Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 14 Issue 3(2022), Pages 1-19.
https://doi.org/10.54671/BMAA-2022-3-1

DIGITAL $\mathscr{L}_S^n$–TOPOLOGICAL SPACES

ADNAN ABDULWAHID, ELGADDAFI ELAMAMI

Abstract. This paper is a recipe for three crucial ingredients: Bitopological spaces, proximity theory and digital image processing. The notion of $\mathscr{L}_S^n$-topological spaces and $\mathscr{L}_S^n$-proximity spaces are introduced as generalizations of topological spaces and proximity spaces respectively. We explicitly compute and visualize descriptive-$\mathscr{L}_S^n$-open sets.

1. Introduction and Preliminaries

1.1. **Introduction.** Proximity theory has been growing rapidly. It leads to various applications of digital image processing. Descriptive proximity plays a crucial role in visualizing patterns that bridge some important geometric and topological concepts such as connectedness, nearness, adjacency of points, parallel edges, and spatially distinct points with matching descriptions. A proximity space is a topological space equipped with a proximity relation [3]. Using proximity spaces and topology enables us to study and discover many important concepts in a beautiful mathematical approach.
Following [4], a *digital image* is a discrete representation of visual field objects that have spatial (layout) and intensity (color or grey tone) information. From an appearance point of view, a *greyscale digital image* (an image containing pixels that are visible as black or white or grey tones (intermediate between black and white)) is represented by a $2D$ light intensity function $I(x, y)$, where x and y are spatial coordinates and the value of I at (x, y) is proportional to the intensity of light that impacted on an optical sensor and recorded in the corresponding picture element (pixel) at that point. If we have a multicolor image, then a pixel at (x, y) is 1×3 array and each array element indicates a red, green or blue brightness of the pixel in a color band (or color channel). A greyscale digital image I is represented by a single $2D$ array of numbers and a color image is represented by a collection of $2D$ arrays, one for each color band or channel. This is how, for example, Matlab represents color images. A pixel is a physical point in a raster image. A bitopological space is a set together with two topologies. Bitopological spaces can be seen as a generalization of topological spaces. The concept of bitopological spaces was first used by Kelly [1]. Bitopological spaces, proximity theory and digital image

1991 *Mathematics Subject Classification.* 54H30, 68U03, 54H05, 54E05, 94A08, 54E55.
Key words and phrases. topology, proximity, descriptive, image, digital.

Submitted June 24, 2022. Published July 26, 2022.
Communicated by Naim L. Braha.

processing are the primary ingredients of this paper.
Throughout this paper, n is a positive integer, $[n] = \{1, \cdots, n\}$ and $S \subsetneqq [n]$. The paper is organized as follows.

In Section 2, $\mathscr{L}_S^n$-topological spaces are introduced, and defining $\mathscr{L}_S^n$-continuous maps gives rise to a category $\mathscr{C}_S^n$ whose objects are $\mathscr{L}_S^n$-topological spaces and whose morphisms are $\mathscr{L}_S^n$-continuous maps. In Section 3, we introduce the notion of $\mathscr{L}_S^n$-proximity spaces as a generalization of proximity spaces, and we construct $\mathscr{L}_S^n$-topological spaces using proximity relations. In Section 4, the concept of descriptive-$\mathscr{L}_S^n$-proximity spaces are introduced, and we explicitly calculate and visualize descriptive-$\mathscr{L}_S^n$-open sets.

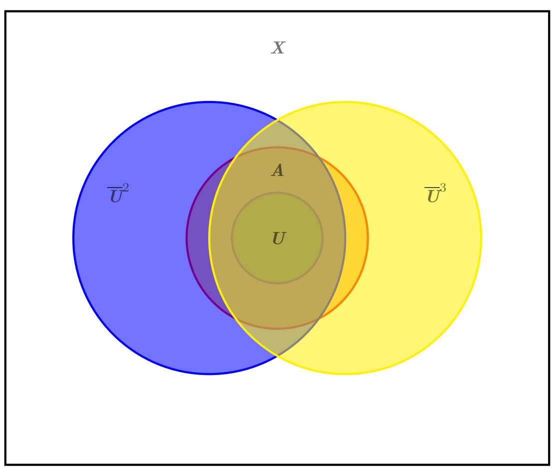
2. $\mathscr{L}_S^n$–Topological Spaces

Definition 2.1. *Let n be a positive integer and $j \in [n] = \{1, \cdots, n\}$ be a fixed positive integer, and let $S \subsetneqq [n]$. Let X be a set, and let $(X, \tau_1), \cdots, (X, \tau_n)$ be topological spaces.*

(1) A set $A \subseteq X$ is called an $\mathscr{L}_j^n$-open set in X if there exists a set $U \in \tau_j$ with

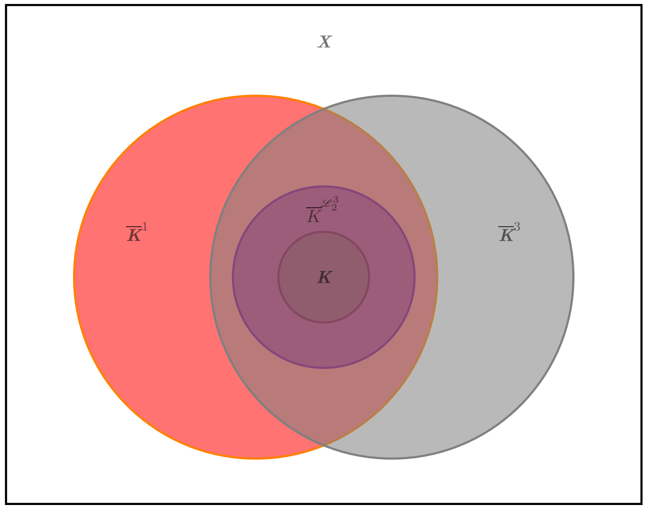
$$U \subseteq A \subseteq \bigcup_{\{i \in [n] \,:\, i \neq j\}} \overline{U}^i,$$

where for any $i \in [n]$, $\overline{U}^j$ is the closure set of U with respect to τ_i. A set $B \subseteq X$ is called an $\mathscr{L}_j^n$-closed set in X if $X \setminus B \in \mathscr{L}_j^n - O(X)$. In this case, we say that $(X, \tau_1, \cdots, \tau_n)$ is $\mathscr{L}_j^n$-topological space (or simply $\mathscr{L}_j^n$-space). The set of all $\mathscr{L}_j^n$-open sets in X is denoted by $\mathscr{L}_j^n - O(X)$ (or $\mathscr{L}_j^n - O((X, \tau_1, \cdots, \tau_n))$ if convenient), and the set of all $\mathscr{L}_j^n$-closed sets in X is denoted by $\mathscr{L}_j^n - C(X)$ (or $\mathscr{L}_j^n - C((X, \tau_1, \cdots, \tau_n))$ if convenient).



$\mathscr{L}_1^3$-open set

(2) The $\mathscr{L}_j^n$-closure of a set $K \subseteq X$, denoted by $\overline{K}^{\mathscr{L}_j^n}$ is the intersection of all $\mathscr{L}_j^n$-closed sets containing K.



The $\mathscr{L}_2^3$-closure of a set $K \subseteq X$

(3) *A set $E \subseteq X$ is called an $\mathscr{L}_S^n$-open set in X if there exists a set $V \in \bigcap_{a\in S} \tau_a$ with*

$$V \subseteq E \subseteq \bigcup_{\{i\in[n]\,:\, i\notin S\}} \overline{V}^i,$$

where for any $i \in [n]$, $\overline{V}^i$ is the closure set of V with respect to τ_i. A set $F \subseteq X$ is called an $\mathscr{L}_S^n$-closed set in X if $X \setminus F \in \mathscr{L}_S^n - O(X)$. In this case, we say that $(X, \tau_1, \cdots, \tau_n)$ is $\mathscr{L}_S^n$-topological space (or simply $\mathscr{L}_S^n$-space).

(4) *The $\mathscr{L}_S^n$-closure of a set $K \subseteq X$, denoted by $\overline{K}^{\mathscr{L}_S^n}$ is the intersection of all $\mathscr{L}_S^n$-closed sets containing K. The set of all $\mathscr{L}_S^n$-open sets in X is denoted by $\mathscr{L}_S^n - O(X)$ (or $\mathscr{L}_S^n - O((X, \tau_1, \cdots, \tau_n))$ if convenient), and the set of all $\mathscr{L}_S^n$-closed sets in X is denoted by $\mathscr{L}_S^n - C(X)$ (or $\mathscr{L}_S^n - C((X, \tau_1, \cdots, \tau_n))$ if convenient).*

Remark 2.2.

