ-GAMMA TYPE HYBRID OPERATORS

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Abstract. In this paper bivariate summation-integral type hybrid operators are constructed and their approximation properties are explored. We investigate the rate of convergence and order of approximation. Local approximation results are obtained using mixed modulus of continuity, Lipschitz-maximal and Peetre's K-functional. Global approximation results are studied using weight functions. Approximation properties in Bögel functional space are also explored.

1. Introduction

The field of approximation theory is motivated by the celebrated Weierstrass approximation theorem [31]. To give an elegant proof of this theorem, Bernstein [3] introduced a sequence of polynomials in 1912, named as Bernstein polynomials as follows:

$$B_n(f;x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right),\quad n\in\mathbb{N},$$

where $\binom{n}{k}$ are binomial coefficients and $f\in C[0,1]$ (class of all continuous functions defined on $[0,1]$).
To approximate in a wider class, *i.e.*, $L_p[0,\infty)$(space of Lebesgue integrable functions on $[0,\infty)$), Kantorovich [16] introduced a sequence of linear positive operators as: for $f\in L_p[0,\infty)$, $1\le p<\infty$, $K_n:L_p([0,\infty))\longrightarrow L_p([0,\infty))$ defined by

$$K_n(f;x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\int_{k/n}^{k+1/n}f(t)dt.$$

Several generalizations and modifications of these operators have been introduced and studied by many researchers (see Acar et al. ([1, 2]), Braha ([6, 7, 9]), Braha and Kadak ([8]), Han and Guo ([12]), Kumar and Pratap ([17]), Mohiuddine et al. ([18, 19, 20]), Mursaleen et al. ([21, 22]), Wafi and Rao ([27, 29, 30]). For

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$j \in \mathbb{N}_0 = \{0\} \bigcup \mathbb{N}, i \geq 0$ and $f \in C[0,\infty)$, Sucu [26] presented a Szász type operators using a generalization of exponential function (see [25]) as follows:

$$S_n^*(f;x) := \frac{1}{e_i(nx)} \sum_{k=0}^{\infty} \frac{(nx)^j}{\gamma_i(j)} f\left(\frac{j+2i\theta_j}{n}\right), \tag{1.1}$$

where

$$e_i(t) = \sum_{j=0}^{\infty} \frac{t^j}{\gamma_i(j)}, \quad t \in [0,\infty).$$

Here $\gamma_i(2j) = \frac{2^{2j} j! \Gamma(j+i+1/2)}{\Gamma(i+1/2)}$ and $\gamma_i(2j+1) = \frac{2^{2j+1} j! \Gamma(j+i+3/2)}{\Gamma(i+1/2)}$ and the recursive relation for γ_j is defined as $\gamma(j+1) = (j+1+2i\theta_j)\gamma_j(j), \quad j \in \mathbb{N}_0,$ with $\theta_j = \begin{cases} 0 & if \quad j \in 2\mathbb{N}, \\ 1 & if \quad j \in 2\mathbb{N}+1 \end{cases}.$

Many generalizations of the operators given by (1.1) are studied by a number of mathematicians (see [13], [14], [15],[23], [28]). Motivated by the idea of Sucu [26], Wafi and Rao [27] gave Szász-Gamma operators based on Dunkl analogue as

$$D_n^f(x) = \frac{1}{e_i(nx)} \sum_{j=0}^{\infty} \frac{(nx)^j}{\gamma_i(j)} \frac{n^{j+2i\theta_j+\lambda+1}}{\Gamma(j+2i\theta_j+\lambda+1)} \int_0^{\infty} t^{j+2i\theta_j+\lambda} e^{-nt} f(t)dt, \tag{1.2}$$

where $\Gamma(t) = \int_0^{\infty} x^t e^{-x} dx$ is the gamma function and $\lambda \geq 0$. In this paper, we introduce a bivariate analogue of the operators given in (1.2) and explore approximation properties in various function spaces.
The contents of rest of the paper is as follows. In Section 2, we introduce a sequence of bivariate operators to the operators given in (1.2) and prove some lemmas which are needed to obtain approximating results in next sections. Approximation theorems are proved in Section 3. Section 4 contains the estimating results in Bögel space.

2. Dunkl analogue of Bivariate-Szász-Gamma-operators

Let $\mathcal{I}^2 = \{(x,y) : 0 \leq x < \infty, \ 0 \leq y < \infty\}$ and $C\left(\mathcal{I}^2\right)$ be the space of all continuous functions defined on $\mathcal{I}^2$ with the norm $||g||_{C(\mathcal{I}^2)} = \sup_{(x,y)\in\mathcal{I}^2} |g(x,y)|$. Then, for $g \in C\left(\mathcal{I}^2\right), \ 0 \leq \mu,\nu \leq 1$ and $n,m \in \mathbb{N}$ we introduce the bivariate operators as

$$D_{n,m}^{\mu,\nu}(g;x,y) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \mathcal{P}_{n,m.k.l}^{\mu,\nu}(x,y) \int_0^{\infty}\int_0^{\infty} \mathcal{Q}_{k,l}^{\mu,\nu}(t_1,t_2) g(t_1,t_2) dt_1 dt_2, \tag{2.1}$$

where $\mathcal{P}_{n,m.k.l}^{\mu,\nu}(x,y) = \mathcal{A}_{n,k}^{\mu}\mathcal{B}_{m,l}^{\nu}$, $\mathcal{Q}_{k,l}^{\mu,\nu}(t_1,t_2) = \mathcal{C}_{n,k}^{\mu}(t_1)\mathcal{D}_{m,l}^{\nu}(t_2)$, with

$$\mathcal{A}_{n,k}^{\mu} = \frac{1}{e_\mu(nx)} \frac{(nx)^k}{\gamma_\mu(k)} \frac{n^{k+2\mu\theta_k+\lambda+1}}{\Gamma(k+2\mu\theta_k+\lambda+1)} \tag{2.2}$$

and

$$\mathcal{B}_{m,l}^{\nu} = \frac{1}{e_\mu(my)} \frac{(my)^l}{\gamma_\nu(l)} \frac{m^{l+2\nu\theta_l+\lambda+1}}{\Gamma(l+2\nu\theta_l+\lambda+1)}, \tag{2.3}$$

$$\mathcal{C}_{n,k}^{\mu}(t_1) = t_1^{k+2\mu\theta_k+\lambda} e^{-nt_1} \ and \ \mathcal{D}_{m,l}^{\nu}(t_2) = t_2^{l+2\nu\theta_l+\lambda} e^{-mt_2}. \tag{2.4}$$

We collect the following lemma from [27].

Lemma 2.1. *For the operators $D_n(.;.,.)$ given by (1.2), the followings are obtained*

$$\begin{aligned}
D_n(e_0;x) &= 1,\\
D_n(e_1;x) &= x+\frac{\lambda+1}{n},\\
D_n(e_2;x) &= x^2+\left(4+2\lambda+2\mu\frac{e_\mu(-nx)}{e_\mu(nx)}\right)\frac{x}{n}+\frac{(\lambda+1)(\lambda+2)}{n^2},\\
D_n(e_3;x) &= x^3+\left(9+3\lambda-2\mu\frac{e_\mu(-nx)}{e_\mu(nx)}\right)\frac{x^2}{n}\\
&+ \left(18+3\lambda(\lambda+5)+4\mu^2+2\mu(8+3\lambda)\frac{e_\mu(-nx)}{e_\mu(nx)}\right)\frac{x}{n^2}\\
&+ \frac{(\lambda^3+6\lambda^2+11\lambda+6)}{n^3},\\
D_n(e_4;x) &= x^4+\left(16+4\lambda+4\mu\frac{e_\mu(-nx)}{e_\mu(nx)}\right)+o\left(\frac{1}{n}\right).
\end{aligned}$$

We have the following.

