Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 16 Issue 2(2024), Pages 40-51
https://doi.org/10.54671/BMAA-2024-2-4

λ^2-STATISTICAL SUMMABILITY OF DOUBLE SEQUENCES IN FUZZY N-NORMED SPACES

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Abstract. This paper's main purpose is to present λ^2-statistically convergent and condition of being λ^2- statistically Cauchy for double sequences in fuzzy n-normed spaces. At the same time, in fuzzy n-normed space, we have introduced the concept of λ^2-summability and Cesaro summability for double sequences in fuzzy n normed spaces. Then, we studied the relation between these concepts and statistical convergence in fuzzy n-normed space.

1. Introduction and Background

Fast first introduced the concept of statistical convergence of sequences[9]. Studies of statistical convergence have been carried out in many branches of mathematics, such as Fourier analysis, Banach spaces, Number theory and Measurement theory. Connor[4], Šalát[36], Fridy[11], Çınar[7] and others have examined the concept of statistical convergence in summability theory. Gunawan[12], Reddy[27], Hazarika and Savaş[14] studied statistical convergence and statistical Cauchy series in n-normed spaces.

The first study on double sequences was done by Bromwich[32]. Hardy[13], Moricz[23], Moricz and Rhoades[25], Tripathy[33] and many other mathematicians studied this subject.

Pringsheim[26] gave the concept of convergence of even series in 1900. Mursaleen and Edely[22] and Móricz[24] independently extended the concept of statistical convergence to double sequences.

Matloka[17] first introduced the concept of convergence in sequences of fuzzy numbers, and some of the fundamental theorems of fuzzy valued sequences were proved. The readers also refer to the recent textbooks [21] and [3] for functional analysis, summability theory, and related topics and the articles [28, 29, 30] and [31] on the classical sets of fuzzy valued sequences and concerning results. Altınok et al.[1] and many others have studied the statistical convergence of fuzzy numbers.

1991 *Mathematics Subject Classification.* 3E72, 40A05.
Key words and phrases. Statistical convergence; λ-statistical convergence; Fuzzy n-normed space; Summability; Double Sequence.

Submitted May 3, 2024. Accepted for publication June 9, 2024. Published Jun 12, 2024.
Communicated by F. Basar.

Mohiuddine et al.[19] examined the statistical convergence of double sequences in fuzzy normed spaces. Finally, Türkmen and Çınar [35] studied statistical convergence in fuzzy normed linear spaces.

In this study, we define λ-statistical convergence for double sequences in fuzzy n-normed spaces and give some of its properties and results. Now, some basic definitions and theorems, such as λ-statistical convergence, fuzzy number, fuzzy norm, fuzzy n-norm, and double series (Bag and Samanta[2], Diamond and Kloeden[5], Kaleva and Seikkala[15], Mizumoto and Tanaka[18]).

2. Preliminaries

Fuzzy number: Fuzzy sets concern a nonempty base set X of elements of interest. The basic concept is that each element $x \in X$ shall be assigned a membership grade $u(x)$, which takes the values in $[0,1]$, $u(x)=1$ to Full Membership, $0<u(x)<1$ to Partial Membership and $u(x)=0$ equal to nonparticipation.

A fuzzy subset of X, according to Zadeh[37],is not empty subset $\{(x,u(x)) : x \in X\}$ of $X \times [0,1]$ for some function $u : X \to [0,1]$. For a fuzzy set, the function u itself is frequently used.

A fuzzy set u on $\mathbb{R}$ is called a fuzzy number when the following properties are present:

i. There exists an $x_0 \in \mathbb{R}$ such that $u(x_0)=1$, that is, u is normal;

ii. For $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$,

$$u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)],$$

that is, u is fuzzy convex;

iii. u is upper semicontinuous;

iv. $[u]_0 = cl\{x \in \mathbb{R} : u(x) > 0\}$ is compact.

Let $L(\mathbb{R})$ be a set of all fuzzy numbers. u is called a non-negative fuzzy number if $u \in L(\mathbb{R})$ and $u(t)=0$ for $t<0$. We denote the set of all non-negative fuzzy numbers by $L^*(\mathbb{R})$. We can say that $u \in L^*(\mathbb{R})$ iff $u_\alpha^- \geq 0$ for each $\alpha \in [0,1]$. Clearly, we have $\widetilde{0} \in L(\mathbb{R})$. For $u \in L(\mathbb{R})$, the α level set of u is defined by

$$[u]_\alpha = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\}, & \text{if } \alpha \in [0,1] \\ suppu, & \text{if } \alpha = 0. \end{cases}$$

A partial order $\preccurlyeq$ on $L(\mathbb{R})$ is defined by $u \preccurlyeq v$ iff $u_\alpha^- \leq v_\alpha^-$ and $u_\alpha^+ \leq v_\alpha^+$ for all $\alpha \in [0,1]$.

