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ON INVERSE SOURCE PROBLEM FOR SOBOLEV EQUATION WITH MITTAG-LEFFLER KERNEL IN L^r SPACE

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Abstract. In this paper, we consider a Sobolev equation with the Atangana-Baleanu-Caputo fractional derivative. We give the explicit fomula of the source term. Under the observations of tha data in L^r spaces, we provide a regularized solution using Fourier truncated method. We give the error estimate between the exact solution and the regularized solution. The main tool is of using some embeddings.

1. Introduction

Let $\Omega \in \mathbb{R}^N (N \geq 1)$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. In this paper, we are interested to study time fractional diffusion equation with fractional derivative as follows

$$\begin{cases} {}_0^{ABC}D_t^{\beta}\big(u(x,t) + m\mathcal{L}u(x,t)\big) + \mathcal{L}u(x,t) = g(t)f(x), & x \in \Omega,\ t \in (0,1), \\ u(x,t) = 0, & x \in \partial\Omega,\ t \in (0,1), \\ u(x,1) = \rho(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Here in the main equation as above, the Atangana - Baleanu fractional derivative ${}_0^{\mathrm{ABC}}D_t^{\beta}u(x,t)$ is defined by

$${}_0^{\mathrm{ABC}}D_t^{\beta}u(x,t) = \frac{\mathcal{M}(\beta)}{1-\beta}\int_0^t \frac{\partial u(x,s)}{\partial s} E_{\beta,1}\Big(\frac{-\beta(t-s)^{\beta}}{1-\beta}\Big)ds, \tag{1.2}$$

where the normalization function $\mathcal{M}(\beta)$ can be any function satisfying the conditions $\mathcal{M}(\beta) = 1 - \beta + \frac{\beta}{\Gamma(\beta)}$, here $\mathcal{M}(0) = \mathcal{M}(1) = 1$ (see Definition 2.1 in [1]) and $E_{\beta,1}$ is the MittagLeffler function. Our main goal in this paper is of finding the source term $f(x)$ from the given data ρ and the measured data at the final time

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$u(x,1) = \rho(x)$, $\rho \in L^2(\Omega)$ such that

$$\|g - g_\epsilon\|_{L^r(0,1)} + \|\rho - \rho_\epsilon\|_{L^r(\Omega)} \leq \epsilon. \tag{1.3}$$

One of the narrow branches of fractional analysis is the theory of fractional diffusion equations. Fractional-time diffusion equations are used to model complex phenomena such as long-term memory or spatial interactions, non-local and local dynamics. For details, please refer to documents [2, 3, 5, 6, 7, 8, 18]. One of the modern trends in fractional analysis is the development of fractional operators with non-singular kernels. The study of these fractional derivatives is important to satisfy the need for modeling applications in various fields, such as fluids, mechanics, viscoelasticity, biology, physics and engineering, see in [3, 4, 17, 9, 10, 11, 12, 13, 21, 22, 23, 24]. Several definitions of fractional derivatives have been given based on non-special nuclei such as Atangana-Baleanu-Caputo fractions and derivatives. Regarding the study of (1.1) problem with Atangana-Baleanu derivative, we list some previous results as follows

- Under the case $F = F(x,t)$, from [19], the kernels of the extended Mittag-Leffler type functions are studied in this study using a partial differential equation model with the new universal fractional derivatives. Analysis and consideration are given to an initial boundary value problem for the anomalous diffusion of fractional order. The Mittag-Leffler kernel fractional derivative, also known as the Atangana-Babeanu fractional derivative in time, is interpreted in the Caputo sense. They discovered findings on the existence, uniqueness, and regularity of the solution.
- Under the case $F = g(t)f(x)$, from [14, 16], the problem of determining inverse source problem for fractional diffusion equation containing Atangana-Baleanu-Caputo fractional derivative. We first establish an explicit formula of the source term from the average data of the function in the time variable. We then show that the inverse source problem is ill-posed in the meaning of Hadamard i.e., the source function is not stable according to the given data. To overcome this instability, we propose a regularized method as in the Fractional Landweber method. We also obtain the upper bounds and find the convergence rate between the regularized solution and sought source function. Estimates are also derived in two cases on selection rules, a priori parameter, and a posterior parameter. Numerical examples are given which illustrate the usefulness of our method.
- Under the case $F = F(x,t,u(x,t))$, in the paper [15], they investigated a nonlinear time fraction Volterra equation with a Mittag-Leffler multiplier in Hilbert space. By applying the properties of the Mittag-Leffler function and the eigenvalue expansion, the existence of a light solution to our problem has been proved. The main tool to prove our results is the use of some Sobolev embeddings and some fixed point theorems.

As we know, the inverse issue for diffusion equation with Atangana-Babeanu fractional derivative where the observed data is in the $L^p(\Omega)$ space with $p \neq 2$ is solved for the first time in this study. One significant challenge is that because the data is not in $L^2(\Omega)$, we cannot utilize Parseval equality directly. We get around these issues by leveraging embedding between $L^p(\Omega)$ and $\mathcal{H}^s(\Omega)$ scale-spaces. We have the regularized solution through the Fourier series truncation method with the observed data $\big(\varphi_\epsilon, g_\epsilon\big) \in L^r(0,T) \times L^r(\Omega)$. After that, the error established between the regularized solution and the exact solution in the Theorem (3.1), by the main analytical technique is to use some embeddings and some evaluations using Hlder inequality.

The structure of our paper is as follows. The existence of mild solution u to (1.1) in Section 1.In the Section 2, we have some preliminaries. The main results in Section 3 is Theorem 3.1, our main tool is Sobolev embeddings.

