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COUPLED RANDOM IMPULSIVE SEMILINEAR DIFFERENTIAL SYSTEM OF EQUATIONS AND APPLICATIONS

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Abstract. In this paper, we use a new technique for the treatment of systems based on the advantage of vector-valued norms in the study of coupled random impulsive semilinear differential systems of equations. Using the fixed point theory technique we prove existence, uniqueness, and stability. Finally, an example is provided to illustrate the relevance of the results.

1. Introduction

The study of impulsive differential equations opens a large space for natural frameworks, for many real-life mathematical phenomena and there has been much research activity concerning the theory of impulsive differential equations (see [9, 14]). When the impulses exist at random points, then the solutions of the differential equations are a stochastic process. It is very different from deterministic impulsive differential equations and also it is different from stochastic differential equations. Thus the random impulsive equations give more realistic solutions than deterministic impulsive equations. In [7], the authors study the existence and uniqueness of theorems and their stability through continuous dependence on initial conditions of random impulsive semilinear differential systems. It is also worth emphasizing that impulsive differential systems and evolution differential systems are used to describe numerous models of real processes and phenomena appearing in the applied sciences, for instance, in physics, related to chemical technology, population dynamics, biotechnology, and economics. That is why, in recent years, they have been the objectives of many investigations. We refer to the monographs by Bainov and Simeonov [1], and Benchohra et al. [3], amongst others, to see several studies on the properties of their solutions. Stability has also attracted the attention of many mathematicians (see [4, 2, 5, 7, 6, 8, 13, 15, 17, 16, 19, 18]).
The classical Banach contraction principle is a very useful tool in nonlinear analysis with many applications to integral and differential equations, optimization theory, and other topics. There are many generalizations of this result, one of them is

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due to A.I. Perov [10] and consists of replacing usual metric spaces with spaces endowed with vector-valued metrics. According to this result, if a space X is a Cartesian product $X = X_1 \times \cdots X_n$ and each component X_i is a complete metric space with the metric d_i, then instead of endowing X with some metric δ generated by $d_1, \cdots, d_n$, for instance any one of the metrics

$$\begin{aligned} \delta^p(x,y) &= (\sum_{i=1}^{n} d_i(x_i,y_i)^p)^{\frac{1}{p}} \ (1 \le p < \infty), \\ \delta^{\infty}(x,y) &= \max\{d_1(x_1,y_1), \cdots, d_n(x_n,y_n)\}, \end{aligned}$$

and applying Banach's contraction principle in the complete metric space (X, δ), better results are obtained if one considers the vector-valued metric

$$d(x,y) = (d_1(x_1,y_1), \cdots, d_n(x_n,y_n))^T$$

and one requires a *generalized contraction* (in Perov's sense) condition in the vector-matrix form

$$d(F(x), F(y)) \le A d(x,y), \quad x, y \in X,$$

where A is a square matrix of type $n \times n$ with nonnegative elements having the spectral radius $\rho(A) < 1$. This approach is very fruitful for the treatment of systems of equations arising from various fields of applied mathematics. The advantage of using vector-valued metrics and norms instead of the usual scalar ones, in connexion with several techniques of nonlinear analysis has been pointed out in [12].

The paper is organized as follows: Some preliminaries are presented in section 2, followed by an existence and uniqueness result of a random impulsive semilinear differential system of equations, in section 3, using Perov's fixed point Theorem where we use a vector-valued approach. It is allowed that the components of the equations of the system behave differently, and thus, more general results can be obtained. In section 4, we study the stability through continuous dependence on initial conditions. Finally, in section 5, an example is presented.

2. Preliminaries

In this section, we recall from the literature some notations, definitions, and auxiliary results which will be used throughout this paper. If $x, y \in \mathbb{R}^n, x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, then, by $x \le y$, we mean $x_i \le y_i$, for all $i = 1, \ldots, n$. Also $\mid x \mid = (\mid x_1 \mid, \ldots, \mid x_n \mid)$,

$$\max(x,y) = (\max(x_1,y_1), \ldots, \max(x_n,y_n))$$

and $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i > 0\}$. If $c \in \mathbb{R}$, then $x \le c$ means $x_i \le c$, for each $i = 1, \ldots, n$.

Definition 2.1. *Let X be a nonempty set. By a vector-valued metric on X, we mean a map $d : X \times X \to \mathbb{R}^n$ with the following properties:*

(i) $d(u,v) \ge 0$ *for all* $u, v \in X$; *if* $d(u,v) = 0$ *then* $u = v$;
(ii) $d(u,v) = d(v,u)$, *for all* $u, v \in X$;
(iii) $d(u,v) \le d(u,w) + d(w,v)$ *for all* $u, v, w \in X$.

