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INEQUALITIES RELATED TO BEREZIN NORM AND BEREZIN NUMBER OF OPERATORS

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Abstract. The Berezin symbol $\widetilde{A}$ of an operator A on the reproducing kernel Hilbert space $\mathcal{H}(\Omega)$ over some set Ω with the reproducing kernel k_λ is defined by

$$\tilde{A}(\lambda) = \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle, \ \lambda \in \Omega.$$

In this paper, we obtain some new inequalities for Berezin symbols of operators on reproducing kernel Hilbert spaces by using classical Hermite-Hadamard inequality and convex functions. Some other related questions are also discussed.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ be the Banach algebra of all bounded linear operators defined on a complex Hilbert space $(\mathcal{H}; \langle \cdot, \cdot \rangle)$ with the identity operator $1_{\mathcal{H}}$ in $\mathcal{B}(\mathcal{H})$. A bounded linear operator A defined on $\mathcal{H}$ is self-adjoint if and only if $\langle Ax, x\rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$.

A functional Hilbert space is the Hilbert space of complex-valued functions on some set $\Omega \subset \mathbb{C}$ that the evaluation functionals $\varphi_\lambda(f) = f(\lambda)$, $\lambda \in \Omega$ are continuous on $\mathcal{H}$. Then, by the Riesz representation theorem there is a unique element $k_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$ and every $\lambda \in \Omega$.

The function k on $\Omega \times \Omega$ defined by $k(z, \lambda) = k_\lambda(z)$ is called the reproducing kernel of $\mathcal{H}$, see [1]. It was shown that $k_\lambda(z)$ can be represented by

$$k_\lambda(z) = \sum_{n=1}^{\infty} \overline{e_n(\lambda)} e_n(z)$$

for any orthonormal basis $\{e_n\}_{n\geq 1}$ of $\mathcal{H}$. The prototypical RKHSs are the Hardy space $H^2(\mathbb{D})$, where $\mathbb{D} = \{z \in C : |z| < 1\}$ is the unit disc, the Bergman space $L_a^2(\mathbb{D})$, the Dirichlet space $D^2(\mathbb{D})$ and the Fock space $F(\mathbb{C})$. For example, for the Hardy-Hilbert space $H^2 = H^2(\mathbb{D})$ over the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ $\{z^n\}_{n\geq 1}$ is an orthonormal basis, therefore the reproducing kernel of H^2 is the function $k_\lambda(z) = \sum\limits_{n=1}^{\infty} \overline{\lambda_n} z^n = \left(1 - \overline{\lambda} z\right)^{-1}$, $\lambda \in \mathbb{D}$.

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For A a bounded linear operator on $\mathcal{H}$, its Berezin symbol (also called Berezin transform) $\widetilde{A}$ is defined on Ω by (see Berezin [5])

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \ (\lambda \in \Omega)$$

where $\widehat{k}_\lambda := \frac{k_\lambda}{\|k_\lambda\|}$ is the normalized reproducing kernel of the space $\mathcal{H}$ and the inner product $<\ ,\ >$ is taken in the space $\mathcal{H}$. It is obvious that the Berezin symbol $\widetilde{A}$ is a bounded function on Ω and $\sup_{\lambda\in\Omega}\left|\widetilde{A}(\lambda)\right|$, which is called the Berezin number of operator A (see Karaev [18, 19]), does not exceed $\|A\|$, i.e.,

$$\operatorname{ber}(A) := \sup_{\lambda\in\Omega}\left|\widetilde{A}(\lambda)\right| \leq \|A\|.$$

The Berezin number of an operator A satisfies the following properties:
(i) $\operatorname{ber}(\alpha A) = |\alpha| \operatorname{ber}(A)$, for all $\alpha \in \mathbb{C}$;
(ii) $\operatorname{ber}(A+B) \leq \operatorname{ber}(A) + \operatorname{ber}(B)$ for all $A, B \in \mathcal{B}(\mathcal{H})$.