2. Preliminaries

Let us recall that the spectral problem

$$\begin{cases} -\mathcal{L}\varphi_n(x) = \xi_n\varphi_n(x), & \text{in } \Omega, \\ \varphi_n(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

admits a family of eigenvalues

$$0 < \xi_1 \le \xi_2 \le \xi_3 \le ... \le \xi_n \le ... \nearrow \infty.$$

For all $r \ge 0$, the operator $\mathcal{L}^r$ (here $\mathcal{L} = -\Delta$) also possesses the following representation:

$$\mathcal{L}^r h = \sum_{n=1}^{\infty} \Big(\int_\Omega h(x)\varphi_n(x)dx \Big) \xi_n^r \varphi_n,$$

$$h \in \mathcal{H}^r(\Omega) = \Big\{ h \in L^2(\Omega) : \sum_{n=1}^{\infty} \Big(\int_\Omega h(x)\varphi_n(x)dx \Big)^2 \xi_n^{2r} < \infty \Big\}. \tag{2.2}$$

Consider on $\mathcal{H}^r$ the norm

$$\|h\|_{\mathcal{H}^r(\Omega)} = \Big(\sum_{n=1}^{\infty} \Big(\int_\Omega h(x)\varphi_n(x)dx \Big)^2 \xi_n^{2r} \Big)^{\frac{1}{2}}, \quad h \in \mathcal{H}(\mathcal{A}^r).$$

Lemma 2.1. *([6]) Let $0 < \beta < 1$, then there exist $0 < \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ such that*

$$\frac{\mathcal{B}_1}{1+y} \le E_{\beta,1}(-y) \le \frac{\mathcal{B}_2}{1+y}, \; E_{\beta,\alpha}(-y) \le \frac{\mathcal{B}_3}{1+y}, \; \text{for all } y \ge 0, \; \alpha \in \mathbb{R}. \tag{2.3}$$

Lemma 2.2. *For $\xi > 0, \beta > 0$, $m \in \mathbb{N}^*$, we have*

$$\begin{aligned} &\frac{d^m}{dt^m} E_{\beta,1}(-\xi t^\beta) = -\xi t^{\beta-m} E_{\beta,\beta-m+1}(-\xi t^\beta), \\ &\frac{d}{dt}\big(t E_{\beta,2}(-\xi t^\beta)\big) = E_{\beta,1}(-\xi t^\beta), \\ &\frac{d}{dt}\big(t^{\beta-1} E_{\beta,\beta}(-\xi t^\beta)\big) = -t^{\beta-2} E_{\beta,\beta-1}(-\xi t^\beta). \end{aligned} \tag{2.4}$$

Lemma 2.3. [6] *For $t > 0$, and $\xi > 0$, and $0 < \beta < 1$, then one has*

$$\partial_t^\beta E_{\beta,1}(-\xi t^\beta) = -\,\xi E_{\beta,1}(-\xi t^\beta),$$

Lemma 2.4. *For $\beta \in (0,1)$, putting $\mathcal{A}_{3,n}(m,\beta) = \dfrac{\mathcal{M}(\beta)}{\xi_n\big(\mathcal{M}(\beta) + \xi_n\sigma_n(1-\beta)\big)}$, it gives*

$$\mathcal{A}_{3,n}(m,\beta) \ge \frac{1}{\xi_n} \frac{m\mathcal{M}(\beta)}{\Big(\frac{\mathcal{M}(\beta)}{\xi_1\sigma_1} + (1-\beta)\Big)}. \tag{2.5}$$

Proof. First of all, we notice that $\xi_n\sigma_n = \frac{\xi_n}{1+m\xi_n} \geq \frac{1}{\frac{1}{\xi_1}+m} = \frac{1}{\xi_1^{-1}+m}$, and we have

$$\mathcal{A}_{3,n}(m,\beta) = \frac{\mathcal{M}(\beta)}{\xi_n\big(\mathcal{M}(\beta)+\xi_n\sigma_n(1-\beta)\big)} \geq \frac{\mathcal{M}(\beta)}{\frac{\xi_n^2}{1+m\xi_n}\Big(\frac{\mathcal{M}(\beta)}{\xi_n\sigma_n}+(1-\beta)\Big)}$$
$$\geq \frac{m\mathcal{M}(\beta)}{\xi_n\Big(\frac{\mathcal{M}(\beta)}{\xi_n\sigma_n}+(1-\beta)\Big)} \geq \frac{1}{\xi_n}\frac{m\mathcal{M}(\beta)}{\Big(\frac{\mathcal{M}(\beta)}{\xi_1\sigma_1}+(1-\beta)\Big)}. \tag{2.6}$$

□

Lemma 2.5. *For $M>0$, by Lemmas 2.1 and 2.2, we have*

$$\int_0^M \Big|t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n t^\alpha\right)\Big|dt = \int_0^M t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n t^\alpha\right)dt$$
$$= -\frac{1}{\lambda_n}\int_0^\eta \frac{d}{dt}E_{\alpha,1}\left(-\lambda_n t^\alpha\right)dt = \frac{1}{\lambda_n}\left(1-E_{\alpha,1}\left(-\lambda_n M^\alpha\right)\right). \tag{2.7}$$

Lemma 2.6. *Let $\beta\in(0,1)$, we have estimate*

$$\frac{1}{\xi_n}\frac{m\mathcal{M}(\beta)}{\Big(\frac{\mathcal{M}(\beta)}{\xi_1\sigma_1}+(1-\beta)\Big)}\left[1-E_{\beta,1}\Big(-\frac{\beta(\xi_1^{-1}+m)^{-1}}{\mathcal{M}(\beta)+\big(\xi_1^{-1}+m\big)^{-1}(1-\beta)}\Big)\right]$$
$$\leq \frac{\sigma_n\beta\mathcal{M}(\beta)}{\big(\mathcal{M}(\beta)+\sigma_n\xi_n(1-\beta)\big)^2}\int_0^1 E_{\beta,\beta}\Big(-\frac{\beta\xi_n\sigma_n(1-s)^\beta}{\mathcal{M}(\beta)+\sigma_n\xi_n(1-\beta)}\Big)(1-s)^{\beta-1}ds \leq \frac{1}{\xi_n}. \tag{2.8}$$