Lemma 2.2. *Let $D^{\mu,\nu}_{n,m}(.;.,.)$ be the operators defined by* (2.1). *Then we get*

$$D^{\mu,\nu}_{n,m}(g;x,y)=\mathcal{U}^{\mu}_{n,k}\left(\mathcal{V}^{\nu}_{m,l}(g;x,y)\right)=\mathcal{V}^{\nu}_{m,l}\left(\mathcal{U}^{\mu}_{n,k}(g;x,y)\right),$$

where

$$\mathcal{U}^{\mu}_{n,k}(g;x,y)=\sum_{k=0}^{\infty}\mathcal{A}^{\mu}_{n,k}(x)\int_0^{\infty}\mathcal{C}^{\mu}_{n,k}(t_1)g(t_1,t_2)dt_1, \tag{2.5}$$

$$\mathcal{V}^{\nu}_{m,l}(g;x,y)=\sum_{l=0}^{\infty}\mathcal{B}^{\nu}_{m,l}(y)\int_0^{\infty}\mathcal{D}^{\nu}_{m,l}(t_2)g(t_1,t_2)dt_2, \tag{2.6}$$

where $\mathcal{A}^{\mu}_{n,k}(x)$, $\mathcal{B}^{\nu}_{m,l}(y)$, $\mathcal{C}^{\mu}_{n,k}(t_1)$ and $\mathcal{D}^{\nu}_{m,l}(t_2)$ are given by eqn (2.2),(2.3) *and* (2.4).

Proof. We easily see that

$$\begin{aligned}
&\mathcal{U}^{\mu}_{n,k}\left(\mathcal{V}^{\nu}_{m,l}(g;x,y)\right)\\
&= \mathcal{U}^{\mu}_{n,k}\left(\sum_{l=0}^{\infty}\mathcal{B}^{\nu}_{m,l}(y)\int_0^{\infty}\mathcal{D}_{m,l}(t_2)g(t_1,t_2)dt_2\right)\\
&= \sum_{l=0}^{\infty}\mathcal{B}^{\nu}_{m,l}(y)\mathcal{U}^{\mu}_{n,k}\left(\int_0^{\infty}\mathcal{D}_{m,l}(t_2)g(t_1,t_2)dt_2\right)\\
&= \sum_{k=0}^{\infty}\sum_{k=0}^{\infty}\mathcal{P}^{\mu,\nu}_{n,m,k,l}(x,y)\int_0^{\infty}\int_0^{\infty}\mathcal{Q}^{\mu,\nu}_{n,m}(t_1,t_2)g(t_1,t_2)dt_1dt_2\\
&= D^{\mu;\nu}_{n,m}(g;x,y).
\end{aligned}$$

Similarly, it is proved that $\mathcal{V}^{\nu}_{m,l}\left(\mathcal{U}^{\mu}_{n,k}(g;x,y)\right)=D^{\mu,\nu}_{n,m}(g;x,y)$. ☐

To prove basic estimates and approximation results, we consider test functions and central moments as $e_{i,j}(t_1,t_2)=t_1^it_2^j,\ (i,j)\in\{0,1,2,3\}$ and $\psi^{i,j}_{x,y}(t_1,t_2)=(t_1-x)^i(t_2-y)^j,\ (i,j)\in\{1,2,3\}$ respectively.

Lemma 2.3. *For the operators $D_{n,m}^{\mu,\nu}(.;.,.)$ given by (2.1), the followings are obtained*

$$\begin{aligned}
D_{n,m}^{\mu,\nu}(e_{0,0};x,y) &= 1,\\
D_{n,m}^{\mu,\nu}(e_{1,0};x,y) &= x+\frac{\lambda+1}{n},\\
D_{n,m}^{\mu,\nu}(e_{0,1};x,y) &= y+\frac{\lambda+1}{m},\\
D_{n}^{\mu,\nu}(e_{2,0};x,y) &= x^2+\left(4+2\lambda+2\mu\frac{e_\mu(-nx)}{e_\mu(nx)}\right)\frac{x}{n}+\frac{(\lambda+1)(\lambda+2)}{n^2},\\
D_{n,m}^{\mu,\nu}(e_{0,2};x,y) &= y^2+\left(4+2\lambda+2\nu\frac{e_\nu(-my)}{e_\nu(my)}\right)\frac{y}{m}+\frac{(\lambda+1)(\lambda+2)}{m^2},\\
D_{n,m}^{\mu,\nu}(e_{3,0};x,y) &= =x^3+\left(9+3\lambda-2\mu\frac{e_\mu(-nx)}{e_\mu(nx)}\right)\frac{x^2}{n}\\
&+ \left(18+3\lambda(\lambda+5)+4\mu^2+2\mu(8+3\lambda)\frac{e_\mu(-nx)}{e_\mu(nx)}\right)\frac{x}{n^2}\\
&+ \frac{(\lambda^3+6\lambda^2+11\lambda+6)}{n^3},\\
D_{n,m}^{\mu,\nu}(e_{0,3};x,y) &= y^3+\left(9+3\lambda-2\nu\frac{e_\nu(-my)}{e_\nu(my)}\right)\frac{y^2}{m}\\
&+ \left(18+3\lambda(\lambda+5)+4\nu^2+2\nu(8+3\lambda)\frac{e_\nu(-my)}{e_\nu(my)}\right)\frac{y}{m^2}\\
&+ \frac{(\lambda^3+6\lambda^2+11\lambda+6)}{m^3}.
\end{aligned}$$

Proof. We use (1.2) and the property $\theta_{\mu+1}=(-1)^\mu+\theta_\mu$. For $i=j=0$, one has

$$\begin{aligned}
D_{n,m}^{\mu,\nu}(e_{0,0};x,y) &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\mathcal{P}_{n,m.k.l}^{\mu,\nu}(x,y)\int_0^\infty\int_0^\infty \mathcal{Q}_{k,l}^{\mu,\nu}(t_1,t_2)dt_1dt_2\\
&= \frac{1}{e_\mu(nx)}\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}\frac{\Gamma(k+2\mu\theta_k+\lambda+1)}{\Gamma(k+2\mu\theta_k+\lambda+1)}\frac{1}{e_\mu(my)}\\
&\quad \sum_{l=0}^{\infty}\frac{(my)^l}{\gamma_\nu(l)}\frac{\Gamma(l+2\nu\theta_l+\lambda+1}{\Gamma(l+2\nu\theta_l+\lambda+1)}\\
&= 1.
\end{aligned}$$

$$\begin{aligned}
D_{n,m}^{\mu,\nu}(e_{1,0};x,y) &= \frac{1}{e_\mu(nx)}\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}\frac{\Gamma(k+2\mu\theta_k+\lambda+2)}{\Gamma(k+2\mu\theta_k+\lambda+1)}\\
&\times \int_0^\infty t_1^{k+2\mu\theta_k+\lambda+1}e^{-nt_1}dt_1\\
&= \frac{1}{e_\mu(nx)}\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}\frac{\Gamma(k+2\mu\theta_k+\lambda+2)}{n\Gamma(k+2\mu\theta_k+\lambda+1)}\\
&= \frac{1}{ne_\mu(nx)}\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}(k+2\mu\theta_k+\lambda+1)
\end{aligned}$$

$$= x + \frac{\lambda+1}{n}.$$

$$\begin{aligned}
D^{\mu,\nu}_{n,m}(e_{0,1};x,y) &= \frac{1}{e_\nu(my)}\sum_{l=0}^{\infty}\frac{(my)^l}{\gamma_\nu(l)}\frac{\Gamma(l+2\nu\theta_l+\lambda+2)}{\Gamma(l+2\nu\theta_l+\lambda+1)}\\
&\times \int_0^\infty t_2^{l+2\nu\theta_l+\lambda+1}e^{-mt_2}dt_2\\
&= \frac{1}{e_\nu(my)}\sum_{l=0}^{\infty}\frac{(my)^l}{\gamma_\nu(l)}\frac{\Gamma(l+2\nu\theta_l+\lambda+2)}{m\Gamma(l+2\nu\theta_l+\lambda+1)}\\
&= \frac{1}{me_\nu(my)}\sum_{l=0}^{\infty}\frac{(my)^l}{\gamma_\nu(l)}(l+2\nu\theta_l+\lambda+1)\\
&= y+\frac{\lambda+1}{m}.
\end{aligned}$$