It is also clear from the definition of Berezin symbol that the range of the Berezin symbol $\widetilde{A}$, which is called the Berezin set of operator A, lies in the numerical range $W(A)$ of operator A, i.e.,

$$\operatorname{Ber}(A) := \operatorname{Range}\left(\widetilde{A}\right) = \left\{\widetilde{A}(\lambda) : \lambda \in \Omega\right\} \subset W(A) := \{\langle Ax, x\rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\}$$

which implies that $\operatorname{ber}(A) \leq w(A) := \sup_{\|x\|=1} |\langle Ax, x\rangle|$ (numerical radius of operator A) (for more information, see [8, 16, 22, 23, 24, 25]). So, many questions, which are well studied for the numerical radius $w(A)$ of operator A, can be naturally asked for the Berezin number $\operatorname{ber}(A)$ of operator A. For example, is it true, or under which additional conditions the following are true:
(I) $\operatorname{ber}(A) \geq \frac{1}{2}\|A\|$;
(II) $\operatorname{ber}(AB) \leq \operatorname{ber}(A)\operatorname{ber}(B)$, where $A, B \in \mathcal{B}(\mathcal{H})$.

If $A = cI$ with $c \neq 0$, then obviously $\operatorname{ber}(A) = |c| > \dfrac{|c|}{2} = \dfrac{\|A\|}{2}$. However, it is known that in general the above inequality (II) is not satisfied (see Karaev [20]).

Berezin set and Berezin number of operators are new numerical characteristics of operators on the RKHS which are introduced by Karaev in [18]. For the basic properties and facts on these new concepts, see [2, 3, 4, 6, 14, 15, 26, 28].

It is well-known that

$$\operatorname{ber}(A) \leq w(A) \leq \|A\| \tag{1.1}$$

and

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\| \tag{1.2}$$

for any $A \in \mathcal{B}(\mathcal{H})$. Also, Berezin number inequalities were given by using the other inequalities in [10, 11, 12, 13, 17, 29, 30].

We also define the following so-called Berezin norm of operators $A \in \mathcal{B}(\mathcal{H})$:

$$\|A\|_{\text{Ber}} := \sup_{\lambda\in\Omega}\|Ak_\lambda\|.$$

It is easy to see that actually $\|A\|_{\text{Ber}}$ determines a new operator norm in $\mathcal{B}(\mathcal{H}(\Omega))$ (since the set of reproducing kernels $\{k_\lambda : \lambda \in \Omega\}$ span the space $\mathcal{H}(\Omega)$). It is also trivial that $\operatorname{ber}(A) \leq \|A\|_{\text{Ber}} \leq \|A\|$.

In the present paper, we present some new general forms by using this bounded function $\tilde{A}$ for the Berezin numbers of some operator convex function. With this theme, we also establish several new convex inequalities involving of the Berezin inequalities.

2. Known Lemmas

To prove our results, we need the following sequence of lemmas.

For a convex function $f : J \to \mathbb{R}$ and for any $a, b \in J$, the well-known Hermite-Hadamard inequality states the following

$$f\left(\frac{a+b}{2}\right) \le \int_0^1 f\left(ta + (1-t)\,b\right) dt \le \frac{f(a) + f(b)}{2}. \tag{2.1}$$

In [21], Kittaneh proved the following well known inequalities.

Lemma 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$ and let $x, y \in \mathcal{H}$ be any vectors. Then*

$$|\langle Ax, y\rangle| \le \sqrt{\langle |A|\,x, x\rangle \langle |A^*|\,y, y\rangle}. \tag{2.2}$$

Mond and Pečarić [27] obtained the following result.

Lemma 2.2. *Let $A \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator with spectrum contained in the interval J and $x \in \mathcal{H}$ be a unit vector. If f is a convex function on J, then*

$$f\left(\langle Ax, x\rangle\right) \le \langle f(A)\,x, x\rangle\,. \tag{2.3}$$

We will also need to recall the definition of an operator convex function.

Definition 2.3. *A function $f : J \to \mathbb{R}$ is said to be an operator convex function if f is continuous and $f\left((A+B)/2\right) \le \left(f(A) + f(B)\right)/2$ for all self-adjoint operators A, $B \in \mathcal{B}(\mathcal{H})$ with spectra in the interval J.*

Recall that a linear map $\Psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is said to be positive if it maps positive operators to positive operators, i.e., if $A \ge 0$ in $\mathcal{B}(\mathcal{H})$, then $\Psi(A) \ge 0$ in $\mathcal{B}(\mathcal{K})$. If in addition $\Psi(1_{\mathcal{H}}) = 1_{\mathcal{K}}$ then Ψ is said to be unital.