(1) For any $i \in [n]$ with $S = \{i\}$, one has $\mathscr{L}_S^n - O(X) = \mathscr{L}_i^n - O(X)$.
(2) Let $S \subsetneq [n]$. If $A, A' \in \mathscr{L}_S^n - O(X)$, then $A \cap A'$ need not be in $\mathscr{L}_S^n - O(X)$ (see Example (2.7)). However, the following proposition shows that the union of a family of $\mathscr{L}_S^n$-open sets in X is $\mathscr{L}_S^n$-open.

Proposition 2.3. *For any $i \in [n]$, let (X, τ_i) be a topological space, and let $S \subsetneq [n]$.*

(1) *Let $\{A_{\alpha\in\Lambda}\}$ be a family of an $\mathscr{L}_S^n$-open sets in X. Then*

$$\bigcup_{\alpha\in\Lambda} A_a \in \mathscr{L}_S^n - O(X).$$

(2) *Let $\{F_{\alpha\in\Lambda}\}$ be a family of an $\mathscr{L}_S^n$-closed sets in X. Then*

$$\bigcap_{\alpha\in\Lambda} E_a \in \mathscr{L}_S^n - C(X).$$

(3) *If $U \in \bigcap_{a\in S} \tau_a$, then $\overline{U}^j \in \mathscr{L}_S^n - O(X)$ for every $j \in [n] \setminus S$.*

Proof.

(1) Let $S \subsetneqq [n]$, and let $\{A_{\alpha\in\Lambda}\}$ be a family of an $\mathscr{L}_S^n$-open sets in X. By Definition (2.1), for any $\alpha \in \Lambda$, there exists a set $U_\alpha \in \bigcap_{a\in S} \tau_a$ with

$$U_\alpha \subseteq A_\alpha \subseteq \bigcup_{i\in[n]\setminus S} \overline{U_\alpha}^i.$$

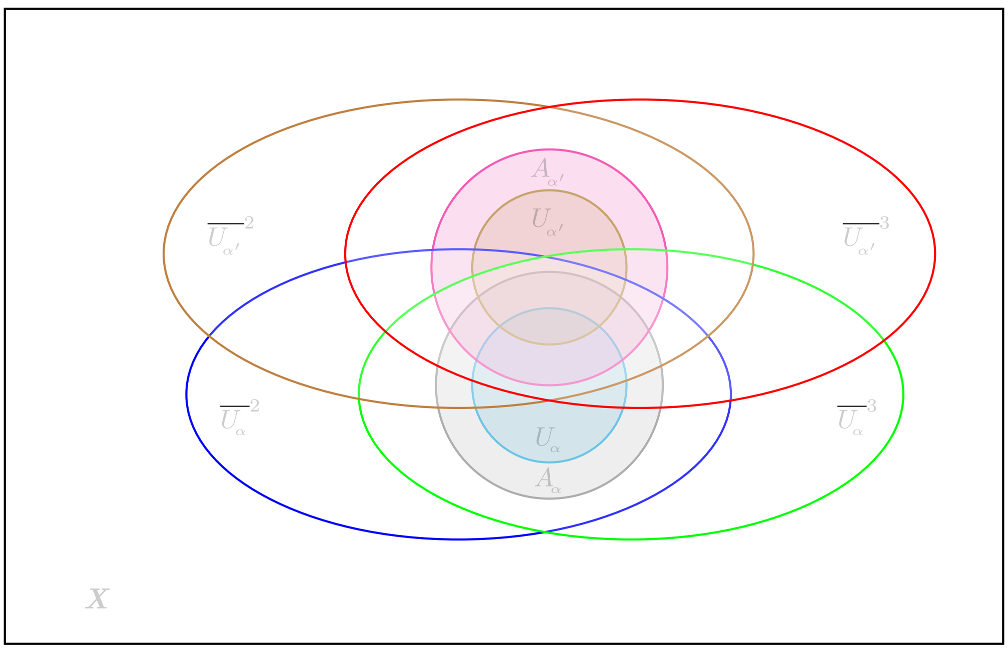
Thus, we have

$$\bigcup_{\alpha\in\Lambda} U_\alpha \subseteq \bigcup_{\alpha\in\Lambda} A_\alpha \subseteq \bigcup_{\alpha\in\Lambda} \bigcup_{i\in[n]\setminus S} \overline{U_\alpha}^i \subseteq \bigcup_{i\in[n]\setminus S} \bigcup_{\alpha\in\Lambda} \overline{U_\alpha}^i \subseteq \bigcup_{i\in[n]\setminus S} \overline{\bigcup_{\alpha\in\Lambda} U_\alpha}^i.$$

As a consequence, we have

$$\bigcup_{\alpha\in\Lambda} E_\alpha \in \mathscr{L}_j^n - O(X).$$

(2) This follows directly from part (1) of the proposition.
(3) This clearly follows from Definition 2.1.



A Union of Two $\mathscr{L}_1^3$-open Sets A_α and $A_{\alpha'}$ is an $\mathscr{L}_1^3$-open Set

□

The following is an immediate consequence of Proposition (2.3).

Corollary 2.4. *For any $i \in [n]$, let (X, τ_i) be a topological space, and let $S \subsetneqq [n]$.*

(1) $\overline{K}^{\mathscr{L}_S^n} \in \mathscr{L}_S^n - C(X)$ *for any* $K \subseteq X$.
(2) $K \in \mathscr{L}_S^n - C(X)$ *if and only if* $K = \overline{K}^{\mathscr{L}_S^n}$.

Proof.

(1) This follows immediately from Definition (2.1) and part (2) of Proposition (2.3).

(2) Suppose that $K \in \mathscr{L}_S^n - C(X)$. By Definition (2.1), $\overline{K}^{\mathscr{L}_j^n}$ is the intersection of all $\mathscr{L}_j^n$-closed sets containing K. So, $K \subseteq \overline{K}^{\mathscr{L}_S^n}$. Since $K \in \mathscr{L}_S^n - C(X)$, the intersection of all $\mathscr{L}_j^n$-closed sets containing K is K itself.Thus, $K = \overline{K}^{\mathscr{L}_S^n}$. If $K = \overline{K}^{\mathscr{L}_S^n}$, then by part (2) of Proposition (2.3), $K \in \mathscr{L}_S^n - C(X)$ as desired.

□

Theorem 2.5. *For any $i \in [n]$, let (X, τ_i) be a topological space, and let n be a positive integer.*

(1) For any $k \in [n]$, one has
$$\tau_k \subseteq \mathscr{L}_k^n - O(X).$$

(2) For any $k, l, m \in [n]$ with $1 \leq k \leq l \leq m \leq n$, we have
$$\mathscr{L}_k^l - O(X) \subseteq \mathscr{L}_k^m - O(X).$$

(3) For any $E \subseteq X$ and a fixed positive integer $j \in [n]$, one has
$$\overline{E}^{\mathscr{L}_j^n} \subseteq \overline{E}^j.$$

(4) For any set $S \subsetneqq [n]$, we have
$$\bigcap_{a \in S} \tau_a \subseteq \mathscr{L}_S^n - O(X).$$

(5) For any set $S \subsetneqq [n]$ and an integer t with $t \geq n$, we have
$$\mathscr{L}_S^n - O(X) \subseteq \mathscr{L}_S^t - O(X).$$

(6) Let $S \subseteq S' \subsetneqq [n]$. Then
$$\mathscr{L}_{S'}^n - O(X) \subseteq \mathscr{L}_S^m - O(X).$$

(7) For any set $S \subsetneqq [n]$, we have
$$\mathscr{L}_S^n - O(X) \subseteq \bigcap_{a \in S} \mathscr{L}_a^n - O(X).$$

(8) For any $S \subsetneqq [n]$ and $E \subseteq X$, one has
$$\overline{E}^{\mathscr{L}_S^n} \subseteq \overline{E}^{\bigcap_{a \in S} \tau_a}.$$

(9) Let $S \subsetneqq [n]$ and $U \in \tau_a$ for any $a \in S$. Then
$$\bigcup_{i \in [n] \,:\, i \notin S} \overline{U}^i \in \mathscr{L}_S^n - O(X).$$

(10) Let $S \subsetneqq [n]$ and F an τ_a-closed for some $a \in S$. Then
$$\bigcap_{i \in [n] \,:\, i \notin S} \overline{F}^i \in \mathscr{L}_S^n - C(X).$$

Proof.