$$\begin{aligned}
D^{\mu,\nu}_{n,m}(e_{2,0};x,y) &= \frac{1}{e_\mu(nx)}\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}\frac{\Gamma(k+2\mu\theta_k+\lambda+2)}{\Gamma(k+2\mu\theta_k+\lambda+1)}\\
&\times \int_0^\infty t_1^{k+2\mu\theta_k+\lambda+2}e^{-nt_1}dt_1\\
&= \frac{1}{e_\mu(nx)}\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}\frac{\Gamma(k+2\mu\theta_k+\lambda+3)}{n^2\Gamma(k+2\mu\theta_k+\lambda+1)}\\
&= \frac{1}{e_\mu(nx)}\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}\frac{(k+2\mu\theta_k+\lambda+2)(k+2\mu\theta_k+\lambda+1)}{n^2}\\
&= \frac{1}{e_\mu(nx)}\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}(k+2\mu\theta_k)^2+(2\lambda+3)(k+2\mu\theta_k)
\end{aligned}$$

$$\begin{aligned}
&+ (\lambda+1)(\lambda+2)\\
&= x^2+\left(4+2\lambda+2\mu\frac{e_\mu(-nx)}{e_\mu(nx)}\right)\frac{x}{n}+\frac{(\lambda+1)(\lambda+2)}{n^2}.
\end{aligned}$$

$$\begin{aligned}
D^{\mu,\nu}_{n,m}(e_{0,2};x,y) &= \frac{1}{e_\nu(my)}\sum_{l=0}^{\infty}\frac{(my)^l}{\gamma_\nu(l)}\frac{\Gamma(l+2\nu\theta_l+\lambda+2)}{\Gamma(l+2\nu\theta_l+\lambda+1)}\\
&\times \int_0^\infty t_2^{l+2\nu\theta_l+\lambda+2}e^{-mt_2}dt_2\\
&= \frac{1}{e_\nu(my)}\sum_{l=0}^{\infty}\frac{(my)^l}{\gamma_\nu(l)}\frac{\Gamma(l+2\nu\theta_l+\lambda+3)}{m^2\Gamma(l+2\nu\theta_l+\lambda+1)}\\
&= \frac{1}{e_\nu(my)}\sum_{l=0}^{\infty}\frac{(my)^l}{\gamma_\nu(l)}\frac{(l+2\nu\theta_l+\lambda+2)(l+2\nu\theta_l+\lambda+1)}{m^2}\\
&= \frac{1}{e_\nu(my)}\sum_{l=0}^{\infty}\frac{(my)^l}{\gamma_\nu(l)}(l+2\nu\theta_l)^2+(2\lambda+3)(l+2\nu\theta_l)
\end{aligned}$$

$$\begin{aligned} &+ \quad (\lambda+1)(\lambda+2) \\ &= \quad y^2 + \left(4 + 2\lambda + 2\nu\frac{e_\nu(-my)}{e_\nu(my)}\right)\frac{y}{m} + \frac{(\lambda+1)(\lambda+2)}{m^2}. \end{aligned}$$

Other parts are proved likewise. □

Lemma 2.4. *Let $D^{\mu,\nu}_{n,m}(.;.,.)$ be defined by (2.1). Then we have*

$$\begin{aligned} D^{\mu,\nu}_{n,m}(\psi^{1,0}_{x,y};x,y) &= \frac{\lambda+1}{n}, \\ D^{\mu,\nu}_{n,m}(\psi^{0,1}_{x,y};x,y) &= \frac{\lambda+1}{m}, \\ D^{\mu,\nu}_{n,m}(\psi^{2,0};x,y) &= \left(2 + 2\mu\frac{e_\mu(-nx)}{e_\mu(nx)}\right)\frac{x}{n} + \frac{(\lambda+1)(\lambda+2)}{n^2}, \\ D^{\mu,\nu}_{n,m}(\psi^{0,2}_{x,y};x,y) &= \left(2 + 2\nu\frac{e_\nu(-my)}{e_\nu(my)}\right)\frac{y}{m} + \frac{(\lambda+1)(\lambda+2)}{m^2}, \\ D^{\mu,\nu}_{n,m}(\psi^{4,0}_{x,y};x,y) &= o\left(\frac{1}{n^2}\right), \\ D^{\mu,\nu}_{n,m}(\psi^{0,4}_{x,y};x,y) &= o\left(\frac{1}{m^2}\right). \end{aligned}$$

Proof. We have

$$\begin{aligned} D^{\mu,\nu}_{n,m}(\psi^{1,0}_{x,y};x,y) &= D^{\mu,\nu}_{n,m}(e_{1,0};x,y) - xD^{\mu,\nu}_{n,m}(e_{0,0};x,y), \\ &= x + \frac{\lambda+1}{n} - x \\ &= \frac{\lambda+1}{n}, \end{aligned}$$

$$\begin{aligned} D^{\mu,\nu}_{n,m}(\psi^{0,1}_{x,y};x,y) &= D^{\mu,\nu}_{n,m}(e_{0,1};x,y) - yD^{\mu,\nu}_{n,m}(e_{0,0};x,y), \\ &= y + \frac{\lambda+1}{m} - y \\ &= \frac{\lambda+1}{m}, \end{aligned}$$

$$\begin{aligned} &D^{\mu,\nu}_{n,m}(\psi^{2,0}_{x,y};x,y) \\ &= \quad D^{\mu,\nu}_{n,m}(e_{2,0};x,y) - 2xD^{\mu,\nu}_{n,m}(e_{1,0};x,y) + x^2 D^{\mu,\nu}_{n,m}(e_{0,0};x,y) \\ &= \quad x^2 + \left(4 + 2\lambda + 2\mu\frac{e_\mu(-nx)}{e_\mu(nx)}\right)\frac{x}{n} + \frac{(\lambda+1)(\lambda+2)}{n^2} \\ &- \quad 2x\left(x + \frac{\lambda+1}{n}\right) + x^2 \\ &= \quad 2 + 2\mu\frac{e_\mu(-nx)}{e_\mu(nx)}\Bigg)\frac{x}{n} + \frac{(\lambda+1)(\lambda+2)}{n^2}, \end{aligned}$$

$$\begin{aligned} &D^{\mu,\nu}_{n,m}(\psi^{0,2}_{x,y};x,y) \\ &= \quad D^{\mu,\nu}_{n,m}(e_{0,2};x,y) - 2yD^{\mu,\nu}_{n,m}(e_{0,1};x,y) + y^2 D^{\mu,\nu}_{n,m}(e_{0,0};x,y) \end{aligned}$$

$$\begin{aligned}
&= y^2+\left(4+2\lambda+2\nu\frac{e_\nu(-my)}{e_\nu(my)}\right)\frac{y}{m}+\frac{(\lambda+1)(\lambda+2)}{m^2}\\
&- 2y\left(y+\frac{\lambda+1}{m}\right)+y^2\\
&= 2+2\nu\frac{e_\nu(-my)}{e_\nu(my)}\Big)\frac{y}{m}+\frac{(\lambda+1)(\lambda+2)}{m^2}.
\end{aligned}$$

Similarly, the remaining parts of the Lemma 2.4 are proved. □

Lemma 2.5. *For the sequence $D^{\mu,\nu}_{n,m}(.;.,.)$ given by (2.1), we have*

$$\begin{aligned}
(1)\quad D^{\mu,\nu}_{n,m}(\psi^{2,0}_{x,y};x,y) &= o\left(\frac{1}{n}\right)(x+1)^2\leq C_1(x+1)^2 \text{ as } n,m\to\infty;\\
(2)\quad D^{\mu,\nu}_{n,m}(\psi^{0,2}_{x,y};x,y) &= o\left(\frac{1}{m}\right)(y+1)^2\leq C_2(y+1)^2 \text{ as } n,m\to\infty;\\
(3)\quad D^{\mu,\nu}_{n,m}(\psi^{4,0}_{x,y};x,y) &= o\left(\frac{1}{n^2}\right)(x+1)^4\leq C_3(x+1)^4 \text{ as } n,m\to\infty;\\
(4)\quad D^{\mu,\nu}_{n,m}(\psi^{0,4}_{x,y};x,y) &= o\left(\frac{1}{m^2}\right)(y+1)^4\leq C_4(y+1)^4 \text{ as } n,m\to\infty.
\end{aligned}$$