The Choi-Davis inequality was introduced in [7], as follows:

Lemma 2.4. *Let $f : J \to \mathbb{R}$ be the operator convex function and A be the self-adjoint operator whose spectrum is the interval J. Then,*

$$f\left(\Psi(A)\right) \le \Psi\left(f(A)\right) \tag{2.4}$$

for the unital positive linear map Ψ.

As a consequence, for example, one has the Kadison inequality

$$\Psi^2(A) \le \Psi\left(A^2\right). \tag{2.5}$$

It is unfortunate that the inequality (2.4) is not valid for convex functions f. However, if f is convex, instead of operator convex, then one has the weaker inequality (see [9])

$$f\left(\langle \Psi(A)\,x, x\rangle\right) \le \langle \Psi\left(f(A)\right) x, x\rangle\,;\; x \in \mathcal{H},\;\; \|x\| = 1. \tag{2.6}$$

3. Main Results

In this section we prove our results for Berezin inequalities, then we show how these results are related to existed ones.

Theorem 3.1. *Let $\mathcal{H}=\mathcal{H}(\Omega)$ be a RKHS. If $A\in\mathcal{B}(\mathcal{H})$ and $f:[0,\infty)\to[0,\infty)$ is an increasing operator convex function, then*

$$f(\mathrm{ber}(A))\leq\left\Vert\int_0^1 f(t|A|+(1-t)|A^*|)\,dt\right\Vert_{\mathrm{ber}}\leq\frac{1}{2}\left\Vert f(|A|)+f(|A^*|)\right\Vert_{\mathrm{ber}}. \tag{3.1}$$

Proof. Let $\widehat{k}_\lambda$ be a normalized reproducing kernel. Then

$$\begin{aligned}
f\left(\left|\left\langle A\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\right)&\leq f\left(\sqrt{\left\langle|A|\,\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\left\langle|A^*|\,\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}\right)\\
&\text{(by the inequality (2.2))}\\
&\leq f\left(\frac{\left\langle|A|\,\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+\left\langle|A^*|\,\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{2}\right)\\
&\text{(by the AM-GM inequality)}\\
&\leq\int_0^1 f\left(t\left\langle|A|\,\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+(1-t)\left\langle|A^*|\,\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)dt \qquad (3.2)\\
&\text{(by the inequality (2.1))}.
\end{aligned}$$

From operator convexity of f, we get

$$\begin{aligned}
f\left(t\left\langle|A|\,\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+(1-t)\left\langle|A^*|\,\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)&=f\left(\left\langle(t|A|+(1-t)|A^*|)\,\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)\\
&\leq\left\langle f(t|A|+(1-t)|A^*|)\,\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\\
&\text{(by the inequality (2.3))}\\
&\leq t\left\langle(f|A|)\,\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+(1-t)\left\langle f(|A^*|)\,\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 f\left(t\left\langle|A|\,\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+(1-t)\left\langle|A^*|\,\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)dt&\leq\left\langle\int_0^1\{f(t|A|+(1-t)|A^*|)\,dt\}\,\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\\
&\leq\frac{1}{2}\left\langle(f(|A|)+f(|A^*|))\,\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle.
\end{aligned} \tag{3.3}$$

By considering the inequalities (3.2) and (3.3), we have

$$\begin{aligned}
f\left(\left|\widetilde{A}(\lambda)\right|\right)&\leq\left\langle\int_0^1\{f(t|A|+(1-t)|A^*|)\,dt\}\,\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\\
&\leq\frac{1}{2}\left\langle(f(|A|)+f(|A^*|))\,\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle
\end{aligned}$$

and

$$\begin{aligned}\sup_{\lambda\in\Omega}\left(\left|\widetilde{A}(\lambda)\right|\right) &\leq \sup_{\lambda\in\Omega}\left\langle \int_0^1 \left\{f\left(t\left|A\right|+(1-t)\left|A^*\right|\right)dt\right\}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle \\ &\leq \sup_{\lambda\in\Omega}\frac{1}{2}\left\langle \left(f\left(\left|A\right|\right)+f\left(\left|A^*\right|\right)\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\end{aligned}$$

which shows that

$$f\left(\operatorname{ber}(A)\right) \leq \left\|\int_0^1 f\left(t\left|A\right|+(1-t)\left|A^*\right|\right)dt\right\|_{\operatorname{ber}} \leq \frac{1}{2}\left\|f\left(\left|A\right|\right)+f\left(\left|A^*\right|\right)\right\|_{\operatorname{ber}}$$

is desired. The theorem is proved. $\square$

By using some arguments of the paper [24], we have the following theorem.