Proof. For $E_{\beta,\beta}(-z)\geq 0$ for $0<\beta<1$ and $z\geq 0$, and using the Lemmas 2.4 and 2.6, we obtain

$$a)\frac{\sigma_n\beta\mathcal{M}(\beta)}{\big(\mathcal{M}(\beta)+\sigma_n\xi_n(1-\beta)\big)^2}\int_0^1 E_{\beta,\beta}\Big(-\frac{\beta\xi_n\sigma_n(1-s)^\beta}{\mathcal{M}(\beta)+\sigma_n\xi_n(1-\beta)}\Big)(1-s)^{\beta-1}ds$$
$$\geq \frac{1}{\xi_n}\frac{m\mathcal{M}(\beta)}{\Big(\frac{\mathcal{M}(\beta)}{\xi_1\sigma_1}+(1-\beta)\Big)}\left[1-E_{\beta,1}\Big(-\frac{\beta(\xi_1^{-1}+m)^{-1}}{\mathcal{M}(\beta)+\big(\xi_1^{-1}+m\big)^{-1}(1-\beta)}\Big)\right]. \tag{2.9}$$

$$b)\frac{\sigma_n\beta\mathcal{M}(\beta)}{\big(\mathcal{M}(\beta)+\sigma_n\xi_n(1-\beta)\big)^2}\int_0^1 E_{\beta,\beta}\Big(-\frac{\beta\xi_n\sigma_n(1-s)^\beta}{\mathcal{M}(\beta)+\sigma_n\xi_n(1-\beta)}\Big)(1-s)^{\beta-1}ds$$
$$= -\frac{\mathcal{M}(\beta)}{\xi_n\big(\mathcal{M}(\beta)+\xi_n\sigma_n(1-\beta)\big)}\int_0^1 \frac{d}{ds}\Big(E_{\beta,1}\big(-\frac{\beta\xi_n\sigma_n(1-s)^\beta}{\mathcal{M}(\beta)+\sigma_n\xi_n(1-\beta)}\big)\Big)d\tau$$
$$= \frac{1}{\xi_n}\Big(1-E_{\beta,1}\big(-\frac{\beta\xi_n\sigma_n}{\mathcal{M}(\beta)+\sigma_n\xi_n(1-\beta)}\big)\Big) \leq \frac{1}{\xi_n}. \tag{2.10}$$

□

Lemma 2.7. [17] *The following statement are true:*

$$\left.\begin{aligned} L^r(\Omega) &\hookrightarrow \mathcal{H}^s(\Omega), \quad \text{if} \quad -\frac{N}{4} < s \le 0, \quad r \ge \frac{2N}{N-4s}, \\ \mathcal{H}^s(\Omega) &\hookrightarrow L^r(\Omega), \quad \text{if} \quad 0 \le s < \frac{N}{4}, \quad r \le \frac{2N}{N-4s}. \end{aligned}\right\} \tag{2.11}$$

3. The inverse source problem (1.1)

Let us first to review the initial value problem as follows

$$\begin{cases} {}_0^{ABC}D_t^{\beta}\big(u(x,t) + m\mathcal{L}u(x,t)\big) + \mathcal{L}u(x,t) = F(x,t), & \text{in } \Omega \times (0,1], \\ u(x,t) = 0, & \text{on} \quad \partial\Omega \times (0,1], \\ u(x,0) = u_0(x), & \text{in} \quad \Omega, \end{cases} \tag{3.1}$$

where u_0 and F are given functions. Let $u(x,t) = \sum_{j=1}^{\infty} u_n(t)\varphi_n(x)$ be the Fourier series in $L^2(\Omega)$ with $u_n(t) = \int_{\Omega} u(x,t)\varphi_n(x)dx$, then we have the fractional integro-differential equation involving the Atangana-Baleanu fractional derivative in the form

$${}_0^{ABC}D_t^{\beta}\big(1 + m\xi_n\big)u_n(t) + \xi_n u_n(t) = F_n(t), \tag{3.2}$$

in [20], the solution (3.1) can be represented as by Fourier series

$$u(x,t) = \sum_{n=1}^{\infty} \Big(\int_{\Omega} u(x,t)\varphi_n(x)dx \Big) \varphi_n(x)$$

and then given by

$$\begin{aligned} u_n(t) = {} & \frac{\mathcal{M}(\beta)}{\mathcal{M}(\beta) + \frac{\xi_n}{1+m\xi_n}\big(1-\beta\big)} E_{\beta,1}\Big(\frac{-\beta \frac{\xi_n}{1+m\xi_n} t^{\beta}}{\mathcal{M}(\beta) + \frac{\xi_n}{1+m\xi_n}\big(1-\beta\big)} \Big) u_{0,n} \\ & + \Big(\frac{1}{1+m\xi_n}\Big) \frac{1-\beta}{\mathcal{M}(\beta) + \frac{\xi_n}{1+m\xi_n}\big(1-\beta\big)} F_n(t) \\ & + \Big(\frac{1}{1+m\xi_n}\Big) \frac{\beta\mathcal{M}(\beta)}{\Big(\mathcal{M}(\beta) + \frac{\xi_n}{1+m\xi_n}\big(1-\beta\big)\Big)^2} \\ & \quad \times \int_0^t E_{\beta,\beta}\Big(\frac{-\beta \frac{\xi_n}{1+m\xi_n} (t-s)^{\beta}}{\mathcal{M}(\beta) + \frac{\xi_n}{1+m\xi_n}\big(1-\beta\big)} \Big) (t-s)^{\beta-1} F_n(\tau) d\tau. \end{aligned} \tag{3.3}$$