3. Order of approximation and rate of convergence

Let ϕ be the weight function $\phi(x,y)=1+x^2+y^2$ and $B_\phi\left(\mathcal{I}^2\right)=\{g:|g(x,y)|\leq C_g\phi(x,y),\ C_g>0\}$ be the space of all bounded functions defined on $\mathcal{I}^2$, endowed with norm $\parallel g \parallel_\phi=\sup_{x,y\in\mathcal{I}^2}\frac{|g(x,y)|}{\phi(x,y)}$. Suppose $C^{(m)}\left(\mathcal{I}^2\right)$ denotes the m-times continuously differentiable functions defined on $\mathcal{I}^2$. We have the following classes of functions:

$$C^m_\phi\left(\mathcal{I}^2\right)=\left\{g:g\in C_\phi\left(\mathcal{I}^2\right) \text{ such that } \lim_{(x,y)\to\infty}\frac{g(x,y)}{\phi(x,y)}=K_g<\infty\right\},$$

$$C^0_\phi\left(\mathcal{I}^2\right)=\left\{h:h\in C^m_\phi\left(\mathcal{I}^2\right) \text{ such that } \lim_{(x,y)\to\infty}\frac{g(x,y)}{\phi(x,y)}=0\right\},$$

$$C_\phi\left(\mathcal{I}^2\right)=\{g:g\in B_\phi\left(\mathcal{I}^2\right)\cap C_\phi\left(\mathcal{I}^2\right)\}.$$

Let $g\in C^0_\phi$ and $\delta_1,\delta_2>0$. Then the weighted modulus of smoothness is defined by

$$\omega_\phi(g;\delta_1,\delta_2)=\sup_{(x,y)\in[0,\infty)}\ \sup_{0\leq|\theta_1|\leq\delta_1,\ 0\leq|\theta_2|\leq\delta_2}\frac{|\ g(x+\theta_1,y+\theta_2)-g(x,y)\ |}{\phi(x,y)\phi(\theta_1,\theta_2)}. \tag{3.1}$$

For any $\eta_1,\eta_2>0$, one obtains

$$\omega_\phi(g;\eta_1\delta_1,\eta_2\delta_2)\leq 4(1+\eta_1)(1+\eta_2)(1+\delta_1^2)(1+\delta_2^2)\omega_\phi(g;\delta_1,\delta_2), \tag{3.2}$$

$$\begin{aligned}
|\ g(t,s)-g(x,y)\ | &\leq \phi(x,y)\phi\left(|\ t-x\ |,|\ s-y\ |\right)\omega_\phi\left(g;|\ t-x\ |,|\ s-y\ |\right)\\
&\leq (1+x^2+y^2)(1+(t-x)^2)(1+(s-y)^2)\\
&\quad\ \omega_\phi\left(g;|\ t-x\ |,|\ s-y\ |\right).
\end{aligned}$$

Theorem 3.1. *For the operators $D^{\mu,\nu}_{n,m}(.;.,.)$ defined by (2.1), we have*

$$\frac{\mid D^{\mu,\nu}_{n,m}(g;x,y)-g(x,y)\mid}{(1+x^2+y^2)} \leq \Psi_{x,y}\left(1+o\left(n^{-1}\right)\right)\left(1+o\left(m^{-1}\right)\right) \omega_\phi\left(g;o\left(n^{-\frac{1}{2}}\right),\left(m^{-\frac{1}{2}}\right)\right),$$

where $\Psi_{n,m}=\left(1+(x+1)+C_1(x+1)^2+\sqrt{C_3}(x+1)^3\right)\left(1+(y+1)+C_2(y+1)^2+\sqrt{C_4}(y+1)^3\right)$ *and* $C_1,\ C_2,\ C_3,\ C_4>0$ *and* $g\in C^0_\phi(I^2)$.

Proof. For all $\delta_n,\delta_m>0$ we have $\mid g(t_1,t_2)-g(x,y)\mid$

$$\begin{aligned}
&\leq 4(1+x^2+y^2)\left(1+(t_1-x)^2\right)\left(1+(s_2-y)^2\right)\\
&\times \left(1+\frac{\mid t_1-x\mid}{\delta_n}\right)\left(1+\frac{\mid s_1-y\mid}{\delta_m}\right)(1+\delta_n^2)(1+\delta_m^2)\omega_\phi\left(g;\delta_n,\delta_m\right)\\
&= 4(1+x^2+y^2)(1+\delta_n^2)(1+\delta_m^2)\\
&\times \left(1+\frac{\mid t_1-x\mid}{\delta_n}+(t_1-x)^2+\frac{1}{\delta_n}\mid t-x\mid(t_1-x)^2\right)\\
&\times \left(1+\frac{\mid t_2-y\mid}{\delta_m}+(t_2-y)^2+\frac{\mid t_2-y\mid}{\delta_m}(t_2-y)^2\right)\omega_\varphi\left(g;\delta_n,\delta_m\right).
\end{aligned}$$

On applying operators $D^{\mu,\nu}_{n,m}(.\ ;.\ ,\ .)$ on both sides of above, we shall obtain $\mid D^{\mu,\nu}_{n,m}(g;x,y)-g(x,y)\mid$

$$\begin{aligned}
&\leq D^{\mu,\nu}_{n,m}\left(\mid g(.,.)-g(x,y)\mid;x,y\right)4(1+x^2+y^2)\\
&\times D^{\mu,\nu}_{n,m}\left(1+\frac{\mid t_1-x\mid}{\delta_n}+(t_1-x)^2+\frac{1}{\delta_n}\mid t_1-x\mid(t-x)^2;x,y\right)\\
&\times D^{\mu,\nu}_{n,m}\left(1+\frac{\mid t_2-y\mid}{\delta_m}+(t_2-y)^2+\frac{\mid s_2-y\mid}{\delta_m}(s_2-y)^2;x,y\right)\\
&\times (1+\delta_n^2)(1+\delta_m^2)\omega_\phi\left(g;\delta_n,\delta_m\right)\\
&= 4(1+x^2+y^2)(1+\delta_n^2)(1+\delta_m^2)\omega_\phi\left(g;\delta_n,\delta_m\right)\\
&\times \left(1+\frac{1}{\delta_n}D^{\mu,\nu}_{n,m}(\mid t_1-x\mid;x,y)+D^{\mu,\nu}_{n,m}((t_1-x)^2;x,y)\right.\\
&+ \left.\frac{1}{\delta_n}D^{\mu,\nu}_{n,m}(\mid t_1-x\mid(t_1-x)^2;x,y\right)\\
&\times \left(1+\frac{1}{\delta_m}D^{\mu,\nu}_{n,m}(\mid t_2-y\mid;x,y)+D^{\mu,\nu}_{n,m}((t_2-y)^2;x,y)\right.\\
&+ \left.\frac{1}{\delta_m}D^{\mu,\nu}_{n,m}(\mid t_2-y\mid(t_2-y)^2;x,y\right),\\
&\leq 4(1+x^2+y^2)(1+\delta_n^2)(1+\delta_m^2)\omega_\phi\left(g;\delta_n,\delta_m\right)\\
&\times \left[1+\frac{1}{\delta_n}\sqrt{D^{\mu,\nu}_{n,m}((t_1-x)^2;x,y)}+D^{\mu,\nu}_{n,m}((t_1-x)^2;x,y)\right.\\
&+ \left.\frac{1}{\delta_n}\sqrt{D^{\mu,\nu}_{n,m}((t_1-x)^2;x,y)}\sqrt{D^{\mu,\nu}_{n,m}((t_1-x)^4;x,y)}\right]
\end{aligned}$$

$$\times \quad \left[1+\frac{1}{\delta_m}\sqrt{D^{\mu,\nu}_{n,m}((t_2-y)^2;x,y)}+D^{\mu,\nu}_{n,m}((t_2-y)^2;x,y)\right.$$
$$\left.+\quad \frac{1}{\delta_m}\sqrt{D^{\mu,\nu}_{n,m}((t_2-y)^2;x,y)}\sqrt{D^{\mu,\nu}_{n,m}((t_2-y)^4;x,y)}\right].$$