We call (X, d) a generalized metric space with $d(x,y) := \begin{pmatrix} d_1(x,y) \\ \cdots \\ d_n(x,y) \end{pmatrix}$, where each d_i, for $i = 1, \ldots, n$, is a metric on X. (Notice that d is generalized metric space on X if and only if $d_i, i = 1, \dots, n$ are metrics on X.) For $r = (r_1, \dots, r_n) \in \mathbb{R}^n_+$, we will denote by

$$\mathrm{B}(x_0, r) = \{x \in X; d(x_0, x) < r\} = \{x \in X; d_i(x_0, x) < r_i, i = 1, \dots, n\},$$

the open ball centered in x_0 with radius r, and

$$\overline{\mathrm{B}(x_0, r)} = \{x \in X; d(x_0, x) \leq r\} = \{x \in X; d_i(x_0, x) \leq r_i, i = 1, \dots, n\},$$

the closed ball centered in x_0 with radius r. We mention that for a generalized metric space, the notions of open subset, closed set, convergence, Cauchy sequence, and completeness are similar to those in usual metric spaces.

Definition 2.2. *A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, all the eigenvalues of M are in the open unit disc.*

Theorem 2.3. [12] *Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$. The following assertions are equivalent:*

(i) *M is convergent towards zero;*
(ii) *$M^k \to 0$ as $k \to \infty$;*
(iii) *The matrix $(I - M)$ is nonsingular and*

$$(I - M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots;$$

(iv) *The matrix $(I-M)$ is nonsingular and $(I-M)^{-1}$ has nonnegative elements.*

Definition 2.4. *Let (X, d) be a generalized metric space. An operator $N : X \to X$ is said to be contractive if there exists a convergent to zero matrix M such that,*

$$d(N(x), N(y)) \leq M\, d(x, y), \text{ for all } x, y \in X.$$

For $n = 1$, we recover the classical Banach contraction fixed point result.

Theorem 2.5. [11] *Let (X, d) be a complete generalized metric space with $d : X \times X \to \mathbb{R}^n$, and let $N : X \to X$ be such that*

$$d(N(x), N(y)) \leq M\, d(x, y),$$

for all $x, y \in X$, and some square matrix M of nonnegative numbers. If the matrix M is convergent to zero, that is $M^k \to 0$ as $k \to \infty$ then, N has a unique fixed point $x_* \in X$,

$$d(N^k(x_0), x_*) \leq M^k (I - M)^{-1}\, d(N(x_0), x_0),$$

for every $x_0 \in X$ and $k \geq 1$.

3. Existence and uniqueness results

Let X be a real separable Hilbert space and Ω a nonempty set. Assume that τ_k is a random variable defined from Ω to $D_k =^{def} (0, d_k)$ for $k = 1, 2, \dots,$ where $0 < d_k < +\infty$.
Furthermore, assume that τ_k follows Erlang distribution, where $k = 1, 2, \dots,$ and let τ_i and τ_j are independent with each other as $i \neq j$ for $i, j = 1, 2, \dots,$. For the sake of simplicity, we denote $\Re_\tau = [\tau, +\infty)$, and $\Re^+ = [0, +\infty)$. We consider a

system of semilinear differential equations with random impulses of the form

$$\begin{cases} x'(t) = Ax(t) + f(t, x_t, y_t), t \neq \xi_k, t \geq t_0, \\ y'(t) = Ay(t) + g(t, x_t, y_t), t \neq \xi_k, t \geq t_0, \\ x(\xi_k) = b_k\left(\tau_k\right) x(\xi_k^-), k = 1, 2, \ldots, \\ y(\xi_k) = a_k\left(\tau_k\right) x(\xi_k^-), k = 1, 2, \ldots, \\ x_{t_0} = \varphi \text{ and } y_{t_0} = \psi, \end{cases} \tag{3.1}$$

where A is the infinitesimal generator of strongly continuous semigroup of bounded linear operators $S\left(t\right)$ with domain $D\left(A\right) \subset X$, the functional $f, g : \Re^+ \times \mathrm{C} \times \mathrm{C} \to X$, with $\mathrm{C} = \mathrm{C}\left(\left[-r, 0\right], X\right)$ is the set of piecewise continuous functions mapping $[-r, 0]$ into X with some given $r > 0$, x_t is a function when t is fixed, defined by $x_t\left(s\right) = x\left(t + s\right)$ for all $s \in \left[-r, 0\right]$, $\xi_0 = t_0$ and $\xi_k = \xi_{k-1} + \tau_k$ for $k = 1, 2, \ldots,$. Here, $t_0 \in \Re_\tau$ is arbitrary given real number. The impulse moments $\{\xi_k\}$ form a strictly increasing sequence, i.e., $t_0 = \xi_0 < \xi_1 < \xi_2 < \ldots < \lim\limits_{k \mapsto \infty} \xi_k = \infty$, $b_k : D_k \to X$ for each $k = 1, 2, \ldots,$ $x\left(\xi_k^-\right) = \lim\limits_{t \mapsto \xi_k^-} x\left(t\right)$ according to their paths with the norm $\|x\|_t = \sup\limits_{t-s \leq s \leq t} |x\left(s\right)|$ for each t satisfying $t \geq t_0$, $\|.\|$ is any given norm in X, y_t has also the same definition with $a_k : D_k \to X$, and finally φ, ψ are functions defined from $[-r, 0]$ to X.