(1) Fix $k \in [n]$, and let $A \in \tau_k$. We have

$$A \subseteq A \subseteq \bigcup_{\{i \in [n]\,:\, i \neq j\}} \overline{A}^i.$$

Thus, $A \in \mathscr{L}_k^n - O(X)$, and hence $\tau_k \subseteq \mathscr{L}_k^n - O(X)$.

(2) Fix $k, l, m \in [n]$ with $1 \leq k \leq l \leq m \leq n$, and let $A \in \mathscr{L}_k^l - O(X)$. By definition, there exists a set $U \in \tau_k$ with

$$U \subseteq A \subseteq \bigcup_{\{i \in [l]\,:\, i \neq k\}} \overline{U}^i \subseteq \bigcup_{\{i \in [m]\,:\, i \neq k\}} \overline{U}^i \text{ (since } [l] \subseteq [m]) \ .$$

So, we have

$$\mathscr{L}_k^l - O(X) \subseteq \mathscr{L}_k^m - O(X).$$

(3) Fix $j \in [n]$, and let $E \subseteq X$. The intersection of all $\mathscr{L}_j^n$-closed sets containing K is subset of the intersection of all τ_j-closed sets containing K. Therefore,

$$\overline{E}^{\mathscr{L}_j^n} \subseteq \overline{E}^j.$$

(4) Let $S \subsetneqq [n]$, and let $V \in \bigcap_{a \in S} \tau_a$. We have $V \in \tau_a$ for every $a \in S$ with

$$V \subseteq V \subseteq \bigcup_{\{i \in [n]\,:\, i \notin S\}} \overline{V}^i.$$

So, $V \in \mathscr{L}_S^n - O(X)$, and hence

$$\bigcap_{a \in S} \tau_a \subseteq \mathscr{L}_S^n - O(X).$$

(5) Let $S \subsetneqq [n]$ and fix a positive integer t with $t \geq n$. Let $E \in \mathscr{L}_S^n - O(X)$. By definition, there exists a set $V \in \bigcup_{a \in S} \tau_a$ with

$$V \subseteq E \subseteq \bigcup_{\{i \in [n]\,:\, i \notin S\}} \overline{V}^i \subseteq \bigcup_{\{i \in [t]\,:\, i \notin S\}} \overline{V}^i \text{ (since } [n] \subseteq [t]) \ .$$

Consequently, we have

$$\mathscr{L}_S^n - O(X) \subseteq \mathscr{L}_S^t - O(X).$$

(6) Let $S \subseteq S' \subsetneqq [n]$, and let $E \in \mathscr{L}_{S'}^n - O(X)$. By definition, there exists $V \in \bigcap_{a \in S'} \tau_a$ with

$$V \subseteq E \subseteq \bigcup_{\{i \in [n]\,:\, i \notin S'\}} \overline{V}^i.$$

Since $\bigcap_{a \in S'} \tau_a \subseteq \bigcap_{b \in S} \tau_b$ and

$$\bigcup_{\{i \in [n]\,:\, i \notin S'\}} \overline{V}^i \subseteq \bigcup_{\{i \in [n]\,:\, i \notin S\}} \overline{V}^i,$$

we have $E \in \mathscr{L}_S'^m - O(X)$. Thus,

$$\mathscr{L}_{S'}^n - O(X) \subseteq \mathscr{L}_S'^m - O(X).$$

(7) Let $S \subsetneqq [n]$, and let $E \in \mathscr{L}_S^n - O(X)$. By definition, there exists a set $V \in \bigcap_{a\in S} \tau_a$ with

$$V \subseteq E \subseteq \bigcup_{\{i\in[n]\,:\,i\notin S\}} \overline{V}^i.$$

So, $V \in \tau_a$ for any $a \in S$. Furthermore, for any $a \in S$, we have

$$V \subseteq E \subseteq \bigcup_{\{i\in[n]\,:\,i\notin S\}} \overline{V}^i \subseteq \bigcup_{\{i\in[n]\,:\,i\neq a\}} \overline{V}^i.$$

Accordingly, $E \in \mathscr{L}_a^n - O(X)$ for every $a \in S$ and hence $\mathscr{L}_S^n - O(X) \subseteq \bigcap_{a\in S} \mathscr{L}_a^n - O(X)$.

(8) This an immediate consequence of part (4) of the theorem.

(9) Note that

$$U \subseteq \bigcup_{\{i\in[n]\,:\,i\notin S\}} \overline{U}^i \subseteq \bigcup_{\{i\in[n]\,:\,i\notin S\}} \overline{U}^i.$$

(10) The proof follows directly from the previous part.

□

The following consequence shows that $\mathscr{L}_S^n$-topological spaces are a generalization of topological spaces.

Theorem 2.6. *For any $i \in [n]$, let (X, τ_i) be a topological space, and let n be a positive integer.*

(1) *If $j \in [n]$ is a fixed positive integer and $\tau_j = \mathscr{D}$ is the discrete topology on X, then*

$$\mathscr{L}_j^n - O(X) = \tau_k = \mathscr{D}.$$

(2) *If $j \in [n]$ is a fixed positive integer and $\tau_k = \mathscr{D}$ is the discrete topology on X for all $k \in [n]$ with $k \neq j$, then*

$$\mathscr{L}_j^n - O(X) = \tau_j.$$

(3) *If $S \subsetneqq [n]$ and $\tau_a = \mathscr{D}$ (the discrete topology on X) for all $a \in S$, we have*

$$\mathscr{L}_S^n - O(X) = \mathscr{D}.$$

(4) *If $S \subsetneqq [n]$ and $\tau_b = \mathscr{D}$ (the discrete topology on X) for all $b \in [n] \setminus S$, we have*

$$\mathscr{L}_S^n - O(X) = \bigcap_{a\in S} \tau_a.$$

(5) *If $j \in [n]$ is a fixed positive integer and $\tau_j = \mathscr{I}$ is the indiscrete topology on X, then*

$$\mathscr{L}_j^n - O(X) = \tau_j = \mathscr{I}.$$

(6) *If $S \subsetneqq [n]$ and $\tau_a = \mathscr{I}$ (the indiscrete topology on X) for all $a \in S$, then*

$$\mathscr{L}_S^n - O(X) = \mathscr{I}.$$

Example 2.7. *Let $X = \{a,b,c\}$, $E = \{a,c\}$, and $S = \{1,3\}$. Consider the following topologies on X:*

$$\begin{aligned}
\tau_1 &= \{\emptyset, \{a\}, X\} \\
\tau_2 &= \{\emptyset, \{b\}, X\}, \\
\tau_3 &= \{\emptyset, \{c\}, X\}, \\
\tau_4 &= \{\emptyset, \{a\}, \{a,b\}, X\}, \\
\tau_5 &= \{\emptyset, \{a\}, \{a,c\}, X\}, \\
\tau_6 &= \{\emptyset, \{a\}, \{b,c\}, X\}, \\
\tau_7 &= \{\emptyset, \{b\}, \{a,c\}, X\}, \\
\tau_8 &= \{\emptyset, \{b\}, \{a,b\}, X\}, \\
\tau_9 &= \{\emptyset, \{b\}, \{b,c\}, X\}, \\
\tau_{10} &= \{\emptyset, \{c\}, \{a,b\}, X\}, \\
\tau_{11} &= \{\emptyset, \{c\}, \{a,c\}, X\}, \\
\tau_{12} &= \{\emptyset, \{c\}, \{b,c\}, X\}.
\end{aligned}$$

A simple calculation shows that

$$\mathscr{L}_2^{12} - O(X) = \{\emptyset, \{b\}, \{a,b\}, \{b,c\}, X\}$$

and

$$\mathscr{L}_6^{12} - O(X) = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, \{b,c\}, X\}.$$