On applying Lemma 2.5 and choosing $\delta_n = o\left(n^{-\frac{1}{2}}\right)$ and $\delta_m = o\left(m^{-\frac{1}{2}}\right)$, the following is obtained $\mid D^{\mu,\nu}_{n,m}(g;x,y)-g(x,y)\mid$

$$\begin{aligned}
&\leq \quad 4(1+x^2+y^2)(1+\delta_n^2)(1+\delta_m^2)\omega_\phi\left(g;\delta_n,\delta_m\right)\\
&\times \quad \left[1+\frac{1}{\delta_n}\sqrt{o\left(\frac{1}{n}\right)(x+1)^2}+o\left(\frac{1}{n}\right)(x+1)^2\right.\\
&+ \quad \left.\frac{1}{\delta_n}\sqrt{o\left(\frac{1}{n}\right)(x+1)^2}\sqrt{o\left(\frac{1}{n}\right)(x+1)^4}\right]\\
&\times \quad \left[1+\frac{1}{\delta_m}\sqrt{o\left(\frac{1}{m}\right)(y+1)^2}+o\left(\frac{1}{m}\right)(y+1)^2\right.\\
&+ \quad \left.\frac{1}{\delta_m}\sqrt{o\left(\frac{1}{m}\right)(y+1)^2}\sqrt{o\left(\frac{1}{m}\right)(y+1)^4}\right]\\
&\leq \quad 4(1+x^2+y^2)(1+\delta_n^2)(1+\delta_m^2)\omega_\varphi\left(g;\delta_n,\delta_m\right)\\
&\times \quad \left[1+(x+1)+C_1(x+1)^2+\sqrt{C_2}(x+1)^3\right]\\
&\qquad \left[1+(y+1)+C_3(y+1)^2+\sqrt{C_4}(y+1)^3\right],
\end{aligned}$$

which completes the proof of the theorem. □

We recollect the following.

Lemma 3.2. [10, 11] *For the sequence of operators $D_{n,m}(.\,;\ .\,,\ .)$, acting $C_\phi \to B_\phi$ defined earlier, there exists some positive constant K such that*

$$\parallel D_{n,m}(\phi;x,y)\parallel_\phi \leq K.$$

Theorem 3.3. [10, 11] *For the positive sequence of operators $D_{n,m}(.\,;\ .\,,\ .)$, acting $C_\phi \to B_\phi$ defined earlier satisfying the following conditions*

$$\begin{aligned}
(1)\quad \lim_{n,m\to\infty}\parallel D_{n,m}(1;x,y)-1\parallel_\phi &= 0;\\
(2)\quad \lim_{n,m\to\infty}\parallel D_{n,m}(t_1;x,y)-x\parallel_\varphi &= 0;\\
(3)\quad \lim_{n,m\to\infty}\parallel D_{n,m}(t_2;x,y)-y\parallel_\varphi &= 0;\\
(4)\quad \lim_{n,m\to\infty}\parallel D_{n,m}((t_1^2+t_2^2);x,y)-(x^2+y^2)\parallel_\varphi &= 0.
\end{aligned}$$

Then for all $g \in C^0_\phi$,

$$\lim_{n,m\to\infty}\parallel D_{n,m}(g;\ .\,,\ .)-g\parallel_\phi = 0,$$

and there exists another function $h \in C_\phi \setminus C_\phi^0$, such that

$$\lim_{n,m\to\infty} \parallel D_{n,m}(h;\ .\ ,\ .) - h \parallel_\phi \geq 1.$$

Theorem 3.4. *If $g \in C_\phi^0\left(\mathcal{I}^2\right)$, then we have*

$$\lim_{n,m\to\infty} \parallel D_{n,m}^{\mu,\nu}(g) - g \parallel_\phi = 0.$$

Proof. We compute

$$\begin{aligned}
&\parallel D_{n,m}^{\mu,\nu}(\phi;x,y) \parallel_\phi \\
&= \sup_{(x,y)\in\mathcal{I}^2} \frac{\mid D_{n,m}^{\mu,\nu}(1+x^2+y^2;x,y) \mid}{1+x^2+y^2} \\
&= 1 + \sup_{(x,y)\in\mathcal{I}^2} \left[\frac{1}{1+x^2+y^2}\left|\left(D_{n,m}^{\mu,\nu}(x^2;x,y) + D_{n,m}^{\mu,\nu}(y^2;x,y)\right)\right|\right] \\
&= 1 + \sup_{(x,y)\in\mathcal{I}^2} \frac{x^2}{1+x^2+y^2} + \left|\frac{1}{n}\left(4+2\lambda+2\mu\frac{e_\mu(-nx)}{e_\mu(nx)}\right)\right| \\
&\quad \sup_{(x,y)\in\mathcal{I}^2} \frac{x}{1+x^2+y^2} \\
&+ \left|\frac{(\lambda+1)(\lambda+2)}{n^2}\right| + \sup_{(x,y)\in\mathcal{I}^2} \frac{y^2}{1+x^2+y^2} + \left|\frac{(\lambda+1)(\lambda+2)}{m^2}\right| \\
&+ \left|\frac{1}{m}\left(4+2\lambda+2\nu\frac{e_\nu(-my)}{e_\nu(my)}\right)\right| \sup_{(x,y)\in\mathcal{I}^2} \frac{y}{1+x^2+y^2} \\
&\leq 1 + \left|\frac{4+2\lambda+2\mu}{n}\right| + \left|\frac{4+2\lambda+2\nu}{m}\right| + \left|\frac{(\lambda+1)(\lambda+2)}{n^2}\right| \\
&+ \left|\frac{(\lambda+1)(\lambda+2)}{m^2}\right|.
\end{aligned}$$

Now for n, m sufficiently large values, there exists a positive constant K such that

$$\parallel D_{n,m}^{\mu,\nu}(\phi;x,y) \parallel_\phi \leq K.$$

In order to prove Theorem 3.4, it is sufficient to show $\lim_{n,m\to\infty} \parallel D_{n,m}^{\mu,\nu}(e_{i,j}) - e_{i,j} \parallel = 0$ for all $e_{i,j} \in \{(0,0),(0,1),(1,0),(2,0),(0,2)\}$. In the light of Lemmas 2.3, 3.2 and Theorem 3.3, the proof follows. □

We define the following.
For $g \in C(\mathcal{I}^2)$ and $\delta, \delta_n, \delta_m > 0$, the modulus of continuity of second-order is defined by

$$\omega\left(g;\delta_n,\delta_m\right) = \sup\{\mid g(t_1,t_2) - g(x,y) \mid : (t_1,t_2),(x,y) \in \mathcal{I}^2\}$$

with $\mid t_1 - x \mid \leq \delta_n$, $\mid t_2 - y \mid \leq \delta_m$ with the partial moduli of continuity defined as

$$\omega_1\left(g;\delta\right) = \sup_{0\leq y\leq 1}\ \sup_{|x_1-x_2|\leq\delta} \{\mid g(x_1,y) - g(x_2,y) \mid\},$$

$$\omega_2\left(g;\delta\right) = \sup_{0\leq x\leq 1}\ \sup_{|y_1-y_2|\leq\delta} \{\mid g(x,y_1) - g(x,y_2) \mid\}.$$

Theorem 3.5. *For any $g \in C(\mathcal{I}^2)$, we have*

$$\mid D_{n,m}^{\mu,\nu}(g;x,y) - g(x,y) \mid \leq 2\big(\omega_1(g;\delta_{x,n}) + \omega_2(g;\delta_{y,m})\big).$$

Proof. Making use of Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&| D^{\mu,\nu}_{n,m}(g;x,y) - g(x,y) | \\
&\leq D^{\mu,\nu}_{n,m}\left(| g(t_1,t_2) - g(x,y) |; x,y\right) \\
&\leq D^{\mu,\nu}_{n,m}\left(| g(t_1,t_2) - g(x,t_2) |; x,y\right) \\
&+ D^{\mu,\nu}_{n,m}\left(| g(x,t_1) - g(x,y) |; x,y\right) \\
&\leq D^{\mu,\nu}_{n,m}\left(\omega_1(g; | t_1 - x |); x,y\right) \\
&+ D^{\mu,\nu}_{n,m}\left(\omega_2(g; | t_2 - y |); x,y\right) \\
&\leq \omega_1(g;\delta_n)\left(1 + \delta_n^{-1} L^{\mu,\nu}_{n,m}(| t_1 - x |; x,y)\right) \\
&+ \omega_2(g;\delta_m)\left(1 + \delta_m^{-1} D^{\mu,\nu}_{n,m}(| t_2 - y |; x,y)\right) \\
&\leq \omega_1(g;\delta_n)\left(1 + \frac{1}{\delta_n}\sqrt{D^{\mu,\nu}_{n,m}((t_1-x)^2; x,y)}\right) \\
&+ \omega_2(g;\delta_m)\left(1 + \frac{1}{\delta_m}\sqrt{D^{\mu,\nu}_{n,m}((t_2-y)^2; x,y)}\right).
\end{aligned}$$