Denote $\{B_t, t \geq 0\}$ the simple counting process generated by $\{\xi_n\}$, that is, $\{B_t \geq n\} = \{\xi_n \leq t\}$, and denote $\mathcal{F}_t$ the $\sigma-$algebra generated by $\{B_t, t \geq 0\}$. Then $(\Omega, P, \{\mathcal{F}_t\})$ is a probability space. Let $L_2 = L_2\left(\Omega, \mathcal{F}_t, X\right)$ denote the Hilbert space of all $\mathcal{F}_t-$measurable square integrable random variables with values in X.

Assume that $T > t_0$ is any fixed time to be determined later and let B denote the Banach space $B\left(\left[t_0 - r, T\right], L_2\right)$, the family of all $\mathcal{F}_t-$ measurable, $C-$valued random variables ψ with the norm

$$\|\psi\|_B = \left(\sup_{t_0 \leq t \leq T} E\|\psi\|_t^2\right)^{1/2}.$$

Let $L_2^0\left(\Omega, B\right)$ denote the family of all $\mathcal{F}_0-$ measurable, $B-$valued random variables φ.

Definition 3.1. *A semigroup $\{S\left(t\right); t \geq t_0\}$ is said to be exponentially stable if there are positive constants $M \geq 1$ and $\gamma > 0$ such that $\|S\left(t\right)\| \leq Mc^{-\gamma(t-t_0)}$ for all $t \geq t_0$, where $\|.\|$ denotes the operator norm in $\mathcal{L}\left(X\right)$ (The Banach algebra of bounded linear operators from X into X). A semigroup $\{S\left(t\right); t \geq t_0\}$ is said to be uniformly bounded if $\|S\left(t\right)\| \leq M$ for all $t \geq t_0$, where $M \geq 1$ is some constant. If $M = 1$, then the semigroup is said to be a contraction semigroup.*

Definition 3.2. *For a given $T \in \left(t_0, +\infty\right)$, a stochastic process,*

$$\left\{u(t) \in B \times B, t_0 - r \leq t \leq T\right\},$$

is called a mild solution to the system (3.1) in $(\Omega \times \Omega, P, \{\mathcal{F}_t\})$, if

(i) *$u(t) \in B \times B$ is $\mathcal{F}_t-$adapted for $t \geq t_0$,*

(ii) $u(t_0+s) = (x(t_0+s), y(t_0+s)) = (\varphi(s), \psi(s)) \in L_2^0(\Omega, B) \times L_2^0(\Omega, B)$, *where* $s \in [-r, 0]$, *and*

$$\begin{cases} x(t) = \displaystyle\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(\tau_i) S(t-t_0)\varphi(0) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} S(t-s) f(s, x_s, y_s)\,\mathrm{d}s \\ \qquad + \displaystyle\int_{\xi_k}^{t} S(t-s)\, f(s, x_s, y_s)\,\mathrm{d}s\Big] I_{(\xi_k,\xi_{k+1})}(t),\ t \in [t_0, T], \\ y(t) = \displaystyle\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(\tau_i) S(t-t_0)\varphi(0) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} S(t-s) g(s, x_s, y_s)\,\mathrm{d}s \\ \qquad + \displaystyle\int_{\xi_k}^{t} S(t-s)\, g(s, x_s, y_s)\,\mathrm{d}s\Big] I_{(\xi_k,\xi_{k+1})}(t),\ t \in [t_0, T], \end{cases} \tag{3.2}$$

where $\prod_{j=m}^{n}(.) = 1$ *as* $m > n$, $\prod_{j=i}^{k} b_j(\tau_j) = b_k(\tau_k) b_{k-1}(\tau_{k-1}) \dots b_i(\tau_i)$, $\prod_{j=i}^{k} a_j(\tau_j) = a_k(\tau_k) a_{k-1}(\tau_{k-1}) \dots a_i(\tau_i)$ *and* $I_A(.)$ *is the index function, i.e.,*

$$I_A(t) = \begin{cases} 1 & if,\ t \in A, \\ 0 & if,\ t \notin A. \end{cases}$$

Now we introduce the following hypotheses:
(H_1) There exit constants $L_1 = L_1(T) > 0, \overline{L_1} = \overline{L_1}(T) > 0, L_2 = L_2(T) > 0, K_1 = K_1(T) > 0, \overline{K_1} = \overline{K_1}(T) > 0, K_2 = K_2(T) > 0$ for $x, \overline{x}, y, \overline{y} \in C$ and for every $t_0 \leqslant t \leqslant T$ such that,

$$E\ \|f(t,x,y) - f(t,\overline{x},\overline{y})\|^2 \leq\ L_1\ E\ \|x - \overline{x}\|_t^2 + \overline{L_1}\ E\ \|y - \overline{y}\|_t^2,$$
$$E\|f(t,x,y)\|^2 \leq\ L_2(1 + E\|x\|_t^2 + E\|y\|_t^2),$$
$$E\ \|g(t,x,y) - g(t,\overline{x},\overline{y})\|^2 \leq\ K_1\ E\ \|x - \overline{x}\|_t^2 + \overline{K_1}\ E\ \|y - \overline{y}\|_t^2,$$