Note that $\tau_1 \cap \tau_3 = \mathscr{I}$. We have

$$\mathscr{L}_S^{12} - C(X) = \mathscr{I},$$

since

$$\mathscr{L}_2^{12} - C(X) = \{\emptyset, \{a,c\}, \{c\}, \{a\}, X\}$$

and

$$\mathscr{L}_6^{12} - C(X) = \{\emptyset, \{b,c\}, \{c\}, \{b\}, \{a\}, X\}.$$

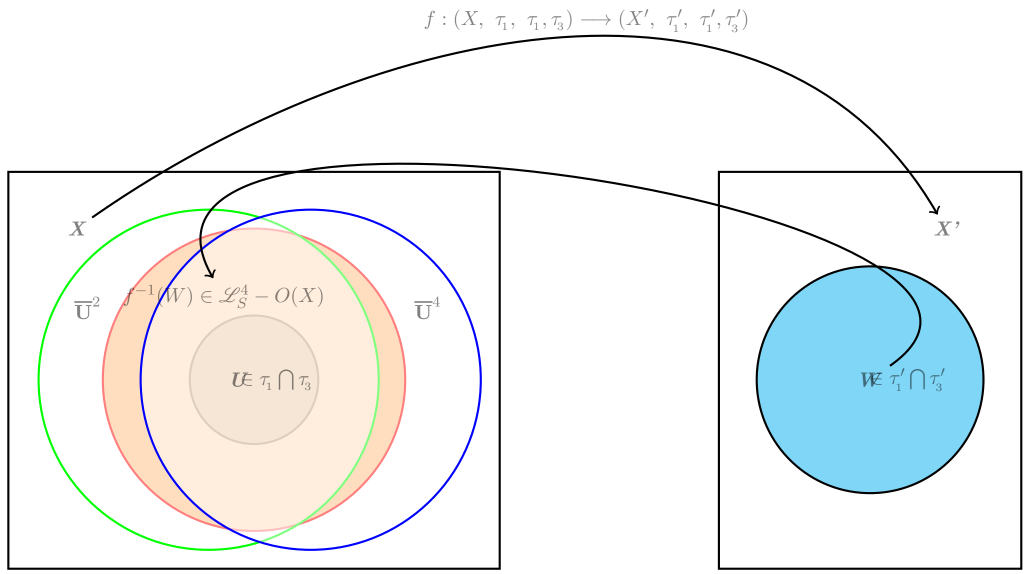
It is clear that

$$\overline{E}^{\mathscr{L}_2^{12}} = \{a,c\} = E, \quad \overline{E}^{\mathscr{L}_6^{12}} = \{a,b,c\} = X, \quad \text{and} \quad \overline{E}^{\mathscr{L}_S^{12}} = \{a,b,c\} = X.$$

One might notice that $\{a,b\}, \{b,c\} \in \mathscr{L}_6^{12} - O(X)$, but $\{a,b\} \cap \{b,c\} = \{b\} \notin \mathscr{L}_6^{12} - O(X)$. In general, if $A, A' \in \mathscr{L}_S^n - O(X)$, then $A \cap A'$ need not be in $\mathscr{L}_S^n - O(X)$.

Definition 2.8. *Let $S \subsetneqq [n]$, and let $(X, \tau_1, \cdots, \tau_n)$ and $(X', \tau_1', \cdots, \tau_n')$ be $\mathscr{L}_S^n$-topological spaces. A map $f : (X, \tau_1, \cdots, \tau_n) \to (X', \tau_1', \cdots, \tau_n')$ is called an $\mathscr{L}_S^n$-continuous map if $f^{-1}(W) \in \mathscr{L}_S^n - O(X)$, for every $W \in \bigcap_{a \in S} \tau_a'$ in X'. The map $f : (X, \tau_1, \cdots, \tau_n) \to (X', \tau_1', \cdots, \tau_n')$ is called an $\mathscr{L}_S^n$- homeomorphism if it is bijective, and f and f^{-1} are $\mathscr{L}_S^n$-continuous maps.*

Theorem 2.9. *Let $S \subsetneqq [n]$, and let $(X, \tau_1, \cdots, \tau_n)$ and $(X', \tau_1', \cdots, \tau_n')$ be $\mathscr{L}_S^n$-topological spaces, and let $f : X \to X'$ be a map.*



$\mathscr{L}_S^4$-Continuity with $S=\{1,3\}$

(1) If $f:(X,\ \tau_1,\cdots,\tau_n)\rightarrow(X',\tau_1',\cdots,\tau_n')$ is an $\mathscr{L}_a^n$-continuous map for any $a\in S$ and

$$\bigcap_{a\in S}\mathscr{L}_a^n-O(X)\subseteq\mathscr{L}_S^n-O(X),$$

then $f:(X,\ \tau_1,\cdots,\tau_n)\rightarrow(X',\tau_1',\cdots,\tau_n')$ is an $\mathscr{L}_S^n$-continuous map.

(2) If $f:(X,\bigcap_{a\in S}\tau_a)\rightarrow(X',\bigcap_{a\in S}\tau_a')$ is a continuous map, then $f:(X,\tau_1,\cdots,\tau_n)\rightarrow(X',\tau_1',\cdots,\tau_n')$ is an $\mathscr{L}_S^n$-continuous map.

(3) If $f:(X,\tau_a)\rightarrow(X',\tau_a')$ is a continuous map for any $a\in S$, then $f:(X,\ \tau_1,\cdots,\tau_n)\rightarrow(X',\tau_1',\cdots,\tau_n')$ is an $\mathscr{L}_S^n$-continuous map.

Proof.

(1) Let $W\in\bigcap_{a\in S}\tau_a'$. It follows that $W\in\tau_a'$ for every $a\in S$. $f:(X,\ \tau_1,\cdots,\tau_n)\rightarrow(X',\tau_1',\cdots,\tau_n')$ is an $\mathscr{L}_a^n$-continuous map for any $a\in S$, $f^{-1}(W)\in\bigcap_{a\in S}\mathscr{L}_a^n-O(X)$.

Since $\bigcap_{a\in S}\mathscr{L}_a^n-O(X)\subseteq\mathscr{L}_S^n-O(X)$, $f^{-1}(W)\in\mathscr{L}_S^n-O(X)$ which completes the proof.

(2) This is clear.

(3) This follows directly from the previous part.

□

The following is an immediate consequence of Theorem (2.9)

Corollary 2.10. *Let* $(X, \tau_1, \cdots, \tau_n)$ *and* $(X', \tau'_1, \cdots, \tau'_n)$ *be* $\mathscr{L}_S^n$*-topological spaces, and let* $f : X \to X'$ *be a map such that* $f : (X, \tau_a) \to (X', \tau'_a)$ *is a homeomorphism for any* $a \in S$*. Then* $f : (X, \tau_1, \cdots, \tau_n) \to (X', \tau'_1, \cdots, \tau'_n)$ *is an* $\mathscr{L}_S^n$*-homeomorphism.*

We have the following analogue of the "usual continuity".

Theorem 2.11. *Let* $S \subsetneqq [n]$*, and let* $(X, \tau_1, \cdots, \tau_n)$ *and* $(X', \tau'_1, \cdots, \tau'_n)$ *be* $\mathscr{L}_S^n$*-topological spaces, and let* $f : X \to X'$ *be a map. Then, the following statements are equivalent:*

(1) $f : (X, \tau_1, \cdots, \tau_n) \to (X', \tau'_1, \cdots, \tau'_n)$ *is* $\mathscr{L}_S^n$*-continuous.*
(2) $f^{-1}(W) \in \mathscr{L}_S^n - C(X)$*, for every* $\bigcap_{a \in S} \tau'_a$*–closed set* W *in* X'*.*
(3) For any subset E *in* X*,*

$$f(\overline{E}^{\mathscr{L}_S^n}) \subseteq \overline{f(E)}^{\bigcap_{a \in S} \tau'_a}.$$

(4) For any subset E' *in* X'*,*

$$\overline{f^{-1}(E)}^{\mathscr{L}_S^n} \subseteq f^{-1}(\overline{E}^{\bigcap_{a \in S} \tau'_a}).$$

It might be noticeable that Definition (2.8) gives rise to a category $\mathscr{C}_S^n$ whose objects are $\mathscr{L}_S^n$-topological spaces and whose morphisms are $\mathscr{L}_S^n$-continuous maps.

3. $\mathscr{L}_S^n$–proximity Spaces

Following [2], we first recall some basic concepts of proximity spaces.