Theorem 3.2. *Let $\mathcal{H}=\mathcal{H}(\Omega)$ be a RKHS. If $A\in B(\mathcal{H})$, then*

$$\operatorname{ber}^2(A) \leq \frac{1}{2}\left\|\left|A\right|^2+\left|A^*\right|^2\right\|_{\operatorname{ber}}. \tag{3.4}$$

Here A^* is the adjoint operator of A, $\left|A\right|^2=A^*A$ and $\left|A^*\right|^2=AA^*$.

Proof. Let $\widehat{k}_\lambda$ normalized reproducing kernel of space $\mathcal{H}$ and $A=B+iC$ be the Cartesian decomposition of A, i.e., $B=\frac{A+A^*}{2}$ and $C=\frac{A-A^*}{2i}$. Then B and C are self-adjoint and

$$A^*A+AA^*=2\left(B^2+C^2\right). \tag{3.5}$$

By the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left|\left\langle A\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^2 &= \left\langle B\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle^2+\left\langle C\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle^2 \\ &\leq \left\|B\widehat{k}_\lambda\right\|^2+\left\|C\widehat{k}_\lambda\right\|^2 \\ &= \left\langle B^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+\left\langle C^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle=\left\langle \left(B^2+C^2\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\end{aligned} \tag{3.6}$$

and

$$\sup_{\lambda\in\Omega}\left|\left\langle A\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^2 \leq \sup_{\lambda\in\Omega}\left\langle \left(B^2+C^2\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle$$

which is equivalent to

$$\left(\operatorname{ber}(A)\right)^2 \leq \left\|B^2+C^2\right\|_{\operatorname{ber}} = \frac{1}{2}\left\|\left|A\right|^2+\left|A^*\right|^2\right\|_{\operatorname{ber}}$$

and completes the proof of the theorem. $\square$

Since the function $f(t)=t^r$, $1\leq r\leq 2$ is an increasing operator convex function, Theorem 3.1 implies the following general form of the inequality (3.4).

Corollary 3.3. *Let $\mathcal{H}=\mathcal{H}(\Omega)$ be a RKHS. If $A\in\mathcal{B}(\mathcal{H})$, then for any $1\leq r\leq 2$,*

$$\operatorname{ber}^r(A) \leq \left\|\left(\int_0^1 t\left|A\right|+(1-t)\left|A^*\right|\right)^r dt\right\|_{\operatorname{ber}} \leq \frac{1}{2}\left\|\left|A\right|^r+\left|A^*\right|^r\right\|_{\operatorname{ber}}.$$

In the next result, we present another version for convex functions.

Theorem 3.4. *Let $\Psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a unital positive linear map and $A \in \mathcal{B}(\mathcal{H})(\Omega)$. If $f : [0,\infty) \to [0,\infty)$ is an increasing convex function, then*

$$\begin{aligned} f\left(\operatorname{ber}^2(\Psi(A))\right) &\le \int_0^1 f\left(\left\|\Psi^{1/2}\left(t|A|^2 + (1-t)|A^*|^2\right)\right\|_{\text{Ber}}^2\right) dt \\ &\le \frac{1}{2}\left\|\Psi\left(f\left(|A|^2\right) + f\left(|A^*|^2\right)\right)\right\|. \end{aligned}$$