and

$$E\|g(t,x,y)\|^2 \leq\ K_2(1 + E\|x\|_t^2 + E\|y\|_t^2).$$

(H_2) $E\left\{\max_{i,k}\left\{\prod_{j=i}^{k}\|b_j(\tau_j)\|\right\}\right\}$ and $E\left\{\max_{i,k}\left\{\prod_{j=i}^{k}\|a_j(\tau_j)\|\right\}\right\}$ are uniformly bounded, that is, there is $C > 0$ such that,

$$E\left\{\max_{i,k}\left\{\prod_{j=i}^{k}\|b_j(\tau_j)\|\right\}\right\} \leqslant C, \quad for\ all\ \ \tau_j \in D_j, j = 1, 2, \dots,$$

and

$$E\left\{\max_{i,k}\left\{\prod_{j=i}^{k}\|a_j(\tau_j)\|\right\}\right\} \leqslant C, \quad for\ all\ \ \tau_j \in D_j, j = 1, 2, \dots.$$

Theorem 3.3. *Assume that* (H_1) *and* (H_2) *are satisfied, then there exists a unique (local) continuous mild solution to* (3.1) *for any initial value* (t_0, φ, ψ) *with* $t_0 \geqslant 0$ *and* $(\varphi, \psi) \in B \times B$.

Proof. Let N be the nonlinear operator defined on $B \times B$ by

$$N(x,y)(t) = (N_1(x,y)(t), N_2(x,y)(t)),$$

where

$$\begin{aligned}
(N_1(x,y))(t+t_0) &= \varphi(t),\ t \in [-r,0]\,,\\
(N_1(x,y))(t) &= \sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k} b_i(\tau_i) S(t-t_0)\varphi(0)\\
&\quad + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} S(t-s) f(s,x_s,y_s)\,\mathrm{d}s\\
&\quad + \int_{\xi_k}^{t} S(t-s) f(s,x_s,y_s)\,\mathrm{d}s\Bigg] I_{(\xi_k,\xi_{k+1})}(t), \quad t \in [t_0,T]\,,
\end{aligned}$$

and

$$\begin{aligned}
(N_2(x,y))(t+t_0) &= \psi(t),\ t \in [-r,0]\,,\\
(N_2(x,y))(t) &= \sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k} a_i(\tau_i) S(t-t_0)\psi(0)\\
&\quad + \sum_{i=1}^{k}\prod_{j=i}^{k} a_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} S(t-s) g(s,x_s,y_s)\,\mathrm{d}s\\
&\quad + \int_{\xi_k}^{t} S(t-s) g(s,x_s,y_s)\,\mathrm{d}s\Bigg] I_{(\xi_k,\xi_{k+1})}(t), \quad t \in [t_0,T]\,.
\end{aligned}$$

The continuity of N_1 and N_2 in $[-r,T]$ is obvious. We shall use Theorem 2.5 to prove that N has a unique fixed point. Now, we have to show that N maps $B \times B$ into itself. We have that

$$\begin{aligned}
\|(N_1(x,y))(t)\|^2 &\le \Bigg[\sum_{k=0}^{+\infty}\Bigg[\|\prod_{i=1}^{k} b_i(\tau_i)\|\|S(t-t_0)\|\|\varphi(0)\|\\
&\quad + \sum_{i=1}^{k}\|\prod_{j=i}^{k} b_j(\tau_j)\|\left\{\int_{\xi_{i-1}}^{\xi_i}\|S(t-s) f(s,x_s,y_s)\,\mathrm{d}s\|\right\}\\
&\quad + \int_{\xi_k}^{t}\|S(t-s) f(s,x_s,y_s)\|\,\mathrm{d}s\Bigg] I_{(\xi_k,\xi_{k+1})}(t)\Bigg]^2\\
&\le 2\Bigg[\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\|b_i(\tau_i)\|^2\|S(t-t_0)\|^2\|\varphi(0)\|^2 I_{(\xi_k,\xi_{k+1})}(t)\Bigg]\\
&\quad + \Bigg[\sum_{k=0}^{+\infty}\Bigg[\sum_{i=1}^{k}\|\prod_{j=i}^{k} b_j(\tau_j)\|\left\{\int_{\xi_{i-1}}^{\xi_i}\|S(t-s) f(s,x_s)\,\mathrm{d}s\|\right\}\\
&\quad + \int_{\xi_k}^{t}\|S(t-s)\|\|f(s,x_s)\|\,\mathrm{d}s\Bigg]\ I_{(\xi_k,\xi_{k+1})}(t)\Bigg]^2\Bigg],
\end{aligned}$$