Definition 3.1. [2] *A binary relation* δ *on the power set of* X *is called an (Efremovic) proximity on* X *if* δ *satisfies the following axioms:*

(i) $A\delta B$ *implies* $B\delta A$
(ii) $A\delta(B \bigcup C)$ *implies* $A\delta B$ *or* $A\delta C$
(iii) $A\delta B$ *implies* $A \neq \varnothing$, $B \neq \varnothing$
(iv) $A\underline{\delta}B$ *implies there exists a subset* E *such that* $A\underline{\delta}E$ *and* $(X \setminus E)\underline{\delta}B$
(v) $A \bigcap B \neq \varnothing$ *implies* $A\delta B$

where

$$\underline{\delta} = (\mathscr{P}(X) \times \mathscr{P}(X)) \setminus \delta.$$

A proximity space is a pair (X, δ)*, where* X *is a set and* δ *is a proximity relation. A proximity space is called separated if the following axiom holds:*

(vi) $\{x\}\delta\{y\}$ *implies* $x = y$*. If* $A\delta B$*, we say* A *is near* B *or* A *and* B *are proximal; otherwise we say* A *and* B *are apart, and we write it as* $A\underline{\delta}B$*. We say* B *is a proximal or* δ*-neighborhood of* A*, and we write it as* $A \ll B$*, if and only if* A *and* $X \setminus B$ *are apart.*

The main properties of this set neighborhood relation, listed below, provide an alternative axiomatic characterization of proximity spaces.

For all subsets $A, B, C, D \subseteq X$

(1) $X \ll X$
(2) $A \ll B$ implies $A \subseteq B$
(3) $A \subseteq B \ll C \subseteq D$ implies $A \ll D$
(4) ($A \ll B$ and $A \ll C$) implies $A \ll B \cap C$
(5) $A \ll B$ implies $X \setminus B \ll X \setminus A$

(6) $A \ll B$ implies that there exists some E such that $A \ll E \ll B$.

A proximity or proximal map is one that preserves nearness, that is, given $f : (X, \delta) \to \left(Y, \delta'\right)$ if $A\delta B$ in X, then $f(A)\delta' f(B)$ in Y [2].

Theorem 3.2. [2] *If a subset A of a proximity space (X, δ) is defined to be closed iff $x\delta A$ implies $x \in A$,then the collection of complements of all closed sets so defined yields a topology $\tau = \tau(\delta)$ on X. Furthermore, the τ-closure $\overline{A}$ of A is given by $\overline{A} = \{x : x\delta A\}$.*

Theorem 3.3. *Let $S = \{a_1, \cdots, a_m\} \subsetneqq [n]$, and write $[n] \setminus S = \{b_1, \cdots, b_{n-m}\}$ and let (X, δ_{b_i}) be a proximity space with a corresponding proximity topology $\tau(\delta_{b_i})$ for any $i \in [n] \setminus S$. Define a relation δ on $\mathscr{P}(X)$ (the power set of X) by $A\delta B$ if $A\delta_{b_t} B$ for some $t \in [n] \setminus S$.*

(1) The relation δ is a proximity relation on $\mathscr{P}(X)$.
(2) For any family of topologies $\{\tau_k : k \in [m]\}$, we have

$$\mathscr{L}_S^n - O((X, \tau_1, \cdots, \tau_m, \tau(\delta_{b_1}), \cdots, \tau(\delta_{b_{n-m}}))) = \mathscr{L}_S^{m+1} - O((X, \tau_1, \cdots, \tau_m, \tau(\delta))).$$

Proof.

(1) The proof is clear.
(2) It suffices to show that

$$\overline{E}^{\tau(\delta)} = \bigcup_{i=1}^{n-m} \overline{E}^{\tau(\delta_{b_i})},$$

for any $E \subseteq X$. Let $E \subseteq X$. Let $x \in \overline{E}^{\tau(\delta)}$. We have $x\delta E$ which implies that $x\delta_{b_t} E$ for some $t \in [n] \setminus S$. Thus, $x \in \overline{E}^{\tau(\delta_{b_i})}$, and hence

$$x \in \bigcup_{i=1}^{n-m} \overline{E}^{\tau(\delta_{b_i})}.$$

It follows that

$$\overline{E}^{\tau(\delta)} \subseteq \bigcup_{i=1}^{n-m} \overline{E}^{\tau(\delta_{b_i})}.$$

Let

$$y \in \bigcup_{i=1}^{n-m} \overline{E}^{\tau(\delta_{b_i})}.$$

So, $y \in \overline{E}^{\tau(\delta_t)}$ for some $t \in [n] \setminus S$. This implies that $y \in \overline{E}^{\tau(\delta)}$. Consequently,

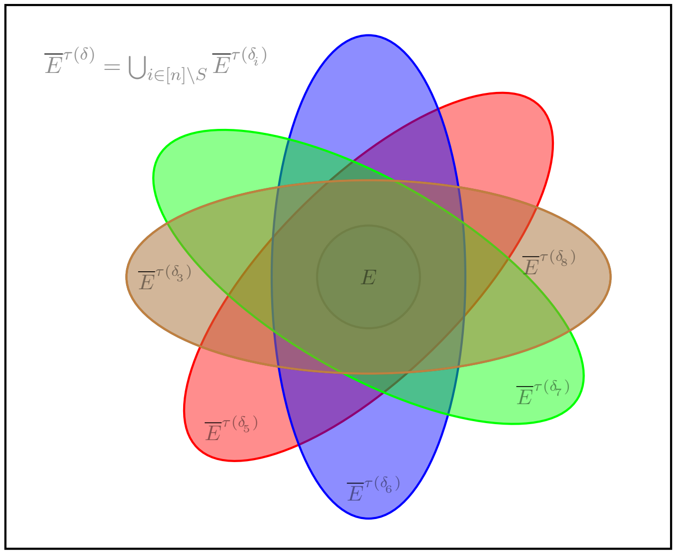
$$\bigcup_{i=1}^{n-m} \overline{E}^{\tau(\delta_{b_t})} \subseteq \overline{E}^{\tau(\delta)}, \text{ and hence } \overline{E}^{\tau(\delta)} = \bigcup_{i=1}^{n-m} \overline{E}^{\tau(\delta_{b_i})}.$$

This completes the proof of assertion *(2)*.

□

Definition 3.4. [2] *If δ_1 and δ_2 are two proximities on a set X, we define*

$$\delta_1 < \delta_2 \textit{ iff } A\delta_1 B \textit{ implies } A\delta_2 B.$$



$\overline{E}^{\tau(\delta)} = \bigcup_{i\in[n]\setminus S} \overline{E}^{\tau(\delta_i)}$ for $S = \{1, 2, 4\}$ and $n = 9$

The above is expressed by saying that δ_1 is finer than δ_2, or δ_2 is coarser than δ_1. The following theorem shows that a finer proximity structure induces a finer topology:

Theorem 3.5. [2] *Let δ_1 and δ_2 be two proximities on a set X. Then $\delta_1 < \delta_2$ implies $\tau(\delta_2) \subseteq \tau(\delta_1)$.*

An analogue of Theorem for $\mathscr{L}_S^n$-spaces can be stated as follows:

Theorem 3.6. *Let $S = \{a_1, \cdots, a_m\} \subsetneq [n]$, and write $[n] \setminus S = \{b_1, \cdots, b_{n-m}\}$ and let (X, δ_i) be a proximity space with a corresponding proximity topology $\tau(\delta_i)$ for any $i \in [n] \setminus S$. If $\delta_1 < \cdots < \delta_{n-m}$, then for any family of topologies $\{\tau_k : k \in [m]\}$, we have*

$$\mathscr{L}_S^n\text{--}O((X, \tau_1, \cdots, \tau_m, \tau(\delta_{b_1}), \cdots, \tau(\delta_{b_{n-m}}))) = \mathscr{L}_S^{m+1}\text{--}O((X, \tau_1, \cdots, \tau_m, \tau(\delta_{b_{n-m}}))).$$

Proof. By Theorem (3.5), we have $\tau(\delta_{b_{n-m}}) \subseteq \cdots \subseteq \tau(\delta_{b_1})$. Thus, for any $E \subseteq X$, we have

$$\bigcup_{i\in[n]\setminus S} \overline{E}^{\tau(\delta_{b_i})} = \overline{E}^{\tau(\delta_{b_{n-m}})}.$$

It follows that

$$\mathscr{L}_S^n\text{--}O((X, \tau_1, \cdots, \tau_m, \tau(\delta_{b_1}), \cdots, \tau(\delta_{b_{n-m}}))) = \mathscr{L}_S^{m+1}\text{--}O((X, \tau_1, \cdots, \tau_m, \tau(\delta_{b_{n-m}}))).$$

□

Definition 3.7. *Let $S \subsetneq [n]$, and let (X, δ_i) be a proximity space for any $i \in [n]$. Let δ be a binary relation on $\mathscr{P}(X)$ (the power set of X) defined by $A\delta B$ if and only if the following axioms are satisfied:*

(i) $x \notin A$ implies $x\delta_a(X \setminus B)$ for any $a \in S$.
(ii) $x\underline{\delta_i}B$ for any $i \in [n] \setminus S$ implies $x \notin A$.