Arithmetic operation $\oplus, \ominus, \odot$ and $\oslash$ on $L(\mathbb{R}) \times L(\mathbb{R})$ are defined by

$$\begin{aligned} (u \oplus v)(t) &= \sup_{s\in\mathbb{R}} \{u(s) \wedge v(t-s)\} \quad t \in \mathbb{R} \\ (u \ominus v)(t) &= \sup_{s\in\mathbb{R}} \{u(s) \wedge v(s-t)\} \quad t \in \mathbb{R} \\ (u \odot v)(t) &= \sup_{\substack{s\in\mathbb{R} \\ s\neq 0}} \{u(s) \wedge v(t/s)\} \quad t \in \mathbb{R} \\ (u \oslash v)(t) &= \sup_{s\in\mathbb{R}} \{u(st) \wedge v(s)\} \quad t \in \mathbb{R} \end{aligned}$$

For $k \in \mathbb{R}^+$, ku is defined as $ku(t) = u(t/k)$ and $0u(t) = \widetilde{0}$, $t \in \mathbb{R}$.

Some arithmetic operations for $\alpha-$level sets are defined as follows:

$u, v \in L(\mathbb{R})$ and $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$ and $[v]_\alpha = [v_\alpha^-, v_\alpha^+]$, $\alpha \in$ 0, 1 . Then

$[u \oplus v]_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+]$ $\quad [u \ominus v]_\alpha = [u_\alpha^- - v_\alpha^+, u_\alpha^+ - v_\alpha^-]$

$[u \odot v]_\alpha = [u_\alpha^-.v_\alpha^-, u_\alpha^+.v_\alpha^+]$ $\quad \left[\widetilde{1} \oslash u\right]_\alpha = \left[\frac{1}{u_\alpha^+}, \frac{1}{u_\alpha^-}\right]$ $u_\alpha^- > 0$

For $u, v \in L(\mathbb{R})$, the supremum metric on $L(\mathbb{R})$ defined as

$D(u,v) = \sup_{0\leq\alpha\leq 1} \max\left\{|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|\right\}.$

It is known that D is a metric on $L(\mathbb{R})$, and $(L(\mathbb{R}), D)$ is a complete metric space.

A sequence $x = (x_k)$ of fuzzy numbers is said to be convergent to the fuzzy number x_0, if for every $\varepsilon > 0$ there exists a positive integer k_0 such that $D(x_k, x_0) < \varepsilon$ for $k > k_0$ and a sequence $x = (x_k)$ of fuzzy numbers convergence to levelwise to x_0 iff $\lim_{k\to\infty} [x_k]_\alpha = [x_0]_\alpha^-$ and $\lim_{k\to\infty} [x_k]_\alpha = [x_0]_\alpha^+$ where $[x_k]_\alpha = \left[(x_k)_\alpha^-, (x_k)_\alpha^+\right]$ and $[x_0]_\alpha = \left[(x_0)_\alpha^-, (x_0)_\alpha^+\right]$ for every $\alpha \in (0,1)$.[18, 17, 16, 10, 8, 5, 2].

λ- statistical convergence:

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ such that $\lambda_n \leq \lambda_n + 1$, $\lambda_1 = 1$. The set of all such sequences will be denoted by Λ. The concept of $\lambda-$ statistical convergence was defined by Mursaleen[20] as follows;

A sequence $x = (x_k)$ is said to be $\lambda-$ statistically convergent or $S_\lambda-$ convergent to L if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - L| \geq \varepsilon\}| = 0,$$

where $I_n = [n - \lambda_n + 1, n]$. In this case, we write $S_\lambda - \lim x = L$ or $x_k \to L(S_\lambda)$ and $S_\lambda = \{x : \exists L \in \mathbb{R}, S_\lambda - \lim x = L\}$.

The concept of λ-statistical convergence of double sequences was defined by Mursaleen [22] as follows;

Let $\gamma = (\gamma_m)$ and $\mu = (\mu_r)$ be two non-decreasing sequences of positive real numbers, each tending to ∞ and such that

$$\begin{aligned} \gamma_{m+1} &\leq \gamma_m + 1,\ \gamma_1 = 1 \text{ and} \\ \mu_{r+1} &\leq \mu_r + 1,\ \mu_1 = 1. \end{aligned}$$

Let $I_m = [m - \gamma_m + 1, m]$ and $J_r = [r - \mu_r + 1, r]$.