Choosing $\delta_n^2 = \delta_{n,x}^2 = D^{\mu,\nu}_{n,m}((t_1-x)^2;x,y)$ and $\delta_m^2 = \delta_{m,y}^2 = D^{\mu,\nu}_{n,m}((t_2-y)^2;x,y)$, we arrive at the desired result. □

Next, we investigate convergence in terms of the Lipschitz class for bivariate functions. For $M > 0$ and $\rho_1, \rho_2 \in [0,1]$, the Lipschitz maximal function space on $E \times E \subset \mathcal{I}^2$ is defined by

$$\begin{aligned}
\mathcal{L}_{\rho_1,\rho_2}(E \times E) &= \Big\{ g : \sup(1+t_1)^{\rho_1}(1+t_2)^{\rho_2}\left(g_{\rho_1,\rho_2}(t_1,t_2) - g_{\rho_1,\rho_2}(x,y)\right) \\
&\leq M \frac{1}{(1+x)^{\rho_1}} \frac{1}{(1+y)^{\rho_2}} \Big\},
\end{aligned}$$

where g is continuous and bounded on $\mathcal{I}^2$, and

$$| g_{\rho_1,\rho_2}(t_1,t_2) - g_{\rho_1,\rho_2}(x,y) |= \frac{| g(t_1,t_2) - g(x,y) |}{| t_1 - x |^{\rho_1} | t_2 - y |^{\rho_2}}; \quad (t_1,t_2),(x,y) \in \mathcal{I}^2. \tag{3.3}$$

Theorem 3.6. *Let $g \in \mathcal{L}_{\rho_1,\rho_2}(E \times E)$. Then for any $\rho_1, \rho_2 \in [0,1]$ there exists $M > 0$ such that*

$$\begin{aligned}
| D^{\mu,\nu}_{n,m}(g;x,y) - g(x,y) | &\leq M\Big\{ \left((d(x,E))^{\rho_1} + \left(\delta_{n,x}^2\right)^{\frac{\rho_1}{2}} \right)\Big((d(y,E))^{\rho_2} \\
&+ \left(\delta_{m,y}^2\right)^{\frac{\rho_2}{2}} \Big) + (d(x,E))^{\rho_1}(d(y,E))^{\rho_2} \Big\},
\end{aligned}$$

where $\delta_{n,x}$ and $\delta_{m,y}$ are given as in the proof of Theorem 3.5.

Proof. Take $| x - x_0 |= d(x,E)$ and $| y - y_0 |= d(y,E)$. For any $(x,y) \in \mathcal{I}^2$, and $(x_0,y_0) \in E \times E$ we let $d(x,E) = \inf\{| x - y |: y \in E\}$. We can write

$$| g(t_1,t_2) - g(x,y) |\leq M \,| g(t_1,t_2) - g(x_0,y_0) | + | g(x_0,y_0) - g(x,y) | \tag{3.4}$$

Applying the operators $D^{\mu,\nu}_{n,m}$, we will get

$$\begin{aligned}
&| D^{\mu,\nu}_{n,m}(g;x,y) - g(x,y) | \\
&\leq D^{\mu,\nu}_{n,m}\left(| g(x,y) - g(x_0,y_0) | + | g(x_0,y_0) - g(x,y) |\right) \\
&\leq M D^{\mu,\nu}_{n,m}\left(| t_1 - x_0 |^{\rho_1} | t_2 - y_0 |^{\rho_2}; x,y\right) \\
&+ M \,| x - x_0 |^{\rho_1} | y - y_0 |^{\rho_2}.
\end{aligned}$$

For $A, B \geq 0$ and $\rho \in [0,1]$, using the inequality $(A+B)^\rho \leq A^\rho + B^\rho$, we have

$$| t_1 - x_0 |^{\rho_1} \leq | t_1 - x |^{\rho_1} + | x - x_0 |^{\rho_1},$$
$$| t_2 - y_0 |^{\rho_1} \leq | t_2 - y |^{\rho_2} + | y - y_0 |^{\rho_2} .$$

Therefore,

$$\begin{aligned} &| D_{n,m}^{\mu,\nu}(g; x, y) - g(x, y) | \\ &\leq M D_{n,m}^{\mu,\nu}\left(| t_1 - x |^{\rho_1} | t_2 - y |^{\rho_2}; x, y\right) \\ &+ M \, | x - x_0 |^{\rho_1} \, D_{n,m}^{\mu,\nu}\left(| t_2 - y |^{\rho_2}; x, y\right) \\ &+ M \, | y - y_0 |^{\rho_2} \, D_{n,m}^{\mu,\nu}\left(| t_1 - x |^{\rho_1}; x, y\right) \\ &+ 2M \, | x - x_0 |^{\rho_1} | y - y_0 |^{\rho_2} \, D_{n,m}^{\mu,\nu}\left(\alpha_{0,0}; x, y\right). \end{aligned}$$

On applying the Hölder's inequality, we get

$$\begin{aligned} &D_{n,m}^{\mu,\nu}\left(| t_1 - x |^{\rho_1} | t_2 - y |^{\rho_2}; x, y\right) \\ &= \mathcal{U}_{n,k}^{\mu}\left(| t_1 - x |^{\rho_1}; x, y\right) \mathcal{V}_{m,l}^{\nu}\left(| t_2 - y |^{\rho_2}; x, y\right) \\ &\leq \left(D_{n,m}^{\mu,\nu}(| t_1 - x |^2; x, y)\right)^{\frac{\rho_1}{2}} \left(D_{n,m}^{\mu,\nu}(\alpha_{0,0}; x, y)\right)^{\frac{2-\rho_1}{2}} \\ &\times \left(D_{n,m}^{\mu,\nu}(| t_2 - y |^2; x, y)\right)^{\frac{\rho_2}{2}} \left(D_{n,m}^{\mu,\nu}(\alpha_{0,0}; x, y)\right)^{\frac{2-\rho_2}{2}}, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} &| D_{n,m}^{\mu,\nu}(g; x, y) - g(x, y) | \\ &\leq M \left(\delta_{n,x}^2\right)^{\frac{\rho_1}{2}} \left(\delta_{m,y}^2\right)^{\frac{\rho_2}{2}} + 2M \left(d(x, E)\right)^{\rho_1} \left(d(y, E)\right)^{\rho_2} \\ &+ M \left(d(x, E)\right)^{\rho_1} \left(\delta_{m,y}^2\right)^{\frac{\rho_2}{2}} + L \left(d(y, E)\right)^{\rho_2} \left(\delta_{n,x}^2\right)^{\frac{\rho_1}{2}}, \end{aligned}$$

and this completes the proof of the theorem. □

Theorem 3.7. *If $g \in C'(\mathcal{I}^2)$, then for all $(x, y) \in \mathcal{I}^2$, the operators $D_{n,m}^{\mu,\nu}(.;.,.)$ satisfy*

$$| D_{n,m}^{\mu,\nu}(g; x, y) - g(x, y) | \leq \| g_x \|_{C(\mathcal{I}^2)} \left(\delta_{n,x}^2\right)^{\frac{1}{2}} + \| g_y \|_{C(\mathcal{I}^2)} \left(\delta_{m,y}^2\right)^{\frac{1}{2}},$$

where $\delta_{n,x}$ and $\delta_{m,y}$ are given by expressions in the proof of Theorem 3.5.