and then,

$$\begin{aligned}
E\|(N_1(x,y))(t)\,\|^2 &\leq 2M^2 E \max_k \left\{ \prod_{i=1}^{k} \|b_i(\tau_i)\|^2 \right\} E\|\varphi(0)\|^2 \\
&+ 2M^2 E \left[\max_{i,k} \left\{ 1, \prod_{j=i}^{k} \|b_j(\tau_j)\| \right\} \right]^2 \times E \left(\int_{t_0}^{t} \|f(s,x_s,y_s)\| \,\mathrm{d}s \, I_{(\xi_k,\xi_{k+1})}(t) \right)^2 \\
&\leq 2M^2C^2 E\|\varphi(0)\|^2 + 2M^2 \max\left\{1, C^2\right\} E \left(\int_{t_0}^{t} \|f(s,x_s,y_s)\| \,\mathrm{d}s \right)^2, \\
E\|(N_1(x,y))\|_t^2 &\leq 2M^2C^2 E\|\varphi(0)\|^2 + 2M^2 \max\left\{1, C^2\right\} (T-t_0) \int_{t_0}^{t} E\|f(s,x_s,y_s)\|^2 \,\mathrm{d}s, \\
\text{and} & \\
E\|(N_1(x,y))\|_t^2 &\leq 2M^2C^2 E\|\varphi(0)\|^2 \\
&+ 2M^2 \max\left\{1, C^2\right\} (T-t_0) \int_{t_0}^{t} L_2 (1 + E\|x\|_s^2 + E\|y\|_s^2) \,\mathrm{d}s \\
&\leq 2\, M^2C^2 E\|\varphi(0)\|^2 + 4M^2 \max\left\{1, C^2\right\} (T-t_0)^2 L_2 \\
&\quad +4M^2 \max\left\{1, C^2\right\} (T-t_0) L_2 \int_{t_0}^{t} (E\|x\|_s^2 + E\|y\|_s^2) \,\mathrm{d}s.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sup_{t\in[t_0,T]} E\|(N_1(x,y))\|_t^2 &\leq 2M^2C^2 E\|\varphi(0)\|^2 + 4M^2 \max\left\{1, C^2\right\} (T-t_0)^2 L_2 \\
&\quad +4M^2 \max\left\{1, C^2\right\} (T-t_0) L_2 \int_{t_0}^{t} \sup_{s\in[t_0,t]} (E\|x\|_s^2 + E\|y\|_s^2) \mathrm{d}s \\
&\leq 2M^2C^2 E\|\varphi(0)\|^2 + 4M^2 \max\left\{1, C^2\right\} (T-t_0)^2 L_2 \\
&\quad +4M^2 \max\left\{1, C^2\right\} (T-t_0)^2 L_2 \sup_{t\in[t_0,T]} (E\|x\|_t^2 + E\|x\|_t^2),
\end{aligned}$$

for all $t \in [-r, T]$. Therefore, N_1 maps $B \times B$ into B. Following the same reasoning and calculation steps, we prove that N_2 maps also $B \times B$ into B. So, the operator N is well defined from $B \times B$ into $B \times B$.
Now, we have to show that N is contractive. Let (x, y) and $(\overline{x}, \overline{y}) \in B \times B$. Then

we have for each $t \in [t_0, T]$,

$$\begin{aligned}
&\|(N_1(x,y))(t) - (N_1(\overline{x},\overline{y}))(t)\|^2 \\
&\leq \left[\sum_{k=0}^{+\infty}\left[\sum_{i=1}^{k}\prod_{j=i}^{k}\|b_j(\tau_j)\| \int_{\xi_{i-1}}^{\xi_i} \|S(t-s)\| \, \|f(s,x_s,y_s) - f(s,,\overline{x}_s,\overline{y}_s)\| \,\mathrm{d}s \right.\right.\\
&\left.\left. + \int_{\xi_k}^{t} \|S(t-s)\| \, \|f(s,x_s,y_s) - f(s,\overline{x}_s,\overline{y}_s)\| \,\mathrm{d}s\right] I_{(\xi_k,\xi_{k+1})}(t)\right]^2 \\
&\leq M^2 \left[\max_{i,j}\left\{1, \prod_{j=i}^{k}\| \, b_j(\tau_j)\|\right\}\right]^2 \\
&\times \left(\int_{t_0}^{t} \|f(s,x_s,y_s) - f(s,,\overline{x}_s,\overline{y}_s)\| \,\mathrm{d}s \, I_{(\xi_k,\xi_{k+1})}(t)\right)^2,
\end{aligned}$$

and

$$\begin{aligned}
\|N_1(x,y) - N_1(\overline{x},\overline{y})\|_t^2 &\leq M^2 \max\left\{1, C^2\right\}(t-t_0)\int_{t_0}^{t} E\|f(s,x_s,y_s) - f(s,\overline{x}_s,\overline{y}_s)\|^2 \,\mathrm{d}s \\
&\leq \; M^2 \max\left\{1, C^2\right\}(T-t_0)\int_{t_0}^{t} (L_1 E \; \|x-y\|_s^2 + \overline{L}_1 E \; \|\overline{x}-\overline{y}\|_s^2) \,\mathrm{d}s.
\end{aligned}$$