Then δ is called an $\mathscr{L}_S^n$–(Efremovic) proximity, or simply $\mathscr{L}_S^n$–proximity, induced by proximity relations $\delta_1, \cdots, \delta_n$. An $\mathscr{L}_S^n$–proximity space induced by $\delta_1, \cdots, \delta_n$ is a pair (X, δ), where X is a set and δ is an $\mathscr{L}_S^n$-proximity relation induced by proximity relations $\delta_1, \cdots, \delta_n$.

Theorem 3.8. *Let (X, δ) be an $\mathscr{L}_S^n$–proximity space induced by $\delta_1, \cdots, \delta_n$. If a subset E of X is defined to be $\mathscr{L}_S^n$–open if and only if $E\delta V$ for some $V \subseteq X$ with $\{x \in X : x\delta_a(X \backslash V)\} = X \backslash V$ for any $a \in S$, then the collection of all $\mathscr{L}_S^n$–open sets so defined yields an $\mathscr{L}_S^n$–topological space (on X) with $\mathscr{L}_S^n{-}O(X) = \mathscr{L}_S^n{-}O(X)(\delta)$.*

Proof.

We will show that the desired $\mathscr{L}_S^n$–topological space is precisely the $\mathscr{L}_S^n$–topological space $(X, \tau(\delta_1), \cdots, \tau(\delta_n))$. We first notice that if $V \subseteq X$ with $\{x \in X : x\delta_a(X \setminus V)\} = X \setminus V$ for any $a \in S$ in X, then $V \in \mathscr{L}_S^n - O(X, \tau(\delta_1), \cdots, \tau(\delta_n))$, and the converse is true as well. Let $E \in \mathscr{L}_S^n - O(X, \tau(\delta_1), \cdots, \tau(\delta_n))$. There exists a set $V \in \bigcap_{a \in S} \tau_a$ with

$$V \subseteq E \subseteq \bigcup_{\{i \in [n] \,:\, i \notin S\}} \overline{V}^i,$$

where for any $i \in [n]$, $\overline{V}^i$ is the closure set of V with respect to τ_i. So, we have

$$\bigcap_{\{i \in [n] \,:\, i \notin S\}} X \setminus \overline{V}^i \subseteq X \setminus E \subseteq X \setminus V.$$

The statement

$$\bigcap_{\{i \in [n] \,:\, i \notin S\}} X \setminus \overline{V}^i \subseteq X \setminus E$$

shows that $x\underline{\delta_i}V$ for any $i \in [n] \setminus S$ implies $x \notin E$, and the statement

$$X \setminus E \subseteq X \setminus V$$

asserts that $x \notin E$ implies $x\delta_a(X \backslash V)$ for any $a \in S$ (since V is $\tau(\delta_a)$–closed for every $a \in S$). Thus, $E\delta V$. Conversely, if $E\delta V$ for some $V \in \bigcap_{a \in S} \tau_a$, then $V \in \bigcap_{a \in S} \tau_a$, and the following statements are satisfied.

(i) $x\underline{\delta_i}V$ for any $i \in [n] \setminus S$ implies $x \notin A$.
(ii) $x \notin E$ implies $x\delta_a(X \setminus V)$ for any $a \in S$.

The statements ((i)) and ((ii)) imply that

$$\bigcap_{\{i \in [n] \,:\, i \notin S\}} X \setminus \overline{V}^i \subseteq X \setminus E$$

and

$$X \setminus E \subseteq X \setminus V$$

respectively. This proves that $E \in \mathscr{L}_S^n - O(X, \tau(\delta_1), \cdots, \tau(\delta_n))$ which completes the proof. □

It might be noticeable that the relation δ defined in Definition (3.7) is not symmetric, in general; that is, $A\delta B$ and $B\delta A$ need not be the same. Nevertheless, we have the following consequence.

Theorem 3.9. *Let (X, δ) be an $\mathscr{L}_S^n$–proximity space induced by $\delta_1, \cdots, \delta_n$, and let $B \subseteq C$.*

(i) *If $A\delta C$, then $A\delta B$.*
(ii) *If $A\delta\overline{B}^{\delta}$, then $A\delta B$, where $\overline{B}^{\delta}$ is the closure with respect to the $\mathscr{L}_S^n$–topological space with $\mathscr{L}_S^n - O(X) = \mathscr{L}_S^n - O(X)(\delta)$.*

Proof.
(i) Let $A\delta C$, and let $x \notin A$. Since $A\delta C$, this implies $x\delta_a(X \setminus C)$ for any $a \in S$. Since $B \subseteq C$, $(X \setminus C) \subseteq (X \setminus B)$. Thus, $x \notin A$ implies $x\delta_a(X \setminus B)$ for any $a \in S$. Let $x\underline{\delta}_i B$ for any $i \in [n] \setminus S$. It follows that
$$x \ll_{\delta_i} X \setminus C$$
for any $i \in [n] \setminus S$. Since $(X \setminus C) \subseteq (X \setminus B)$,
$$x \ll_{\delta_i} X \setminus B$$
for any $i \in [n] \setminus S$. Consequently, $A\delta B$. implies $x \notin A$.
(ii) This follows directly from part (i) and Theorem (3.8).

□

Definition 3.10. *Let (X, δ) and (X, δ') be $\mathscr{L}_S^n$–proximity spaces induced by $\delta_1, \cdots, \delta_n$ and $\delta'_1, \cdots, \delta'_n$ respectively. We define*
$$\delta < \delta' \text{ iff } A\delta' B \text{ implies } A\delta B$$
for any subsets A and B of X.

Theorem 3.11. *Let (X, δ) and (X, δ') be an $\mathscr{L}_S^n$–proximity spaces induced by $\delta_1, \cdots, \delta_n$ and $\delta'_1, \cdots, \delta'_n$ respectively, and let $\delta'_a < \delta_a$ for any $a \in S$. Then $\delta < \delta'$ implies $\mathscr{L}_S^n - O(X)(\delta) \subseteq \mathscr{L}_S^n - O(X)(\delta')$.*

Proof. Let $\delta < \delta'$, and let $E \in \mathscr{L}_S^n - O(X)(\delta)$. It follows from Theorem (3.8) that $E\delta V$ for some $V \in \bigcap_{a\in S} \tau(\delta_a)$. Since $\delta'_a < \delta_a$ for any $a \in S$, by Theorem (3.5), we have $\tau(\delta_a) \subseteq \tau(\delta_a)$ for any $a \in S$, and hence $\bigcap_{a\in S} \tau(\delta_a) \subseteq \bigcap_{a\in S} \tau(\delta'_a)$. Consequently, we have $E\delta V$ with $V \in \bigcap_{a\in S} \tau(\delta'_a)$. Theorem (3.8) asserts that $E \in \mathscr{L}_S^n - O(X)(\delta')$. Therefore, $\mathscr{L}_S^n - O(X)(\delta) \subseteq \mathscr{L}_S^n - O(X)(\delta')$. □

Definition 3.12. *Let (X, δ) and (Y, δ') be an $\mathscr{L}_S^n$–proximity spaces induced by $\delta_1, \cdots, \delta_n$ and $\delta'_1, \cdots, \delta'_n$ respectively. A map $f : (X, \delta) \to (Y, \delta')$ is said to be an $\mathscr{L}_S^n$–proximity map if any $B \subseteq Y$ with $\{y \in Y : y\delta'_a(Y \setminus B)\} = Y \setminus B$ for any $a \in S$ implies $f^{-1}(B)\delta V$, for some $V \subseteq X$ with $\{x \in X : x\delta_a(X \setminus V)\} = X \setminus V$ for any $a \in S$ in X.*