Proof. Let $g \in C'(\mathcal{I}^2)$. For fixed $(x, y) \in \mathcal{I}^2$, we have

$$g(t_1, t_2) - g(x, y) = \int_x^{t_1} g_r(r, t_2) dr + \int_y^{t_2} g_s(x, s) ds.$$

Application of $D_{n,m}^{\mu,\nu}$ on both sides gives

$$\begin{aligned} &D_{n,m}^{\mu,\nu}\left(g(t_1, t_2); x, y\right) - g(x, y) \\ &= D_{n,m}^{\mu,\nu}\left(\int_x^{t_1} g_r(r, t_2) dr; x, y\right) + D_{n,m}^{\mu,\nu}\left(\int_y^{t_2} g_s(x, s) ds; x, y\right). \end{aligned} \quad (3.5)$$

By the sup-norm on $\mathcal{I}^2$ one can write

$$\left| \int_x^{t_1} g_r(r, t_2) dr \right| \leq \int_x^{t_1} | g_r(r, t_2) dr | \leq \| g_x \|_{C(\mathcal{I}^2)} | t_1 - x | \quad (3.6)$$

and

$$\left| \int_y^{t_2} g_s(x, s) ds \right| \leq \int_y^{t_2} | g_\mu(x, s) ds | \leq \| g_y \|_{C(\mathcal{I}^2)} | t_2 - y | \quad (3.7)$$

Utilizing (3.5), (3.6) and (3.7), we obtain

$$\begin{aligned} \mid D^{\mu,\nu}_{n,m}\left(g(x,y);x,y\right)-g(x,y)\mid &\leq D^{\mu,\nu}_{n,m}\left(\left|\int_x^{t_1} g_r(r,t_2)dr\right|;x,y\right)\\ &+ D^{\mu,\nu}_{n,m}\left(\left|\int_y^{t_2} g_s(x,s)ds\right|;x,y\right)\\ &\leq \parallel g_x\parallel_{C(\mathcal{I}^2)} D^{\mu,\nu}_{n,m}\left(\mid t_1-x\mid;x,y\right)\\ &+ \parallel g_y\parallel_{C(\mathcal{I}^2)} D^{\mu,\nu}_{n,m}\left(\mid t_2-y\mid;x,y\right)\\ &\leq \parallel g_x\parallel_{C(\mathcal{I}^2)} \left(D^{\mu,\nu}_{n,m}((t_1-x)^2;x,y)D^{\mu,\nu}_{n,m}(1;x,y)\right)^{\frac{1}{2}}\\ &+ \parallel g_y\parallel_{C(\mathcal{I}^2)} \left(D^{\mu,\nu}_{n,m}((t_2-y)^2;x,y)D^{\mu,\nu}_{n,m}(1;x,y)\right)^{\frac{1}{2}}\\ &= \parallel g_x\parallel_{C(\mathcal{I}^2)}\left(\delta^2_{n,x}\right)^{\frac{1}{2}}+\parallel g_y\parallel_{C(\mathcal{I}^2)}\left(\delta^2_{m,y}\right)^{\frac{1}{2}}, \end{aligned}$$

and the proof is completed. □

Theorem 3.8. *For any $g \in C(\mathcal{I}^2)$, if we define an auxiliary operator such that*

$$T^{\mu,\nu}_{n,m}(g;x,y) = D^{\mu,\nu}_{n,m}(g;x,y)+g(x,y)-g\Bigg(\mathcal{U}^{\mu}_{n,k}(e_{1,0};x,y),\mathcal{V}^{\nu}_{m,l}(e_{0,1};x,y)\Bigg),$$

where, from eqns (2.5), (2.6) *and Lemma 2.3, $\mathcal{U}^{\mu}_{n,k}(e_{1,0};x,y)=D^{\mu,\nu}_{n,m}(e_{1,0};x,y)=x+\frac{\lambda+1}{n}$ and $\mathcal{V}^{\nu}_{m,l}(e_{0,1};x,y)=D^{\mu,\nu}_{n,m}(e_{0,1};x,y)=y+\frac{\lambda+1}{m}$. Then for all $g\in C'(\mathcal{I}^2)$, operators $T^{\mu,\nu}_{n,m}(.;.,.)$ satisfy $\left|T^{\mu,\nu}_{n,m}(g;t_1,t_2)-g(x,y)\right|$*

$$\leq \left\{\delta^2_{n,x}+\delta^2_{m,y}+\left(\frac{\lambda+1}{n}\right)^2+\left(\frac{\lambda+1}{m}\right)^2\right\}\parallel g\parallel_{C^2(\mathcal{I}^2)}.$$

Proof. In the view of operators $T^{\mu,\nu}_{n,m}(.;.,.)$ and Lemma 2.3, we obtain $T^{\mu,\nu}_{n,m}(1;x,y)=1$, $T^{\mu,\nu}_{n,m}(t_1-x;x,y)=0$ and $T^{\mu,\nu}_{n,m}(t_2-y;x,y)=0$. For any $g\in C'(\mathcal{I}^2)$, the Taylor's series gives us

$$\begin{aligned} g(t_1,t_2)-g(x,y) &= \frac{\partial g(x,y)}{\partial x}(t_1-x)+\int_{u_1}^{t}(t_1-\alpha)\frac{\partial^2 g(\lambda,y)}{\partial\alpha^2}\mathrm{d}\alpha\\ &+ \frac{\partial g(x,y)}{\partial t_2}(t_2-y)+\int_y^{t_2}(t_2-\phi)\frac{\partial^2 g(x,\phi)}{\partial\phi^2}\mathrm{d}\phi. \end{aligned}$$

On applying $T^{\mu,\nu}_{n,m}$, we see that
$T^{\mu,\nu}_{n,m}\left(g(t_1,t_2);x,y\right)-T^{\mu,\nu}_{n,m}\left(g(x,y)\right.$

$$\begin{aligned} &= T^{\mu,\nu}_{n,m}\left(\int_x^{t_1}(t_1-\alpha)\frac{\partial^2 g\alpha,y)}{\partial\alpha^2}\mathrm{d}\alpha;x,y\right)\\ &+ T^{\mu,\nu}_{n,m}\left(\int_y^{t_2}(t_2-\phi)\frac{\partial^2 g(x,\phi)}{\partial\phi^2}\mathrm{d}\phi;x,y\right)\\ &= D^{\mu,\nu}_{n,m}\left(\int_x^{t_1}(t_1-\alpha)\frac{\partial^2 g(\alpha,y)}{\partial\alpha^2}\mathrm{d}\alpha;x,y\right) \end{aligned}$$

$$
\begin{aligned}
&+ \quad D_{n,m}^{\mu,\nu}\left(\int_y^{t_2}(t_2-\phi)\frac{\partial^2 g(x,\phi)}{\partial\phi^2}\mathrm{d}\phi; x,y\right)\\
&- \quad \int_x^{x+\frac{\lambda+1}{n}}\left(x+\frac{1+\lambda}{n}-\alpha\right)\frac{\partial^2 g(\lambda,y)}{\partial\alpha^2}\mathrm{d}\alpha\\
&- \quad \int_y^{y+\frac{\lambda+1}{m}}\left(y+\frac{1+\lambda}{m}-\phi\right)\frac{\partial^2 g(x,\phi)}{\partial\phi^2}\mathrm{d}\phi.
\end{aligned}
$$

From hypothesis we easily obtain

$$
\left|\int_x^{t_1}(t_1-\alpha)\frac{\partial^2 g(\alpha,y)}{\partial\alpha^2}\mathrm{d}\alpha\right| \le \int_x^{t}\left|(t_1-\alpha)\frac{\partial^2 g(\alpha,y)}{\partial\alpha^2}\mathrm{d}\alpha\right| \le \| g \|_{C^2(\mathcal{I}^2)} (t_1-x)^2,
$$

$$
\left|\int_y^{t_2}(t_2-\phi)\frac{\partial^2 g(x,\phi)}{\partial\phi^2}\mathrm{d}\phi\right| \le \int_y^{t_2}\left|(t_2-\phi)\frac{\partial^2 g(x,\phi)}{\partial\phi^2}\mathrm{d}\phi\right| \le \| g \|_{C^2(\mathcal{I}^2)} (t_2-y)^2,
$$

$$
\left|\int_x^{x+\frac{\lambda+1}{n}}\left(x+\frac{\lambda+1}{n}-\alpha\right)\frac{\partial^2 g(\alpha,y)}{\partial\alpha^2}\mathrm{d}\alpha\right| \le \| g \|_{C^2(\mathcal{I}^2)} \left(\frac{\lambda+1}{n}\right)^2
$$

$$
\left|\int_y^{y+\frac{\lambda+1}{m}}\left(y+\frac{\lambda+1}{m}-\phi\right)\frac{\partial^2 g(\phi,x)}{\partial\phi^2}\mathrm{d}\phi\right| \le \| g \|_{C^2(\mathcal{I}^2)} \left(\frac{\lambda+1}{m}\right)^2.
$$