Taking the supremum over t, we get,

$$\|N_1(x,y) - N_1(\overline{x},\overline{y})\|_B^2 \leq M^2 \max\left\{1, C^2\right\}(T-t_0)^2 (L_1\|x-y\|_B^2 + \overline{L}_1\|\overline{x}-\overline{y}\|_B^2).$$

Therefore, $\|N_1(x,y) - N_1(\overline{x},\overline{y})\|_B \leq \rho(T)(\sqrt{L_1}\|x-y\|_B + \sqrt{\overline{L}_1}\|\overline{x}-\overline{y}\|_B)$, with $\rho(T) = M\max\{1,C\}(T-t_0)$. Similarly, we have $\|N_2(x,y) - N_2(\overline{x},\overline{y})\|_B \leq \rho(T)(\sqrt{K_1}\|x-y\|_B + \sqrt{\overline{K}_1}\|\overline{x}-\overline{y}\|_B)$. Therefore,

$$\begin{aligned}
\|N(x,y) - N(\overline{x},\overline{y})\|_{B\times B} &= \begin{pmatrix} \|N_1(x,y) - N_1(\overline{x},\overline{y})\|_B \\ \|N_2(x,y) - N_2(\overline{x},\overline{y})\|_B \end{pmatrix} \\
&\leq Q\begin{pmatrix} \|x-y\|_B \\ \|\overline{x}-\overline{y}\|_B \end{pmatrix},
\end{aligned}$$

with $Q = \rho(T)\begin{pmatrix} \sqrt{L_1} & \sqrt{\overline{L_1}} \\ \sqrt{L_2} & \sqrt{\overline{L_2}} \end{pmatrix}$. Then we can take a suitable $0 < T_1 < T$ sufficient small such Q is a convergent to zero matrix, and hence N is contractive on $B_{T_1} \times B_{T_1}$, (B_{T_1} denotes B with T substituted by T_1). Thus, by the Perov fixed point theorem we obtain a unique fixed point $u \in B_{T_1} \times B_{T_1}$ for the operator N, and hence, $Nu = u$ is a mild solution of (3.1). This procedure can be repeated to extend the solution to the entire interval $[-r, T]$ in finitely many similar steps, thereby completing the proof for the existence and uniqueness of the mild solution on the whole interval $[-r, T]$. □

Theorem 3.4. *Let $f : \Re^+ \times C \times C \to X$ be a map satisfying the assumptions (H_1) and (H_2). Then, there exists a unique, global, continuous solution $u = (x,y)$ to* (3.1) *for any initial value (t_0, φ, ψ) with $t_0 \geq 0$ and $(\varphi, \psi) \in B \times B$.*

Proof. This assertion follows immediately since T is arbitrary in the proof of Theorem (3.3) □

4. Stability

Theorem 4.1. *Let $u(t) = (x(t), y(t))$ and $\overline{u}(t) = (\bar{x}(t), \bar{y}(t))$ be mild solutions of the system* (3.1) *with initial values $(\varphi(0), \psi(0))$ and $(\overline{\varphi(0)}, \overline{\psi(0)}) \in B \times B$. If the assumptions of Theorem* 3.4 *are satisfied, then the mild solution of the system* (3.1) *is stable in the mean square.*

Proof. We have that u and $\bar{u}$ are two mild solutions of the system (3.1) for $t \in [t_0, T]$. Then,

$$\begin{aligned} x(t) - \overline{x}(t) &= \sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k} b_i(\tau_i)\, S(t-t_0)\left[\varphi(0) - \overline{\varphi(0)}\right] \\ &+ \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_i-1}^{\xi_i} S(t-s)\,[f(s,x_s,y_s) - f(s,\overline{x_s},\overline{y_s})]\,ds \\ &+ \int_{\xi_k}^{t} S(t-s)\,[f(s,x_s,y_s) - f(s,\overline{x_s},\overline{y_s})]\,ds\Bigg] I_{[\xi_k,\xi_{k+1})}(t). \end{aligned}$$