Let us denote $\sigma_n = (1+m\xi_n)^{-1}$, this implies that

$$\begin{aligned} u(x,t) = &\sum_{n=1}^{\infty} \frac{\mathcal{M}(\beta)}{\mathcal{M}(\beta)+\xi_n\sigma_n\big(1-\beta\big)} E_{\beta,1}\Big(\frac{-\beta\xi_n\sigma_n t^\beta}{\mathcal{M}(\beta)+\xi_n\sigma_n\big(1-\beta\big)}\Big)\big\langle u(0),\varphi_n\big\rangle\varphi_n(x), \\ &+\sum_{n=1}^{\infty} \frac{\sigma_n\big(1-\beta\big)}{\mathcal{M}(\beta)+\xi_n\sigma_n\big(1-\beta\big)}\big\langle F(\cdot,t),\varphi_n\big\rangle\varphi_n(x) \\ &+\sum_{n=1}^{\infty} \frac{\sigma_n\beta\mathcal{M}(\beta)}{\Big(\mathcal{M}(\beta)+\xi_n\sigma_n\big(1-\beta\big)\Big)^2} \\ &\times\int_0^t E_{\beta,\beta}\Big(\frac{-\beta\xi_n\sigma_n(t-\tau)^\beta}{\mathcal{M}(\beta)+\xi_n\sigma_n\big(1-\beta\big)}\Big)(t-s)^{\beta-1}\big\langle F(\cdot,s),\varphi_n\big\rangle ds\varphi_n(x)\cdot \end{aligned} \tag{3.4}$$

Let us now return the problem of identifying the source term. Let $t=1, u(x,0)=0, F(x,t)=g(t)f(x)$, and $F_n(1)=0$, we get

$$\begin{aligned} \int_\Omega \rho(x)\varphi_n(x)dx = &\int_\Omega f(x)\varphi_n(x)dx \ \frac{\sigma_n\beta\mathcal{M}(\beta)}{\Big(\mathcal{M}(\beta)+\xi_n\sigma_n\big(1-\beta\big)\Big)^2} \\ &\times\int_0^1 E_{\beta,\beta}\Big(\frac{-\beta\xi_n\sigma_n(1-s)^\beta}{\mathcal{M}(\beta)+\xi_n\sigma_n\big(1-\beta\big)}\Big)(1-s)^{\beta-1}g(s)ds. \end{aligned} \tag{3.5}$$

To make the formula even more compact, we put

$$\begin{aligned} \mathcal{A}_{1,n}(m,\beta) &= \frac{\sigma_n\beta\mathcal{M}(\beta)}{\Big(\mathcal{M}(\beta)+\xi_n\sigma_n(1-\beta)\Big)^2}, \\ \mathcal{A}_{2,n}(m,\beta) &= \frac{\beta\xi_n\sigma_n}{\mathcal{M}(\beta)+\xi_n\sigma_n(1-\beta)}. \end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we receive

$$\begin{aligned} \int_\Omega \rho(x)\varphi_n(x)dx = &\int_\Omega f(x)\varphi_n(x)dx \\ &\times \mathcal{A}_{1,n}(m,\beta)\int_0^1 E_{\beta,\beta}\Big(-\mathcal{A}_{2,n}(m,\beta)(1-s)^\beta\Big)(1-s)^{\beta-1}g(s)ds. \end{aligned} \tag{3.7}$$

From (3.7), it gives

$$\int_\Omega f(x)\varphi_n(x)dx = \frac{\int_\Omega \rho(x)\varphi_n(x)dx}{\mathcal{A}_{1,n}(m,\beta)\int_0^1 E_{\beta,\beta}\Big(-\mathcal{A}_{2,n}(m,\beta)(1-s)^\beta\Big)(1-s)^{\beta-1}g(s)ds}\cdot \tag{3.8}$$

Through some basic transformations, we get

$$f(x) = \sum_{n=1}^{\infty} \frac{\int_\Omega \rho(x)\varphi_n(x)dx}{\mathcal{A}_{1,n}(m,\beta)\int_0^1 E_{\beta,\beta}\Big(-\mathcal{A}_{2,n}(m,\beta)(1-s)^\beta\Big)(1-s)^{\beta-1}g(s)ds}\varphi_n(x). \tag{3.9}$$

As $n \to \infty$,i.e., $\Big(\mathcal{A}_{1,n}(m,\beta) \int_0^1 E_{\beta,\beta}\big(-\mathcal{A}_{2,n}(m,\beta)(1-s)^\beta\big)(1-s)^{\beta-1}g(s)ds\Big)^{-1}$ $\to \infty$, see in the Lemma 2.6. Thus, it can be concluded from formula (3.9) that the small perturbation of $\rho_\varepsilon(x)$ will cause a great change of $f(x)$. Thus our problem (1.1) is ill-posed. Next, we will give the conditional stability results of the source term $f(x)$.