Proof. Let $\widehat{k}_\lambda$ be a normalized reproducing kernel. Putting $a = \left\langle \Psi(A^*A)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle$ and $b = \left\langle \Psi(AA^*)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle$ in the inequality (2.1), we have

$$\begin{aligned} & f\left(\frac{\left\langle \Psi(A^*A)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \left\langle \Psi(AA^*)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle}{2}\right) \\ & \le \int_0^1 f\left(t\left\langle \Psi(A^*A)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + (1-t)\left\langle \Psi(AA^*)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right) dt \qquad (3.7) \\ & \le \frac{f\left(\left\langle \Psi(A^*A)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right) + f\left(\left\langle \Psi(AA^*)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right)}{2} \\ & \le \frac{1}{2}\left\langle \Psi\left(f(A^*A) + f(AA^*)\right)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\ & \text{(by the inequality (2.6)).} \end{aligned}$$

Hence

$$\int_0^1 f\left(\left\langle \Psi\left(tA^*A + (1-t)AA^*\right)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right) dt \le \frac{1}{2}\left\langle \Psi\left(f(A^*A) + f(AA^*)\right)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle. \tag{3.8}$$

Now since f is increasing and convex, we get

$$\begin{aligned} f\left(\frac{\left\langle \Psi(A^*A)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \left\langle \Psi(AA^*)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle}{2}\right) &= f\left(\frac{\left\langle \Psi(A^*A + AA^*)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle}{2}\right) \\ &= f\left(\left\langle \Psi\left((B^2 + C^2)\right)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right) \text{ (by (3.5))} \\ &= f\left(\left\langle \Psi(B^2)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \left\langle \Psi(C^2)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right) \\ &\ge f\left(\left\langle \Psi^2(B)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \left\langle \Psi^2(C)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right) \\ &\quad \text{(by the inequality (2.5))} \\ &\ge f\left(\left\langle \Psi(B)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^2 + \left\langle \Psi(C)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^2\right) \\ &\quad \text{(by the Cauchy-Schwarz inequality)} \\ &= f\left(\left|\left\langle \Psi(A)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|^2\right) \text{ (by (3.6))} \end{aligned} \tag{3.9}$$

and this together with (3.7), (3.8) and (3.9) imply

$$\begin{aligned} f\left(\left|\widetilde{\Psi(A)}(\lambda)\right|^2\right) &\leq \int_0^1 f\left(\left\langle \Psi\left(tA^*A+(1-t)AA^*\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)dt \\ &\leq \frac{1}{2}\left\langle \Psi\left(f\left(A^*A\right)+f\left(AA^*\right)\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle. \end{aligned}$$

By taking supremum over $\lambda\in\Omega$, we have

$$\begin{aligned} f\left(\mathrm{ber}^2\left(\Psi(A)\right)\right) &\leq \sup_{\lambda\in\Omega}\left(\int_0^1 f\left(\left\|\Psi^{1/2}\left(t|A|^2+(1-t)|A^*|^2\right)\widehat{k}_\lambda\right\|^2\right)\right)dt \\ &= \int_0^1 f\left(\left\|\Psi^{1/2}\left(t|A|^2+(1-t)|A^*|^2\right)\right\|_{\mathrm{Ber}}^2\right)dt \\ &\leq \frac{1}{2}\left\|\Psi\left(f\left(|A|^2\right)+f\left(|A^*|^2\right)\right)\right\| \end{aligned}$$

which give the desired results of the theorem. □

Corollary 3.5. *Let $\Psi:\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{H})$ be a unital positive linear map. If $A\in\mathcal{B}(\mathcal{H})(\Omega)$, then for any $r\geq 1$,*

$$\mathrm{ber}^{2r}\left(\Psi(A)\right)\leq\int_0^1\left\|\Psi^{1/2}\left(t|A|^2+(1-t)|A^*|^2\right)\right\|_{\mathrm{Ber}}^{2r}dt\leq\frac{1}{2}\left\|\Psi\left(|A|^{2r}+|A^*|^{2r}\right)\right\|$$

and

$$\mathrm{ber}^{2r}(A)\leq\int_0^1\left\|\left(t|A|^2+(1-t)|A^*|^2\right)^{1/2}\right\|_{\mathrm{Ber}}^{2r}dt\leq\frac{1}{2}\left\|\left(|A|^{2r}+|A^*|^{2r}\right)\right\|.$$