For any set $K \subseteq \mathbb{N} \times \mathbb{N}$, the number

$$\delta_\lambda^2(K) = \lim_{m,r\to\infty} \frac{1}{\lambda_{mr}} |\{(k,l) : (k,l) \in K \cap I_m \times J_r\}|$$

is said to be the λ^2-density of K, provided the limit exists, where $\lambda_{mr} = \gamma_m \mu_r$.

We now ready to define the λ^2-statistical convergence.

A double sequence $x = (x_{kl})$ in X is said to be λ^2-statistically convergent to $L \in X$ or S_λ^2-convergent if for each $\varepsilon > 0$

$$P - \lim_{m,r\to\infty} \frac{1}{\lambda_{mr}} |\{(k,l) \in I_m \times J_r : |x_{kl} - L| \geq \varepsilon\}| = 0.$$

In this case, we write $x_{kl} \overset{S_\lambda^2}{\to} L$ or $x_{kl} \to L\left(S_\lambda^2\right)$ or $S_\lambda^2 - \lim x_{kl} = L$. Throughout the paper, we will denote $\lambda_{mr} = \gamma_m \mu_r$ and the collection of such sequences will be denoted by Λ_2. Also we will get $I_m = [m - \gamma_m + 1, m]$ and $J_r = [r - \mu_r + 1, r]$

Convergence in fuzzy normed spaces:

Şençimen[6] defined convergence in fuzzy normed spaces as follows, using Kaleva[15] and Felbin[10]:

Let $(X, \|.\|)$ be an FNS. A sequence $(x_n)_{n=1}^{\infty}$ in X is convergent to $x \in X$ with respect to the fuzzy norm on X and we denote by $x_n \overset{FN}{\rightarrow} x$, provided that $(D) - \lim\limits_{n\to\infty} \|x_n\| = \widetilde{0}$ i.e. for every $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that $D\left(\|x_n - x\|, \widetilde{0}\right) < \varepsilon$ for all $n \in \mathbb{N}$. This means that for every $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that

$$\sup_{\alpha \epsilon [0,\ 1]} \|x_n - x\|_{\alpha}^{+} = \|x_n - x\|_{0}^{+} < \varepsilon$$

for all $n \geq N(\varepsilon)$.

Mohiuddine[19], in his article Statistical convergence of double sequences in fuzzy normed spaces gave the convergence of double sequences in fuzzy normed spaces as follows;

Let $(X, \|\bullet\|)$ be a FNS. Then a double sequence $x = (x_{mn})$ said to be convergent to $L \in X$ with respect to the fuzzy norm on X if for every $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that

$$D(\|x_{mn} - L\|, \widetilde{0}) < \varepsilon$$

for all $m, n \geq N$.

In this case we write $x_{mn} \overset{FN}{\rightarrow} L$. This means that for every $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that

$$\sup_{\alpha \epsilon [0,\ 1]} \|x_{mn} - L\|_{\alpha}^{+} = \|x_{mn} - L\|_{0}^{+} < \varepsilon$$

for all $m, n \geq N(\varepsilon)$.

λ-Statistical convergence of double sequences in fuzzy normed space

Türkmen[34] defined λ-statistical convergence of double sequences in fuzzy normed spaces as follows:

Let $(X, \|\bullet\|)$ be a fuzzy normed space. A double sequence $x = (x_{kl})$ said to be $\lambda-$ statistically convergent or $FS_{\lambda}^{2}-$ convergent to $L \in X$ with respect to the fuzzy norm on X if for every $\varepsilon > 0$

$$\lim_{m,r\to\infty} \frac{1}{\lambda_{mr}} \left|\left\{(k,l) \in I_m \times J_r : D\left(\|x_{kl} - L\|, \widetilde{0}\right) \geq \varepsilon\right\}\right| = 0.$$

It can be represented as $x_{kl} \overset{FS_{\lambda}^{2}}{\rightarrow} L$, $x_{kl} \rightarrow L\left(FS_{\lambda}^{2}\right)$, or $FS_{\lambda}^{2} - \lim x_{kl} = L$.