Theorem 3.1. *Let us take $(g_\epsilon, \rho_\epsilon) \in L^r(0,1) \times L^r(\Omega)$ such that $g_\epsilon(t) > G_2 > 0$ for any $0 \le t \le 1$ for any $\frac{1}{\beta} < r < 2$ and condition*

$$\big\|g_\epsilon - g\big\|_{L^r(0,1)} + \big\|\rho_\epsilon - \rho\big\|_{L^r(\Omega)} \le \epsilon. \tag{3.10}$$

Assume that $f \in \mathcal{H}(\mathcal{A}^{j+k})$ for $k > 0$ and $0 < j < \frac{N}{4}$. With the Fourier truncation method, we have

$$f_\epsilon^{\mathcal{C}_\epsilon}(x) = \sum_{\xi_n \le \mathcal{C}_\epsilon} \frac{\int_\Omega \rho_\epsilon(x)\varphi_n(x)dx}{\mathcal{A}_{1,n}(m,\beta) \int_0^1 E_{\beta,\beta}\big(-\mathcal{A}_{2,n}(m,\beta)(1-s)^\beta\big)(1-s)^{\beta-1}g_\epsilon(s)ds}\varphi_n(x). \tag{3.11}$$

Then we have

$$\begin{aligned}\big\|f_\epsilon^{\mathcal{C}_\epsilon} - f\big\|_{L^{\frac{2N}{N-4j}}(\Omega)} &\lesssim |\mathcal{C}_\epsilon|^{-k}\big\|f\big\|_{\mathcal{H}^{j+k}(\Omega)} + \mathcal{C}_\epsilon\epsilon\big|\mathcal{A}_4(\mathcal{B}_3, r, \beta, G_2, m, \xi_1)\big|\|f\|_{\mathcal{H}^j(\Omega)} \\ &\quad + \mathcal{A}_5(\xi_1, \sigma_1, m, \beta)\,(\mathcal{C}_\epsilon)^{j+1+\frac{N}{2r}-\frac{N}{4}}\,\epsilon,\end{aligned} \tag{3.12}$$

whereby $\mathcal{C}_\epsilon$ satisfies that

$$\lim_{\epsilon\to 0}\mathcal{C}_\epsilon\epsilon = \lim_{\epsilon\to 0}\Big((\mathcal{C}_\epsilon)^{j+1+\frac{N}{2r}-\frac{N}{4}}\,\epsilon\Big) = 0, \quad \lim_{\epsilon\to 0}\mathcal{C}_\epsilon = +\infty. \tag{3.13}$$

Remark. *We can take $\mathcal{C}_\epsilon$ satisfying (3.13) as follows*

$$\mathcal{C}_\epsilon = \epsilon^{\overline{\frac{s-1}{j+1+\frac{N}{2r}-\frac{N}{4}}}},\ 0 < s < 1.$$

Then the error order $\big\|f_\epsilon^{\mathcal{C}_\epsilon} - f\big\|_{L^{\frac{2N}{N-4j}}(\Omega)}$ is of order

$$\max\Big\{\epsilon^{\frac{k(j+1+\frac{N}{2r}-\frac{N}{4}}{s-1}}, \epsilon^{\frac{s+j+\frac{N}{2s}-\frac{N}{4}}{j+1+\frac{N}{2s}-\frac{N}{4}}}, \epsilon^s\Big\}.$$

Proof. Using the triangle inequality, we have

$$\big\|f_\epsilon^{\mathcal{C}_\epsilon} - f\big\|_{\mathcal{H}^j(\Omega)} \le \underbrace{\big\|\mathcal{F}_{2,\epsilon} - f\big\|_{\mathcal{H}^j(\Omega)}} + \underbrace{\big\|\mathcal{F}_{1,\epsilon} - \mathcal{F}_{2,\epsilon}\big\|_{\mathcal{H}^j(\Omega)}} + \underbrace{\big\|\mathcal{F}_{1,\epsilon} - f_\epsilon\big\|_{\mathcal{H}^j(\Omega)}}, \tag{3.14}$$

where we denote some following functions

$$\mathcal{F}_{1,\epsilon}(x) = \sum_{\xi_n \le \mathcal{C}_\epsilon} \frac{\int_\Omega \rho(x)\varphi_n(x)dx}{\mathcal{A}_{1,n}(m,\beta) \int_0^1 E_{\beta,\beta}\big(-\mathcal{A}_{2,n}(m,\beta)(1-s)^\beta\big)(1-s)^{\beta-1}g_\epsilon(s)ds}\varphi_n(x), \tag{3.15}$$

and

$$\mathcal{F}_{2,\epsilon}(x) = \sum_{\xi_n \le \mathcal{C}_\epsilon} \frac{\int\limits_\Omega \rho(x)\varphi_n(x)dx}{\mathcal{A}_{1,n}(m,\beta) \int\limits_0^1 E_{\beta,\beta}\big(-\mathcal{A}_{2,n}(m,\beta)(1-s)^\beta\big)(1-s)^{\beta-1}g(s)ds}\varphi_n(x). \tag{3.16}$$

Let us next consider some terms on the right hand side of (3.14).
Step 1. Estimate of $\left\|\mathcal{F}_{2,\epsilon} - f\right\|_{\mathcal{H}^j(\Omega)}$.
Let us recall the function f as follows.

$$f(x) = \sum_{n=1}^{\infty} \frac{\int\limits_\Omega \rho(x)\varphi_n(x)dx}{\mathcal{A}_{1,n}(m,\beta) \int\limits_0^1 E_{\beta,\beta}\big(-\mathcal{A}_{2,n}(m,\beta)(1-s)^\beta\big)(1-s)^{\beta-1}g(s)ds}\varphi_n(x).$$

This expression together with the fomula (3.16) gives us the claim of the following difference

$$\begin{aligned} &f(x) - \mathcal{F}_{2,\epsilon}(x) \\ &= \sum_{\xi_n \ge \mathcal{C}_\epsilon} \frac{\int\limits_\Omega \rho(x)\varphi_n(x)dx}{\mathcal{A}_{1,n}(m\beta) \int\limits_0^1 E_{\beta,\beta}\big(-\mathcal{A}_{2,n}(m,\beta)(1-s)^\beta\big)(1-s)^{\beta-1}g(s)ds}\varphi_n(x) \\ &= \sum_{\xi_n \ge \mathcal{C}_\epsilon} \Big(\int_\Omega f(x)\varphi_n(x)dx\Big)\varphi_n(x). \end{aligned} \tag{3.17}$$