Theorem 3.13. *Let (X, δ) and (Y, δ') be an $\mathscr{L}_S^n$–proximity spaces induced by $\delta_1, \cdots, \delta_n$ and $\delta'_1, \cdots, \delta'_n$ respectively, and let $f : X \to Y$ be a map. Then $f : (X, \delta) \to (Y, \delta')$ is an $\mathscr{L}_S^n$–proximity map iff $f : (X, \tau(\delta_1), \cdots, \tau(\delta_n)) \to (Y, \tau(\delta'_1), \cdots, \tau(\delta'_n))$ is $\mathscr{L}_S^n$–continuous.*

Proof. Suppose that $f : (X, \delta) \to (Y, \delta')$ is an $\mathscr{L}_S^n$–proximity map. Let $Y \setminus W \in \bigcap_{a\in S} \tau(\delta'_a)$. So, W is $\bigcap_{a\in S} \tau(\delta'_a)$–closed and hence $\tau(\delta'_a)$–closed for any $a \in S$. It follows that the set $\{y \in Y : y\delta'_a(Y \setminus W)\} = Y \setminus W$ for any $a \in S$. Since $f : (X, \delta) \to (Y, \delta')$ is an $\mathscr{L}_S^n$–proximity map, $f^{-1}(W)\delta V$, for some $V \subseteq X$ with $\{x \in X : x\delta_a(X \setminus V)\} = X \setminus V$ for any $a \in S$ in X. As a consequence, $f^{-1}(W)\delta V$, for some $V \subseteq X$. By Theorem (3.8), $f^{-1}(W) \in \mathscr{L}_S^n - O(X, \tau(\delta_1), \cdots, \tau(\delta_n))$. Thus, $f : (X, \tau(\delta_1), \cdots, \tau(\delta_n)) \to (Y, \tau(\delta'_1), \cdots, \tau(\delta'_n))$ is $\mathscr{L}_S^n$–continuous. For the other direction, suppose that $f : (X, \tau(\delta_1), \cdots, \tau(\delta_n)) \to (Y, \tau(\delta'_1), \cdots, \tau(\delta'_n))$ is $\mathscr{L}_S^n$–continuous. Let $B \subseteq Y$ with $\{y \in Y : y\delta'_a(Y \setminus B)\} = Y \setminus B$ for any $a \in S$. This implies that B is $\tau(\delta'_a)$–closed for any $a \in S$ and hence $Y \setminus B \in \tau(\delta'_a)$ for any $a \in S$. So, $Y \setminus B \in \bigcap_{a \in S} \tau(\delta'_a)$. Since $f : (X, \tau(\delta_1), \cdots, \tau(\delta_n)) \to (Y, \tau(\delta'_1), \cdots, \tau(\delta'_n))$ is $\mathscr{L}_S^n$–continuous, $f^{-1}(W) \in \mathscr{L}_S^n - O(X, \tau(\delta_1), \cdots, \tau(\delta_n))$. By Theorem (3.8), $f^{-1}(W)\delta V$, for some $V \subseteq X$, and this completes the proof. □

4. Descriptive $\mathscr{L}_S^n$–proximity Spaces

Following [3] and [4], we recall some basic concepts of digital topology. A *probe* Φ maps a member of a set to a value in $\mathbb{R}$ (reals). Probe function values define feature vectors useful in comparing, clustering and classifying members of a set. One can find open sets in digital images. Let $\Phi(x)$ denote a feature vector for the object x, i.e., a vector of feature values that describe x. A feature vector provides a description of an object and subsets of X. Let Φ denote a set of n real-valued probe functions $\Phi : X \to \mathbb{R}$ representing features such as greylevel intensity, colour, shape or texture of a point x (picture element) in a digital image X, i.e.,

$$\Phi = \{\phi_1, \cdots, \phi_n\}.$$

And let $\Phi(x)$ denote a feature vector containing numbers representing feature values extracted from x. Then, for a set of n probe functions, a feature vector has the following form:

$$\Phi(x) = \{\phi_1(x), \cdots, \phi_n(x)\},$$

where $\phi_i(x)$ is the ith feature value. To obtain a descriptive proximity relation (denoted by δ_Φ), one first chooses a set of probe functions, which provides a basis for describing points in a set. Let $A, B \in \mathscr{P}(X)$. Let $\mathcal{Q}(A), \mathcal{Q}(B)$ denote sets of descriptions of points in A, B, respectively. That is,

$$\mathcal{Q}(A) = \{\Phi(a) : a \in A\},\ \mathcal{Q}(B) = \{\Phi(b) : b \in B\}.$$

The expression $A\delta_\Phi B$ reads A is descriptively near B. The relation δ_Φ is called a descriptive proximity relation. Similarly, $A\underline{\delta}_\Phi B$ denotes that A is descriptively far (remote) from B. The descriptive proximity of A and B is defined by

$$A\delta_\Phi B \text{ if and only if } \mathcal{Q}(A) \cap \mathcal{Q}(B) \neq \emptyset. \tag{4.1}$$

The descriptive intersection $\underset{\Phi}{\cap}$ of A and B is defined by

$$A \underset{\Phi}{\cap} B = \{x \in A \cup B : \mathcal{Q}(x) \in \mathcal{Q}(A) \text{ and } \mathcal{Q}(x) \in \mathcal{Q}(B)\}. \tag{4.2}$$

The descriptive proximity relation δ_Φ is defined by

$$\delta_\Phi = \{(A, B) \in (\mathscr{P}(X) \times \mathscr{P}(X)) : clA \underset{\Phi}{\cap} clB \neq \emptyset\}. \tag{4.3}$$

Whenever sets A and B have no points with matching (or almost near) descriptions, the sets are descriptively far from each other (denoted by $A\underline{\delta}_\Phi B$), where

$$\underline{\delta}_\Phi = (\mathscr{P}(X) \times \mathscr{P}(X)) \setminus \delta_\Phi. \tag{4.4}$$

In general, a binary relation δ_Φ is a *descriptive EF-proximity*, provided the following axioms are satisfied for $A, B, C \in \mathscr{P}(X)$.

(EF$_\Phi$.1) A descriptively close to B implies $A \neq \emptyset$, $B \neq \emptyset$.

(EF$_\Phi$.2) $A \underset{\Phi}{\cap} B$ implies A is descriptively close to B.

(EF$_\Phi$.3) A descriptively close to B implies B descriptively close to A (descriptive

symmetry).
(EF_Φ.4) A descriptively close to $(B \cup C)$, if and only if, A descriptively close to B or A descriptively close to C.
(EF_Φ.5) Descriptive Efremovic axiom: A descriptively far from B implies A descriptively far from C and B descriptively far from $X \setminus C$ for some $C \in \mathscr{P}(X)$.
The descriptive proximity relation δ_Φ reads descriptively close to (descriptively near). The structure (X, δ_Φ) is a *descriptive EF-proximity space* (or, briefly, *descriptive EF space*, or even *descriptive space*). The remoteness proximity relation δ_Φ reads descriptively far from (or descriptively remote from or descriptively not close to). For basic concepts of descriptive spaces and digital topology, we refer the reader to [3] and [4].