By using the hypotheses (H_1) and (H_2), we get

$$\begin{aligned} & E\|x-\overline{x}\|_t^2 \\ &\le\; 2\sum_{k=0}^{+\infty}\left[\prod_{i=1}^{k}\|b_i(\tau_i)\|^2\,\|S(t-t_0)\|^2 E\|\varphi(0)-\overline{\varphi(0)}\|^2 I_{[\xi_k,\xi_{k+1})}(t)\right] \\ &+ 2E\Bigg[\sum_{k=0}^{+\infty}\Bigg[\sum_{i=1}^{k}\prod_{j=i}^{k}\|b_j(\tau_j)\|\int_{\xi_i-1}^{\xi_i}\|S(t-s)\|\|f(s,x_s,y_s)-f(s,\overline{x_s},\overline{y_s})\|ds \\ &+ \int_{\xi_k}^{t}\|S(t-s)\|\|f(s,x_s,y_s)-f(s,\overline{x_s},\overline{y_s})\|ds\Bigg] I_{[\xi_k,\xi_{k+1})}(t)\Bigg]^2 \\ &\le 2M^2E\left\{\max_k\left\{\prod_{i=1}^{k}\|b_i(\tau_i)\|^2\right\}\right\} E\|\varphi(0)-\overline{\varphi(0)}\|^2 \\ &+ 2M^2\,E\left[\max_{i,k}\left\{1,\prod_{j=i}^{k}\|b_j(\tau_j)\|\right\}\right]^2 E\left(\int_{t_0}^{t}\|f(s,x_s,y_s)-f(s,\overline{x_s},\overline{y_s})\|ds\, I_{[\xi_k,\xi_{k+1})}(t)\right)^2 \\ &\le 2M^2C^2\,E\|\varphi(0)-\overline{\varphi(0)}\|^2 \\ &+ 2M^2\max\left\{1,C^2\right\}(t-t_0)\int_{t_0}^{t} E\|f(s,x_s,y_s)-f(s,\overline{x_s},\overline{y_s})\|^2\,ds, \end{aligned}$$

and

$$\begin{aligned} & \sup_{t\in[t_0,T]} E\|x-\overline{x}\|_t^2 \\ &\le 2M^2C^2\,E\|\varphi(0)-\overline{\varphi}(0)\|^2 \\ &+ 2M^2\max\left\{1,C^2\right\}(t-t_0)\int_{t_0}^{t}(L_1\sup_{t\in[t_0,t]} E\|x-\overline{x}\|_s^2 + \bar{L}_1\sup_{t\in[t_0,t]} E\|y-\overline{y})\|_s^2\,ds. \end{aligned}$$

By applying Grownwall's inequality, we have

$$\begin{aligned}\sup_{t\in[t_0,T]} E\|x-\overline{x}\|_t^2 \leq{}& M^2C^2\, E\|\varphi(0)-\overline{\varphi}(0)\|^2 \exp\left(2M^2\max\left\{1,C^2\right\}(T-t_0)^2\, L_1\right)\\ &+ M^2C^2\, E\|\varphi(0)-\overline{\varphi}(0)\|^2 \exp\left(2M^2\max\left\{1,C^2\right\}(T-t_0)^2\, \overline{L}_1\right)\\ \leq{}& \Gamma E\|\varphi(0)-\overline{\varphi}(0)\|^2,\end{aligned}$$

where,

$$\Gamma_1 = M^2C^2\left(\exp\left(2M^2\max\left\{1,C^2\right\}(T-t_0)^2\, L_1\right)+\exp\left(2M^2\max\left\{1,C^2\right\}(T-t_0)^2\, \overline{L}_1\right)\right).$$

It is not difficult to see that we have also, by applying Grownwall's inequality, that

$$\begin{aligned}\sup_{t\in[t_0,T]} E\|y-\overline{y}\|_t^2 \leq{}& M^2C^2\, E\|\psi(0)-\overline{\psi}(0)\|^2 \exp\left(2M^2\max\left\{1,C^2\right\}(T-t_0)^2\, K_1\right)\\ &+ M^2C^2\, E\|\psi(0)-\overline{\psi}(0)\|^2 \exp\left(2M^2\max\left\{1,C^2\right\}(T-t_0)^2\, \overline{K}_1\right)\\ \leq{}& \Gamma E\|\psi(0)-\overline{\psi}(0)\|^2,\end{aligned}$$

where

$$\Gamma_2 = M^2C^2\left(\exp\left(2M^2\max\left\{1,C^2\right\}(T-t_0)^2\, K_1\right)+\exp\left(2M^2\max\left\{1,C^2\right\}(T-t_0)^2\, \overline{K}_1\right)\right).$$

As conclusion,

$$\begin{aligned}\|u-\overline{u}\|_{B\times B}^2 \leq{}& \begin{pmatrix}\|x-\overline{x}\|_B^2\\ \|y-\overline{y}\|_B^2\end{pmatrix}\\ \leq{}& \max\left(\Gamma_1,\Gamma_2\right)\begin{pmatrix}\|\varphi(0)-\overline{\varphi}(0)\|_B^2\\ \|\psi(0)-\overline{\psi}(0)\|_B^2\end{pmatrix},\end{aligned}$$

and the mild solution u of the system (3.1) is stable in the mean square. □

5. Application

Let $\tilde{\Omega}\subset\Re^n$ be a bounded domain with smooth boundary $\partial\tilde{\Omega}$. Consider the coupled impulsive system of equations:

$$\left\{\begin{aligned} u_t(x,t) &= u_{xx}(x,t)+\int_{-\tau}^{t}\mu(\theta)u(t+\theta,x)d\theta+\int_{-\tau}^{t}\overline{\mu}(\theta)v(t+\theta,x)d\theta, t\neq\xi_k, t\geqslant\tau,\\ v_t(x,t) &= v_{xx}(x,t)+\int_{-\tau}^{t}\eta(\theta)u(t+\theta,x)d\theta+\int_{-\tau}^{t}\overline{\eta}(\theta)v(t+\theta,x)d\theta, t\neq\xi_k, t\geqslant\tau,\\ u(x,\xi_k) &= q(k)\tau_k u(x,\xi_k^-),\ \ v(x,\xi_k)= p(k)\tau_k v(x,\xi_k^-), \qquad a.s.\ x\in\tilde{\Omega},\\ u(x,t) &= \varphi(x,t),\ v(x,t)= \psi(x,t), \qquad a.s.\ x\in\tilde{\Omega}, -r\leqslant t\leqslant 0,\\ u(x,t) &= v(x,t)= 0 \qquad a.s.\ x\in\partial\tilde{\Omega}.\end{aligned}\right. \tag{5.1}$$

Here, τ_k is a random variable defined on $D_k\equiv(0,d_k)$ for $k=1,2,\cdots$, where $0<d_k<+\infty$ and $\mu,\eta:[-r,0]\to\Re$ are positive functions. Furthermore, assume that τ_k follow Erlang distribution, where $k=1,2,\cdots$ and τ_i and τ_j are independent with each other as $i\neq j$ for $i,j=1,2,\cdots$; q and p are functions of k; $\xi_0=t_0$; $\xi_k=\xi_{k-1}+\tau_k$ for $k=1,2,\cdots$ and $t_0\in\Re^+$ is an arbitrarily given real number.
Let $X=L^2(\tilde{\Omega})$ and define A, an operator on X, by $Au=\frac{\partial^2 u}{\partial x^2}$ with the domain

$$D(A)=\left\{x\in X/\,u\,\text{and}\,\frac{\partial u}{\partial x}\,\text{are absolutely continuous},\ \frac{\partial^2 u}{\partial x^2}\in X,\ u=0\text{ on }\partial\tilde{\Omega}\right\}.$$

It is well known that A generates a strongly continuous semigroup $S(t)$ which is compact, analytic, and self-adjoint. Moreover, the operator A can be expressed as

$$Au = \sum_{n=1}^{\infty} n^2 \langle u, u_n \rangle u_n, \quad u \in D(A),$$

where $u_n(\varsigma) = (\frac{2}{n})^{\frac{1}{2}} \sin(n\varsigma), \quad n = 1, 2, \cdots$, is the orthonormal set of eigenvectors of A and for every $u \in X, S(t)u = \sum_{n=1}^{\infty} exp(-n^2 t)\langle u, u_n \rangle u_n$, which satisfies $\|S(t)\| \leqslant exp(-\pi^2(t - t_0)), t \geqslant t_0$. Hence, $S(t)$ is a contraction semigroup.
We assume the following conditions hold:
(i) The functions $\mu(.)$ and η are continuous in $\Re$ with

$$\int_{-\tau}^{0} \mu(\theta)^2 d\theta < \infty \text{ and } \int_{-\tau}^{0} \eta(\theta)^2 d\theta < \infty,$$

(ii) $E\left[\max_{i,k} \left\{\prod_{j=i}^{k} \| q(j)(\tau_j) \|^2\right\}\right] < \infty$ and $E\left[\max_{i,k} \left\{\prod_{j=i}^{k} \| p(j)(\tau_j) \|^2\right\}\right] < \infty$.
Assuming that conditions (i) and (ii) are verified, then the problem (5.1) can be modeled as the abstract random impulsive differential equation (3.1) by defining

$$f(t, u_t, v_t) = \int_{-\tau}^{t} \mu(\theta) u(t + \theta, x) d\theta + \int_{-\tau}^{t} \overline{\mu}(\theta) v(t + \theta, x) d\theta \text{ and } b_k(\tau_k) = q(k)\tau_k,$$

and

$$g(t, u_t, v_t) = \int_{-\tau}^{t} \eta(\theta) u(t + \theta, x) d\theta + \int_{-\tau}^{t} \overline{\eta}(\theta) v(t + \theta, x) d\theta, \text{ and } a_k(\tau_k) = p(k)\tau_k.$$

Proposition 5.1. *Assume that the hypotheses (H_1) and (H_2) hold, then the system (5.1) has a unique, global mild solution u.*

Proof. Condition (i) implies that (H_1) holds with $L_1 = \int_{-\tau}^{t} \mu^2(\theta)\, d\theta, \overline{L}_1 = \int_{-\tau}^{t} \overline{\mu}^2(\theta)\, d\theta$, $K_1 = \int_{-\tau}^{t} \eta^2(\theta)\, d\theta$ and $\overline{K}_1 = \int_{-\tau}^{t} \overline{\eta}^2(\theta)\, d\theta$ and (H_2) follows from condition (ii). □

6. CONCLUSION

In this paper, we have used a new technique for the treatment of systems based on the advantage of vector-valued norms in the study of coupled random impulsive semilinear differential systems of equations. Other kinds of these random impulsive differential systems of equations can be studied by this technique.

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