The norm on $\mathcal{H}^j(\Omega)$ of (3.17) is calculated through the Parseval equality as follows

$$\begin{aligned} \left\|f - \mathcal{F}_{2,\epsilon}\right\|^2_{\mathcal{H}^j(\Omega)} &= \sum_{\xi_n \ge \mathcal{C}_\epsilon} \xi_n^{2j}\Big(\int_\Omega f(x)\varphi_n(x)dx\Big)^2 \\ &= \sum_{\xi_n \ge \mathcal{C}_\epsilon} \xi_n^{-2k}\xi_n^{2j+2k}\Big(\int_\Omega f(x)\varphi_n(x)dx\Big)^2. \end{aligned}$$

It is easy to see that $\xi_n^{-2k} \le |\mathcal{C}_\epsilon|^{-2k}$ if $\xi_n > \mathcal{C}_\epsilon$ and $k > 0$. Hence, we have

$$\begin{aligned} \left\|f - \mathcal{F}_{2,\epsilon}\right\|^2_{\mathcal{H}^j(\Omega)} &\le |\mathcal{C}_\epsilon|^{-2k} \sum_{\xi_n \ge \mathcal{C}_\epsilon} \xi_n^{2j+2k}\Big(\int_\Omega f(x)\varphi_n(x)dx\Big)^2 \\ &= \left|\mathcal{C}_\epsilon\right|^{-2k}\left\|f\right\|^2_{\mathcal{H}^{j+k}(\Omega)}, \end{aligned} \tag{3.18}$$

It gives that

$$\left\|f - \mathcal{F}_{2,\epsilon}\right\|_{\mathcal{H}^j(\Omega)} \le |\mathcal{C}_\epsilon|^{-k}\left\|f\right\|_{\mathcal{H}^{j+k}(\Omega)}. \tag{3.19}$$

Step 2. Estimate of $\big\|\mathcal{F}_{1,\epsilon}-\mathcal{F}_{2,\epsilon}\big\|_{\mathcal{H}^j(\Omega)}$.
Based on two formulas (3.15) and (3.16), we have

$$\begin{aligned}
&\mathcal{F}_{1,\epsilon}(x)-\mathcal{F}_{2,\epsilon}(x)\\
&=\sum_{\xi_n\le\mathcal{C}_\epsilon}\frac{\int\limits_0^1 E_{\beta,\beta}\big(-\mathcal{A}_{2,n}(m,\beta)(1-s)^\beta\big)(1-s)^{\beta-1}\big(g_\epsilon(s)-g(s)\big)ds}{\int\limits_0^1 E_{\beta,\beta}\big(-\mathcal{A}_{2,n}(m,\beta)(1-s)^\beta\big)(1-s)^{\beta-1}g_\epsilon(s)ds}\\
&\quad\times\frac{\int\limits_\Omega\rho(x)\varphi_n(x)dx}{\mathcal{A}_{1,n}(m,\beta)\int\limits_0^1 E_{\beta,\beta}\big(-\mathcal{A}_{2,n}(m,\beta)(1-s)^\beta\big)(1-s)^{\beta-1}g(s)ds}\varphi_n(x)\cdot
\end{aligned}\tag{3.20}$$

We follows from (3.20) that

$$\begin{aligned}
&\mathcal{F}_{1,\epsilon}(x)-\mathcal{F}_{2,\epsilon}(x)\\
&=\sum_{\xi_n\le\mathcal{C}_\epsilon}\frac{\int\limits_0^1 E_{\beta,\beta}\big(-\mathcal{A}_{2,n}(m,\beta)(1-s)^\beta\big)(1-s)^{\beta-1}\big(g_\epsilon(s)-g(s)\big)ds}{\int\limits_0^1 E_{\beta,\beta}\big(-\mathcal{A}_{2,n}(m,\beta)(1-s)^\beta\big)(1-s)^{\beta-1}g_\epsilon(s)ds}\\
&\qquad\times\bigg(\int_\Omega f(x)\varphi_n(x)dx\bigg)\varphi_n(x).
\end{aligned}\tag{3.21}$$

By taking the norm of (3.21) in space $\mathcal{H}^j(\Omega)$ and using Parseval' s equality, we provide that

$$\begin{aligned}
&\big\|\mathcal{F}_{1,\varepsilon}-\mathcal{F}_{2,\varepsilon}\big\|^2_{\mathcal{H}^j(\Omega)}\\
&=\sum_{\xi_n\le\mathcal{C}_\epsilon}\left[\frac{\int\limits_0^1 E_{\beta,\beta}\big(-\mathcal{A}_{2,n}(m,\beta)(1-s)^\beta\big)(1-s)^{\beta-1}\big(g_\epsilon(s)-g(s)\big)ds}{\int\limits_0^1 E_{\beta,\beta}\big(-\mathcal{A}_{2,n}(m,\beta)(1-s)^\beta\big)(1-s)^{\beta-1}g_\epsilon(s)ds}\right]^2\\
&\qquad\times\xi_n^{2j}\bigg(\int_\Omega f(x)\varphi_n(x)dx\bigg)^2.
\end{aligned}\tag{3.22}$$

From (3.22), noting that $r>\beta^{-1}$ and $r^*=1+\dfrac{1}{r-1}$, using Hölder inequality and Lemma 3.10, we have

$$\begin{aligned}
&\bigg|\int_0^1 E_{\beta,\beta}\big(-\mathcal{A}_{2,n}(m,\beta)(1-s)^\beta\big)(1-s)^{\beta-1}\big(g_\epsilon(\tau)-g(\tau)\big)d\tau\bigg|\\
&\le\bigg[\int_0^1\big|g_\epsilon(s)-g(s)\big|^r ds\bigg]^{\frac1r}\bigg[\int_0^1\big|E_{\beta,\beta}\big(-\mathcal{A}_{2,n}(1-s)^\beta\big)(1-s)^{\beta-1}\big|^{r^*}ds\bigg]^{\frac{1}{r^*}}\\
&\le\big\|g_\epsilon-g\big\|_{L^r(0,1)}\Big(\mathcal{B}_3^{\frac{r}{r-1}}\frac{r-1}{\beta r-1}\Big)^{\frac{r-1}{r}}=\|g_\epsilon-g\|_{L^r(0,1)}\mathcal{B}_3\Big(\frac{r-1}{\beta r-1}\Big)^{\frac{r-1}{r}}\cdot
\end{aligned}\tag{3.23}$$