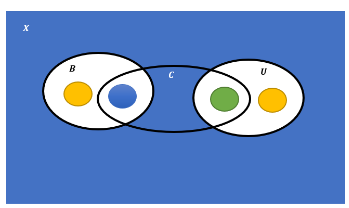
Definition 4.1. *Let $S \subsetneqq [n]$ and $\Phi^{(i)}$ be a set of probe functions representing features of picture points in X for any $i \in [n]$. Let $(X, \delta^{(i)}_{\Phi^{(i)}})$ be a descriptive proximity space for any $i \in [n]$. Then the $\mathscr{L}^n_S$–descriptive proximity space induced by $\delta^{(1)}_{\Phi^{(1)}}, \cdots, \delta^{(n)}_{\Phi^{(n)}}$ is the $\mathscr{L}^n_S$–proximity space (X, δ_Φ) induced by $\delta^{(1)}_{\Phi^{(1)}}, \cdots, \delta^{(n)}_{\Phi^{(n)}}$.*

The following is an immediate consequence of Theorem (3.8),

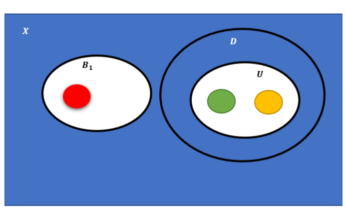
Theorem 4.2. *Let (X, δ_Φ) be an $\mathscr{L}^n_S$–descriptive proximity space induced by $\delta^{(1)}_{\Phi^{(1)}}, \cdots, \delta^{(n)}_{\Phi^{(n)}}$, where $\Phi^{(i)}$ is a set of probe functions representing features of picture points in X for any $i \in [n]$. Then (X, δ_Φ) induces an $\mathscr{L}^n_S$–topological space on X with $\mathscr{L}^n_S - O(X)(\delta_\Phi) = \mathscr{L}^n_S - O(X, \tau(\delta^{(1)}_{\Phi^{(1)}}), \cdots, \tau(\delta^{(n)}_{\Phi^{(n)}}))$.*

Example 4.3. *We use Theorem (2.5) to calculate*

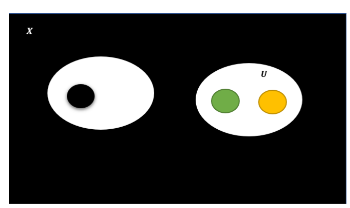
$$A = \bigcup_{\{i \in [6] \,:\, i \neq 2\}} \overline{U}^i \in \mathscr{L}^6_2 - O(X).$$



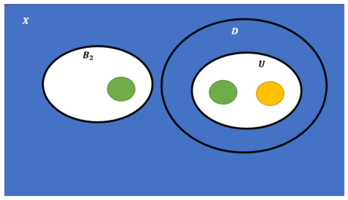
$U \in \tau(\delta^{(2)}_{\Phi^{(2)}})$



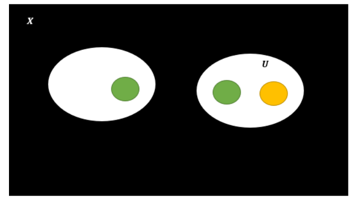
$(X, \tau(\delta^{(1)}_{\Phi^{(1)}}))$



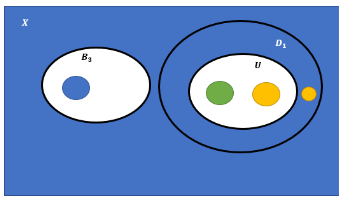
$\overline{U}^1$



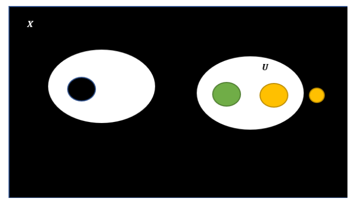
$(X, \tau(\delta^{(3)}_{\Phi^{(3)}}))$



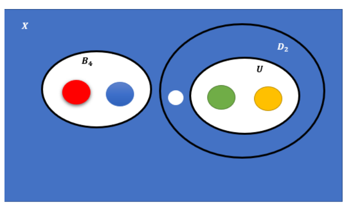
$\overline{U}^3$



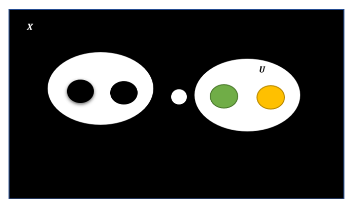
$(X, \tau(\delta^{(4)}_{\Phi^{(4)}}))$



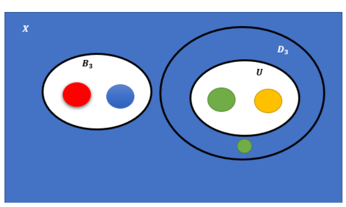
$\overline{U}^4$



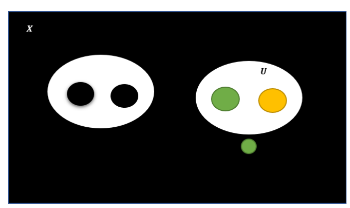
$(X, \tau(\delta^{(5)}_{\Phi^{(5)}}))$



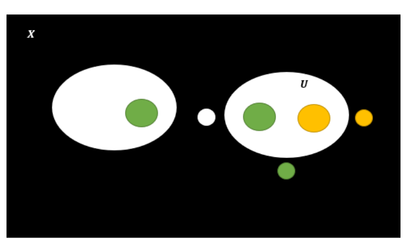
$\overline{U}^5$



$(X, \tau(\delta^{(6)}_{\Phi^{(6)}}))$



$\overline{U}^6$

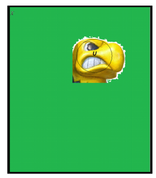


$A \in \mathscr{L}_2^6 - O(X)(\delta_\Phi)$

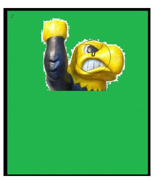
The proximity relations $\delta^{(i)}_{\Phi^{(i)}}$ *are defined as in (4.1) for every* $i \in [6]$. *Here, a black color corresponds to* $0 =$ *lowest intensity, and a white color represents* $255 =$ *highest intensity.*

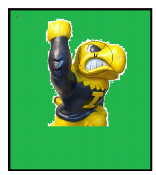
Example 4.4. *We use an image of Iowa "Hawkeyes Herky" and Theorem (2.5) to calculate*

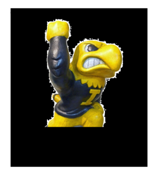
$$A = \bigcup_{\{i\in[6]\,:\,i\neq 4\}} \overline{U}^i \in \mathscr{L}^6_4 - O(X).$$



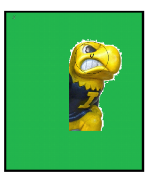
$U \in \tau(\delta^{(4)}_{\Phi})$

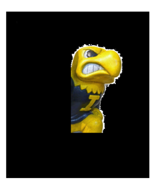


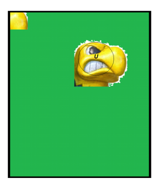


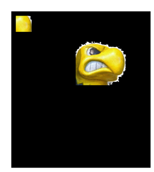


$(X, \tau(\delta^{(1)}_{\Phi^{(1)}}))$ $\overline{U}^1$ $(X, \tau(\delta^{(2)}_{\Phi^{(2)}}))$ $\overline{U}^2$

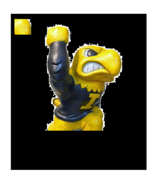








$(X, \tau(\delta^{(3)}_{\Phi^{(3)}}))$ $\overline{U}^3$ $(X, \tau(\delta^{(5)}_{\Phi^{(5)}}))$ $\overline{U}^5$



$(X, \tau(\delta^{(6)}_{\Phi^{(6)}}))$ $\overline{U}^6$ $A \in \mathscr{L}^6_4 - O(X)(\delta_\Phi)$

As in the previous example, the proximity relations $\delta^{(i)}_{\Phi^{(i)}}$ *are defined as in (4.1) for every* $i \in [6]$, *a black color corresponds to* $0 =$ *lowest intensity, and a white color represents* $255 =$ *highest intensity.*

References

[1] J.C Kelly. Bitopological Spaces. Proceedings of the London Mathematical Society.13, 71-89. (1963).

[2] S. A. Naimpally, B. D. Warrack. Proximity Spaces. Cambridge Tracts in Mathematics and Mathematical Physics. 59 (1970). Cambridge: Cambridge University Press. ISBN 0-521-07935-7.

[3] J. F. Peters. Computational Proximity. Excursions in the Topology of Digital Images. Intelligent Systems Reference Library, 102. Springer, [Cham], 2016. doi: 10.1007/978-3-319-30262-1 (2016).

[4] J. F. Peters. Topology of Digital Images. Visual Pattern Discovery in Proximity Spaces, Intelligent Systems Reference Library, 63, Berlin: Springer (2014).

Adnan Hashim Abdulwahid
Department of Mathematics, Centro de Tecnologia Texas A&M University–Texarkana, 7101 University Ave Texarkana, TX 75503, USA
E-mail address: AAbdulwahid@tamut.edu

Elgaddafi Elamami
Mathematics and Computer Science Department, Southern Arkansas University, 100 E. University MSC 9255, Magnolia, Arkansas, 71753, USA
E-mail address: eelamami@saumag.edut