Thus, we obtain $| T_{n,m}^{\mu,\nu}(g;t_1,t_2)-g(x,y) |$

$$
\begin{aligned}
&\le \quad \Bigg\{D_{n,m}^{\mu,\nu}((t_1-x)^2;x,y)+D_{n,m}^{\mu,\nu}((t_2-y)^2;x,y)\\
&+ \quad \left(\frac{\lambda+1}{n}\right)^2+\left(\frac{\lambda+1}{m}\right)^2\Bigg\} \| g \|_{C^2(\mathcal{I}^2)},
\end{aligned}
$$

Choosing $\delta_n^2=\delta_{n,x}^2=D_{n,m}^{\mu,\nu}((t_1-x)^2;x,y)$ and $\delta_m^2=\delta_{m,y}^2=D_{n,m}^{\mu,\nu}((t_2-y)^2;x,y)$, we arrive at the desired result. □

4. Approximation in Bögel space

We recall some definitions and notations from [24]. Consider a function $g:\mathcal{I}_1\times\mathcal{I}_2\to\mathbb{R}$ and suppose for all $(t_1,t_2),\ (x,y)\in\mathcal{I}_1\times\mathcal{I}_2$ and $\Delta^*_{(t_1,t_2)}g(x,y)$ denotes the bivariate mixed difference operator defined as follows:

$$
\Delta^*_{(t_1,t_2)}g(x,y)=g(t_1,t_2)-g(t_1,y)-g(x,t_2)+g(x,y).
$$

We observe that at any point $(x,y)\in\mathcal{I}_1\times\mathcal{I}_2$, $\lim_{(t_1,t_2)\to(x,y)}\Delta^*_{(t_1,t_2)}g(x,y)=0$. The set of all continuous functions and B-continuous functions defined on I^2 will be denoted by $C(\mathcal{I}^2)$ and $C_\varphi(\mathcal{I}^2)$ respectively.
For $X=(t_1,t_2),\ Y=(x,y)\in\mathcal{I}^2$, the Lipschitz class of functions in terms of B-continuous functions is defined by

$$
Lip_M^\xi=\left\{g\in C(\mathcal{I}^2)/\Delta^*_{(x,y)}g(X,Y)\ |\le M \parallel X-Y\parallel^\xi\right\},
$$

where M is a positive constant and $0<\xi\le 1$, and the Euclidean norm by $\parallel X-Y\parallel=\sqrt{(t_1-x)^2+(t_2-y)^2}$.
For more details on space of Bögel functions one is referred to [4, 5].

To investigate more of operators in (2.1), we define the following sequence of operators:

$$K_{n,m}^{\mu,\nu}\left(g(t_1,t_2);x,y\right) = D_{n,m}^{\mu,\nu}\Big(g(t_1,y)+g(x,t_2)-g(t_1,t_2);x,y\Big) \quad (4.1)$$

$$=\sum_{k=0}^{\infty}\sum_{k=0}^{\infty}\mathcal{P}_{n,m,k,l}^{\mu,\nu}(x,y)\int_0^{\infty}\int_0^{\infty}\mathcal{Q}_{n,m}(t_1,t_2)P_{x,y}(t_1,t_2)g(t_1,t_2)dt_1dt_2,$$

where $P_{x,y}(t_1,t_2)=\big(g(t_1,y)+g(x,t_2)-g(t_1,t_2)\big)$.
We prove the following theorems.

Theorem 4.1. *For all $g\in C_\varphi(\mathcal{I}^2)$, the following estimate is obtained*

$$\mid K_{n,m}^{\mu,\nu}\left(g(t_1,t_2);x,y\right)-g(x,y)\mid\leq 4\omega_B\left(g;\delta_{n,x},\delta_{m,y}\right),$$

where $\delta_{n,x}$ and $\delta_{m,y}$ are given as in the proof of Theorem 3.5.

Proof. Let $(t_1,t_2),(x,y)\in\mathcal{I}^2$. For all $n,m\in\mathbb{N}$ and $\delta_n,\delta_m>0$, one gets

$$\begin{aligned}\mid\Delta_{(x,y)}^{*}g(t_1,t_2)\mid &\leq \omega_B\left(g;\mid t_1-x\mid\mid t_2-y\mid\right)\\ &\leq \left(1+\frac{t_1-x}{\delta_n}\right)\left(1+\frac{t_2-y}{\delta_m}\right)\omega_B\left(g;\delta_n,\delta_m\right).\end{aligned}$$

Making use of Cauchy-Schwarz inequality, from (4.1), we get
$\mid K_{n,m}^{\mu,\nu}\left(g(t_1,t_2);x,y\right)-g(x,y)\mid$

$$\begin{aligned}&\leq D_{n,m}^{\mu,\nu}\left(\mid\Delta_{(x,y)}^{*}g(t_1,t_2s)\mid;x,y\right)\\ &\leq \Bigg(D_{n,m}^{\mu,\nu}(\psi_{0,0};x,y)+\frac{1}{\delta_n}\left(D_{n,m}^{\mu,\nu}((t_1-x)^2;x,y)\right)^{\frac12}\\ &+\frac{1}{\delta_m}\left(D_{n,m}^{\mu,\nu}((t_2-y)^2;x,y)\right)^{\frac12}\\ &+\frac{1}{\delta_n}\left(D_{n,m}^{\mu,\nu}((t_1-x)^2;x,y)\right)^{\frac12}\\ &\times\frac{1}{\delta_m}\left(D_{n,m}^{\mu,\nu}((t_2-y)^2;x,y)\right)^{\frac12}\Bigg)\omega_B\left(g;\delta_n,\delta_m\right),\end{aligned}$$

and then in view of Theorem 3.5 the result is obtained. □

Theorem 4.2. *For all $g\in Lip_M^{\xi}$, the operators $K_{n,m}^{\mu,\nu}$ satisfy*

$$\mid K_{n,m}^{\mu,\nu}\left(g(X,Y);x,y\right)-g(x,y)\mid\leq M\{\delta_{n,x}^2+\delta_{m,y}^2\}^{\frac{\xi}{2}},$$

with $\delta_{n,x}$ and $\delta_{m,y}$ defined as before.

Proof. We write

$$\begin{aligned}K_{n,m}^{\mu,\nu}\left(g(X,Y);x,y\right) &= D_{n,m}^{\mu,\nu}\left(g(x,Y)+g(X,y)-g(X,t_2);x,y\right)\\ &= D_{n,m}^{\mu,\nu}\left(g(x,y)-\Delta_{(x,y)}^{*}g(X,t_2);x,y\right)\\ &= g(x,y)-D_{n,m}^{\mu,\nu}\left(\Delta_{(x,y)}^{*}g(X,t_2);x,y\right).\end{aligned}$$

Then,

$$\begin{aligned} \mid K^{\mu,\nu}_{n,m}\left(g((X,Y);x,y)-g(x,y)\right| &\leq D^{\mu,\nu}_{n,m}\left(\mid \Delta^{*}_{(x,y)}g(X,Y)\mid ;x,y\right) \\ &\leq MD^{\mu,\nu}_{n,m}\left(\parallel X-Y\parallel^{\xi};x,y\right) \\ &\leq M\{D^{\mu,\nu}_{n,m}\left(\parallel X-Y\parallel^{2};x,y\right)\}^{\frac{\xi}{2}} \\ &\leq M\{D^{\mu,\nu}_{n,m}\left((t_1-x)^2;x,y\right) \\ &+ D^{\mu,\nu}_{n,m}\left((t_2-y)^2;x,y\right)\}^{\frac{\xi}{2}}, \end{aligned}$$

and the theorem is proved. □

5. Conclusions

In this manuscript we introduced a bivariate Szász-Gamma hybrid operators and studied their approximation properties using modulus of continuity in the space of Lipschitz-maximal type of functions and through the Peetre's K-functional. Error estimates are obtained in Bögel function spaces.

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