This implies that

$$\Big|\int_0^1 E_{\beta,\beta}\big(-\mathcal{A}_{2,n}(m,\beta)(1-s)^\beta\big)(1-s)^{\beta-1}\big(g_\epsilon(s)-g(s)\big)ds\Big| \le \mathcal{B}_3\Big(\frac{r-1}{\beta r-1}\Big)^{\frac{r-1}{r}}\epsilon. \tag{3.24}$$

It is easy to see that

$$\frac{1}{\mathcal{A}_{2,n}(m,\beta)} = \frac{\mathcal{M}(\beta)+\xi_n\sigma_n(1-\beta)}{\beta\xi_n\sigma_n} \ge \frac{\mathcal{M}(\beta)}{\beta\xi_n\sigma_n} \ge \frac{1}{\xi_n}\Big(\frac{\mathcal{M}(\beta)}{\beta}\Big). \tag{3.25}$$

Next, the function $g_\epsilon \ge G_2$, and using the Lemma 2.6, we have

$$\begin{aligned}
&\int_0^1 E_{\beta,\beta}\big(-\mathcal{A}_{2,n}(m,\beta)(1-s)^\beta\big)(1-s)^{\beta-1}g_\epsilon(s)ds \\
&\ge G_2\int_0^1 E_{\beta,\beta}\big(-\mathcal{A}_{2,n}(1-s)^\beta\big)(1-s)^{\beta-1}ds \\
&= \frac{G_2}{\xi_n}\ \frac{\mathcal{M}(\beta)}{\beta}\Big[1-E_{\beta,1}\Big(-\frac{\beta(\xi_1^{-1}+m)^{-1}}{\mathcal{M}(\beta)+\big(\xi_1^{-1}+m\big)^{-1}(1-\beta)}\Big)\Big].
\end{aligned} \tag{3.26}$$

From (3.24) and (3.26), we assert that

$$\begin{aligned}
&\textit{The right hand side of } (3.22) \\
&\le\ \xi_n\epsilon\ \underbrace{\mathcal{B}_3\Big(\frac{r-1}{\beta r-1}\Big)^{\frac{r-1}{r}}\frac{\beta}{G_2\mathcal{M}(\beta)}\Bigg[1-E_{\beta,1}\Big(-\frac{\beta(\xi_1^{-1}+m)^{-1}}{\mathcal{M}(\beta)+\big(\xi_1^{-1}+m\big)^{-1}(1-\beta)}\Big)\Bigg]^{-1}}_{\mathcal{A}_4(\mathcal{B}_3,r,\beta,G_2,m,\xi_1)}.
\end{aligned} \tag{3.27}$$

Combining (3.22) and (3.27), we find that

$$\begin{aligned}
\big\|\mathcal{F}_{1,\epsilon}-\mathcal{F}_{2,\epsilon}\big\|^2_{\mathcal{H}^j(\Omega)} &\le \big|\mathcal{A}_4(\mathcal{B}_3,r,\beta,G_2,m,\xi_1)\big|^2\epsilon^2 \\
&\quad\times\sum_{\xi_n\le\mathcal{C}_\epsilon}\xi_n^{2j+2}\Big(\int_\Omega f(x)\varphi_n(x)dx\Big)^2.
\end{aligned} \tag{3.28}$$

The finite sum on the right above can be bounded as follows

$$\begin{aligned}
\sum_{\xi_n\le\mathcal{C}_\epsilon}\xi_n^{2j+2}\Big(\int_\Omega f(x)\varphi_n(x)dx\Big)^2 &\le |\mathcal{C}_\epsilon|^2\sum_{\xi_n\le\mathcal{C}_\epsilon}\xi_n^{2j}\Big(\int_\Omega f(x)\varphi_n(x)dx\Big)^2 \\
&\le |\mathcal{C}_\epsilon|^2\big\|f\big\|^2_{\mathcal{H}^j(\Omega)}.
\end{aligned} \tag{3.29}$$

Therefore, we follows from (3.28) that

$$\big\|\mathcal{F}_{1,\epsilon}-\mathcal{F}_{2,\epsilon}\big\|_{\mathcal{H}^j(\Omega)} \le \mathcal{C}_\epsilon\epsilon\Big|\mathcal{A}_4(\mathcal{B}_3,r,\beta,G_2,m,\xi_1)\Big|\ \big\|f\big\|_{\mathcal{H}^j(\Omega)}. \tag{3.30}$$

Step 3. Estimate of $\left\|\mathcal{F}^1_\epsilon - f_\epsilon\right\|_{\mathcal{H}^j(\Omega)}$. We derive that

$$\begin{aligned}&\mathcal{F}_{1,\epsilon}(x) - f_\epsilon(x)\\&= \sum_{\xi_n \le \mathcal{C}_\epsilon} \Big[\mathcal{A}_{1,n}(m,\beta)\int_0^1 E_{\beta,\beta}\big(-\mathcal{A}_{2,n}(m,\beta)(1-s)^\beta\big)(1-s)^{\beta-1} g_\epsilon(s)ds\Big]^{-1}\\&\qquad\qquad \times \Big(\int_\Omega \big(\rho_\epsilon(x)-\rho(x)\big)\varphi_n(x)dx\Big)\varphi_n(x). \qquad (3.31)\end{aligned}$$