This means that for every $\varepsilon > 0$, the natural density of the set $K(\varepsilon) = \left\{(k,l) \in I_m \times J_r : \|x_{kl} - L\|_{0}^{+} \geq \varepsilon\right\}$ is zero, In brief, for every $\varepsilon > 0$, and almost all k and l, it is $\|x_{kl} - L\|_{0}^{+} < \varepsilon$. So it is $FS_{\lambda}^{2} - \lim x = L$. The set of all λ-statistically convergent sequences concerning the fuzzy norm on X will be denoted by FS_{λ}^{2} and $FS_{\lambda}^{2} = \left\{x = (x_{kl}) : \exists\ L, FS_{\lambda}^{2} - \lim x = L\ \right\}$.

Let X be a real linear space of dimension d, where $2 \leq d < \infty$. Let $\|\cdot, \cdot, ..., \cdot\| : X^n \rightarrow L^*(\mathbb{R})$ and the mappings $L; R$ (respectively, left norm and right norm) : $[0,1] \times [0,1] \rightarrow [0,1]$ be symmetric, nondecreasing in both arguments and satisfy $L(0,0) = 0$ and $R(1,1) = 1$ then the quadruple $(X, \|\cdot, \cdot, ..., \cdot\|, L, R)$ is called fuzzy n normed linear space (briefly $(X, \|\cdot, \cdot, ..., \cdot\|)$ $FnNS$) and $\|\cdot, \cdot, ..., \cdot\|$ is a fuzzy n norm if the following axioms are satisfied for every $y, w_1, w_2, ..., w_n \in X$ and $s, t \in \mathbb{R}$,

fnN$_1$:$\|w_1, w_2, ..., w_n\| = \widetilde{0}$ if and only if $w_1, w_2, ..., w_n$ are linearly dependent vectors,
fnN$_2$:$\|w_1, w_2, ..., w_n\|$ is invariant under any permutation of $w_1, w_2, ..., w_n$,
fnN$_3$:$\|\alpha w_1, w_2, ..., w_n\| = |\alpha| \|w_1, w_2, ..., w_n\|$ for all $\alpha \in \mathbb{R}$,
fnN$_4$:$\|w_1 + y, w_2, ..., w_n\| (s + t) \geq L (\|w_1, w_2, ..., w_n\| (s), \|y, w_2, ..., w_n\| (t))$ whenever $s \leq \|w_1, w_2, ..., w_n\|_1^-$, $t \leq \|y, w_2, ..., w_n\|_1^-$ and $s + t \leq \|w_1 + y, w_2, ..., w_n\|_1^-$,
fnN$_5$:$\|w_1 + y, w_2, ..., w_n\| (s + t) \leq R (\|w_1, w_2, ..., w_n\| (s), \|y, w_2, ..., w_n\| (t))$ whenever $s \geq \|w_1, w_2, ..., w_n\|_1^-$, $t \geq \|y, w_2, ..., w_n\|_1^-$ and $s + t \geq \|w_1 + y, w_2, ..., w_n\|_1^-$,
where

$[\|w_1, w_2, ..., w_n\|]_\alpha = \left[\|w_1, w_2, ..., w_n\|_\alpha^-, \|w_1, w_2, ..., w_n\|_\alpha^+\right]$ for $w_1, w_2, ..., w_n \in X, 0 \leq \alpha \leq 1$ and $\inf_{\alpha \in [0,1]} \|w_1, w_2, ..., w_n\|_\alpha^- > 0$. Hence the norm $\|\cdot, \cdot, ..., \cdot\|$ is called fuzzy n norm on X and pair $(X, \|\cdot, \cdot, ..., \cdot\|)$ is called fuzzy n normed space.

Let $(X, \|\cdot, \cdot, ..., \cdot\|)$ be fuzzy n normed space. A sequence $\{x_k\}$ in X is said to be convergent to an element $x \in X$ with respect to the fuzzy n norm on X if for every $\varepsilon > 0$ and for every $w_1, w_2, ..., w_{n-1} \in X \setminus \{0\}$, $\exists$ a number $N = N(w_1, w_2, ..., w_{n-1}, \varepsilon)$ such that $D\left(\|w_1, w_2, ..., w_{n-1}, x_k - x\|, \widetilde{0}\right) < \varepsilon$,$\forall k \geq N$ or equivalently

$(D) - \lim_{k\to\infty} \|w_1, w_2, ..., w_{n-1}, x_k - x\| = \widetilde{0}$.