Proposition 3.6. *Let $\Psi:\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{H})$ be a unital positive linear map and $\mathcal{H}=\mathcal{H}(\Omega)$ be a RKHS. If $A\in\mathcal{B}(\mathcal{H})$ and $f:[0,\infty)\to[0,\infty)$ is an increasing operator convex function, then*

$$\begin{aligned} f\left(\mathrm{ber}^2\left(\Psi(A)\right)\right) &\leq \left\|\Psi\left(\int_0^1 f\left(t|A|^2+(1-t)|A^*|^2\right)dt\right)\right\| \\ &\leq \frac{1}{2}\left\|\Psi\left(f\left(|A|^2\right)+f\left(|A^*|^2\right)\right)\right\|. \end{aligned}$$

In particular, for any $1\leq r\leq 2$

$$\mathrm{ber}^{2r}(|A|)\leq\left\|\int_0^1\left(t|A|^2+(1-t)|A^*|^2\right)^r dt\right\|\leq\frac{1}{2}\left\||A|^{2r}+|A^*|^{2r}\right\|.$$

Proof. Let $\widehat{k}_\lambda\in\mathcal{H}$ be a normalized reproducing kernel. We have

$$\begin{aligned} & f\left(\left\langle\Psi\left(t|A|^2+(1-t)|A^*|^2\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right) \\ & \leq \left\langle\Psi\left(f\left(t|A|^2+(1-t)|A^*|^2\right)\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle \\ & = \left\langle\Psi\left(tf\left(|A|^2\right)+(1-t)f\left(|A^*|^2\right)\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle \\ & = t\left\langle\Psi\left(f\left(|A|^2\right)\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+(1-t)\left\langle\Psi\left(f\left(|A^*|^2\right)\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle \end{aligned}$$

and

$$\int_0^1 f\left(\left\langle \Psi\left(t\left|A\right|^2+(1-t)\left|A^*\right|^2\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)dt \leq \left\langle \Psi\left(\int_0^1 f\left(t\left|A\right|^2+(1-t)\left|A^*\right|^2\right)dt\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle$$
$$\leq \frac{1}{2}\left\langle \Psi\left(f\left(\left|A\right|^2\right)+f\left(\left|A^*\right|^2\right)\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle.$$

By taking supremum over $\lambda\in\Omega$, we have

$$\begin{aligned}
&\sup_{\lambda\in\Omega}\left(\int_0^1 f\left(\left\langle \Psi\left(t\left|A\right|^2+(1-t)\left|A^*\right|^2\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)dt\right)\\
&\leq \sup_{\lambda\in\Omega}\left\langle \Psi\left(\int_0^1 f\left(t\left|A\right|^2+(1-t)\left|A^*\right|^2\right)dt\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\\
&\leq \sup_{\lambda\in\Omega}\frac{1}{2}\left\langle \Psi\left(f\left(\left|A\right|^2\right)+f\left(\left|A^*\right|^2\right)\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle.
\end{aligned}$$

So, from this by using Theorem 3.4, we deduce that

$$\begin{aligned}
f\left(\mathrm{ber}^2\left(\Psi\left(A\right)\right)\right) &\leq \left\|\Psi\left(\int_0^1 f\left(t\left|A\right|^2+(1-t)\left|A^*\right|^2\right)dt\right)\right\|\\
&\leq \frac{1}{2}\left\|\Psi\left(f\left(\left|A\right|^2\right)+f\left(\left|A^*\right|^2\right)\right)\right\|,
\end{aligned}$$

as desired. □

In the next result, we present two operator version.

Theorem 3.7. *Let $\mathcal{H}=\mathcal{H}\left(\Omega\right)$ be a RKHS and $A\in\mathcal{B}\left(\mathcal{H}\right)$. If $f:[0,\infty)\to[0,\infty)$ is an increasing convex function, then*

$$\begin{aligned}
f\left(\mathrm{ber}\left(B^*A\right)\right) &\leq \int_0^1 f\left(\left\|\left(t\left|A\right|^2+(1-t)\left|B\right|^2\right)^{1/2}\right\|_{\mathrm{Ber}}^2\right)dt\\
&\leq \frac{1}{2}\left\|f\left(\left|A\right|^2\right)+f\left(\left|B\right|^2\right)\right\|.
\end{aligned}$$