By taking the norm of both sides of the above expression in space $\mathcal{H}^j(\Omega)$, and using Parseval' s equality, we obtain that

$$\begin{aligned}&\left\|\mathcal{F}_{1,\epsilon} - f_\epsilon\right\|^2_{\mathcal{H}^j(\Omega)}\\&= \sum_{\xi_n \le \mathcal{C}_\epsilon} \Big[\mathcal{A}_{1,n}(m,\beta)\int_0^1 E_{\beta,\beta}\big(-\mathcal{A}_{2,n}(m,\beta)(1-s)^\beta\big)(1-s)^{\beta-1} g_\epsilon(s)ds\Big]^{-2}\\&\qquad \times \xi_n^{2j}\Big(\int_\Omega \big(\rho_\epsilon(x)-\rho(x)\big)\varphi_n(x)dx\Big)^2. \qquad (3.32)\end{aligned}$$

By looking back the inequality (3.26), we get

$$\begin{aligned}&\left\|\mathcal{F}_{1,\epsilon} - f_\epsilon\right\|^2_{\mathcal{H}^j(\Omega)}\\&\le \left(\frac{m\mathcal{M}(\beta)}{\frac{\mathcal{M}(\beta)}{\xi_1\sigma_1}+(1-\beta)}\Big[1 - E_{\beta,1}\Big(-\frac{\beta(\xi_1^{-1}+m)^{-1}}{\mathcal{M}(\beta)+\big(\xi_1^{-1}+m\big)^{-1}(1-\beta)}\Big)\Big]\right)^{-1}\\&\qquad \times \sum_{\xi_n \le \mathcal{C}_\epsilon} \xi_n^{2j+2}\Big(\int_\Omega \big(\rho_\epsilon(x)-\rho(x)\big)\varphi_n(x)dx\Big)^2. \qquad (3.33)\end{aligned}$$

Form (3.33), one has

$$\begin{aligned}&\sum_{\xi_n \le \mathcal{C}_\epsilon} \xi_n^{2j+2}\Big(\int_\Omega \big(\rho_\epsilon(x)-\rho(x)\big)\varphi_n(x)dx\Big)^2\\&\qquad = \sum_{\xi_n \le \mathcal{C}_\epsilon} \xi_n^{2j+2+\frac{N}{r}-\frac{N}{2}}\xi_n^{\frac{Nr-2N}{2r}}\Big(\int_\Omega \big(\rho_\epsilon(x)-\rho(x)\big)\varphi_n(x)dx\Big)^2\\&\qquad \le (\mathcal{C}_\epsilon)^{2j+2+\frac{N}{r}-\frac{N}{2}}\sum_{\xi_n \le \mathcal{C}_\epsilon}\xi_n^{\frac{Nr-2N}{2r}}\Big(\int_\Omega \big(\rho_\epsilon(x)-\rho(x)\big)\varphi_n(x)dx\Big)^2\\&\qquad = (\mathcal{C}_\epsilon)^{2j+2+\frac{N}{r}-\frac{N}{2}}\left\|\rho_\epsilon-\rho\right\|^2_{\mathcal{H}^{\frac{Nr-2N}{4r}}(\Omega)}. \qquad (3.34)\end{aligned}$$

Since $1<r<2$, with $L^r(\Omega)\hookrightarrow \mathcal{H}^{\frac{Nr-2N}{4r}}(\Omega)$. Therefore, we get

$$\left\|\rho_\epsilon-\rho\right\|_{\mathcal{H}^{\frac{Nr-2N}{4r}}(\Omega)} \le \mathcal{C}_1(N,r)\left\|\rho_\epsilon-\rho\right\|_{L^r(\Omega)} \le \mathcal{C}_1(N,r)\epsilon. \qquad (3.35)$$

By summarizing all three evaluations (3.33), (3.34) and (3.35), we derive that

$$\left\|\mathcal{F}_{1,\epsilon} - f_\epsilon\right\|_{\mathcal{H}^j(\Omega)} \le \mathcal{A}_5(\xi_1,\sigma_1,m,\beta)\,(\mathcal{C}_\epsilon)^{j+1+\frac{N}{2r}-\frac{N}{4}}\,\epsilon, \qquad (3.36)$$

whereby

$$\mathcal{A}_5(\xi_1,\sigma_1,m,\beta) = \left(\frac{m\mathcal{M}(\beta)}{\frac{\mathcal{M}(\beta)}{\xi_1\sigma_1}+(1-\beta)} \Big[1 - E_{\beta,1}\Big(-\frac{\beta(\xi_1^{-1}+m)^{-1}}{\mathcal{M}(\beta)+\big(\xi_1^{-1}+m\big)^{-1}(1-\beta)} \Big) \Big] \right)^{-1}. \quad (3.37)$$

From (3.19) to (3.37), we can conclude that

$$\big\| f_\epsilon^{\mathcal{C}_\epsilon} - f \big\|_{\mathcal{H}^j(\Omega)} \le |\mathcal{C}_\epsilon|^{-k} \big\| f \big\|_{\mathcal{H}^{j+k}(\Omega)} + \mathcal{C}_\epsilon \epsilon \big| \mathcal{A}_4(\mathcal{B}_3, r, \beta, G_2, m, \xi_1) \big| \|f\|_{\mathcal{H}^j(\Omega)} + \mathcal{A}_5(\xi_1,\sigma_1,m,\beta) \, (\mathcal{C}_\epsilon)^{j+1+\frac{N}{2r}-\frac{N}{4}} \, \epsilon. \quad (3.38)$$

By using Lemma 2.7, since $0 < j < \frac{N}{4}$, we know that $\mathcal{H}^j(\Omega) \hookrightarrow L^{\frac{2N}{N-4j}}(\Omega)$, which yields to the desired result (3.38). ☐

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