Proof. Let $\widehat{k}_\lambda\in\mathcal{H}$ be a normalized reproducing kernel. Replacing $a=\left\langle\left|A\right|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle$ and $b=\left\langle\left|B\right|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle$ in the inequality (2.1), we get

$$\begin{aligned}
f\left(\frac{\left\langle\left|A\right|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+\left\langle\left|B\right|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{2}\right) &\leq \int_0^1 f\left(\left\langle\left(t\left|A\right|^2+(1-t)\left|B\right|^2\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)dt\\
&\leq \frac{f\left(\left\langle\left|A\right|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)+f\left(\left\langle\left|B\right|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)}{2}.
\end{aligned} \tag{3.10}$$

By using AM-GM inequality and Schwarz inequality, we have

$$\begin{aligned}
f\left(\left|\widetilde{B^*A}(\lambda)\right|\right) &= f\left(\left|\left\langle B^*A\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\right) = f\left(\left|\left\langle A\widehat{k}_\lambda,B\widehat{k}_\lambda\right\rangle\right|\right)\\
&\le f\left(\left\|A\widehat{k}_\lambda\right\|\left\|B\widehat{k}_\lambda\right\|\right)\\
&= f\left(\sqrt{\left\langle |A|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\left\langle |B|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}\right)\\
&\le f\left(\frac{\left\langle |A|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+\left\langle |B|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{2}\right). \qquad (3.11)
\end{aligned}$$

On the other hand, (2.6) implies

$$\frac{f\left(\left\langle |A|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)+f\left(\left\langle |B|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)}{2}\le\frac{1}{2}\left\langle\left(f\left(|A|^2\right)+f\left(|B|^2\right)\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle \qquad (3.12)$$

Combining (3.10), (3.11) and (3.12), we get

$$\begin{aligned}
f\left(\left|\widetilde{B^*A}(\lambda)\right|\right) &\le \int_0^1\left(\left\langle f\left(t|A|^2+(1-t)|B|^2\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)dt\\
&\le\frac{1}{2}\left\langle\left(f\left(|A|^2\right)+f\left(|B|^2\right)\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle
\end{aligned}$$

and

$$\begin{aligned}
\sup_{\lambda\in\Omega} f\left(\left|\widetilde{B^*A}(\lambda)\right|\right) &\le \sup_{\lambda\in\Omega}\int_0^1 f\left(\left\langle\left(t|A|^2+(1-t)|B|^2\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)dt\\
&= \sup_{\lambda\in\Omega}\int_0^1 f\left(\left\|\left(t|A|^2+(1-t)|B|^2\right)^{1/2}\widehat{k}_\lambda\right\|^2\right)dt\\
&\le \sup_{\lambda\in\Omega}\frac{1}{2}\left\langle\left(f\left(|A|^2\right)+f\left(|B|^2\right)\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
f(\mathrm{ber}(B^*A)) &\le \int_0^1 f\left(\left\|\left(t|A|^2+(1-t)|B|^2\right)^{1/2}\right\|_{\mathrm{Ber}}^2\right)dt\\
&\le\frac{1}{2}\left\|f\left(|A|^2\right)+f\left(|B|^2\right)\right\|.
\end{aligned}$$

and completes the proof of the theorem. □

From Proposition 3.6, we have the following result.

Proposition 3.8. *Let $\mathcal{H}=\mathcal{H}(\Omega)$ be a RKHS and $A\in\mathcal{B}(\mathcal{H})$. If $f:[0,\infty)\to[0,\infty)$ is an increasing operator convex function, then*

$$f(\mathrm{ber}(B^*A))\le\left\|\int_0^1 f\left(\left(t|A|^2+(1-t)|B|^2\right)\right)dt\right\|_{\mathrm{ber}}\le\frac{1}{2}\left\|f\left(|A|^2\right)+f\left(|B|^2\right)\right\|_{\mathrm{ber}}.$$

In particular, for any $1\le r\le 2\mathrm{ber}^r(B^*A)\le\left\|\int_0^1\left(t|A|^2+(1-t)|B|^2\right)^r dt\right\|_{\mathrm{ber}}\le\frac{1}{2}\left\||A|^{2r}+|B|^{2r}\right\|_{\mathrm{ber}}$.

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