Let $(X, \|\cdot, \cdot, ..., \cdot\|)$ be fuzzy n normed space. A sequence $\{x_k\}$ in X is said to be statistically convergent to an element $x \in X$ with respect to the fuzzy n norm on X if for every $\varepsilon > 0$ and for every $w_1, w_2, ..., w_{n-1} \in X \setminus \{0\}$, we have

$\delta\left(\left\{k \in \mathbb{N} : D\left(\|w_1, w_2, ..., w_{n-1}, x_k - x\|, \widetilde{0}\right) \geq \varepsilon\right\}\right) = 0.$

3. λ^2- STATISTICAL CONVERGENCE

In this section, we define the λ^2-statistical convergence of double sequences and λ^2-statistically Cauchy in fuzzy n normed linear spaces. We've also gotten some basic properties of this notion in fuzzy n normed spaces.

Definition 3.1. Let $(X, \|\cdot, ..., \cdot\|)$ be a fuzzy n normed space. A double sequence $x = (x_{kl})$ in X is said to be λ^2-statistically convergent to $L \in X$ with respect to fuzzy n norm on X or FnS_λ^2-convergent if for each $\varepsilon > 0$ and for every $w_2, w_3, ..., w_n \in X \setminus \{0\}$,

$$\lim_{m,r\to\infty} \frac{1}{\lambda_{mr}} \left|\left\{(k,l) \in I_m \times J_r : D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \geq \varepsilon\right\}\right| = 0,$$

So, we have written $x_{kl} \overset{FnS_\lambda^2}{\to} L$ or $x_{kl} \to L\left(FnS_\lambda^2\right)$ or $FnS_\lambda^2 - \lim x_{kl} = L$. This implies that for each $\varepsilon > 0$, the set

$$K(\varepsilon) = \left\{(k,l) \in I_m \times J_r : \|x_{kl} - L, w_2, w_3, ..., w_n\|_0^+ \geq \varepsilon\right\}$$

has a natural density zero, namely, for each $\varepsilon > 0$, $\|x_{kl} - L, w_2, w_3, ..., w_n\|_0^+ < \varepsilon$ for almost all k and l.

In this case, we have written $FnS_\lambda^2 - \lim x = L$. The set of all λ^2-statistically convergent sequences with respect to fuzzy n norm on X will be denoted by FnS_λ^2 and $FnS_\lambda^2 = \left\{x = (x_{kl}) : \exists L, FnS_\lambda^2 - \lim x = L\right\}$. In this case, we have written throughout the paper (x_{kl}) is FnS_λ^2-convergent to $L \in X$ means that (x_{kl}) is λ^2-statistically convergent to $L \in X$ with respect to the fuzzy n norm on X and $w_2, w_3, ..., w_n \in X \setminus \{0\}$.

If $\gamma_m = m$ and $\mu_r = r$ for all m, r then the space $FnS_\lambda^2(X)$ is reduced to the space $FnS_2(X)$ and since $\delta_2(K) \leq \delta_\lambda^2(K)$ we have $FnS_\lambda^2(X) \subset FnS_2(X)$.

Lemma 3.2. Let $(X, \|\cdot, ..., \cdot\|)$ be a fuzzy n normed space and $x = (x_{kl})$ be a double sequence in X. Then for each $\varepsilon > 0$, the following statements are equivalent:
(i) $FnS_\lambda^2 - \lim\limits_{k,l\to\infty} x_{kl} = L$.
(ii) $\delta_\lambda^2\left(\left\{(k,l) : (k,l) \in I_m \times J_r, \|x_{kl} - L, w_2, w_3, ..., w_n\|_0^+ \geq \varepsilon\right\}\right) = 0$
(iii) $\delta_\lambda^2\left(\left\{(k,l) : (k,l) \in I_m \times J_r, \|x_{kl} - L, w_2, w_3, ..., w_n\|_0^+ < \varepsilon\right\}\right) = 1$
(iii)$FnS_\lambda^2 - \lim\limits_{k,l\to\infty} \|x_{kl} - L, w_2, w_3, ..., w_n\|_0^+ = 0$

Proof. The proofs are clear from the definitions. □

Theorem 3.3. Let $(X, \|\cdot, ..., \cdot\|)$ be a fuzzy n normed space and $\lambda \in \Lambda_2$. If a sequence (x_{kl}) is a FnS_λ^2-convergent, then FnS_λ^2-limit is unique.

Proof. Suppose that $FnS_\lambda^2 - \lim x_{kl} = L_1$ and $FnS_\lambda^2 - \lim x_{kl} = L_2$ and $L_2 - L_1 = 2\varepsilon > 0$. We define the following sets as

$$A(\varepsilon) = \left\{(k,l) \in I_m \times J_r : \|x_{kl} - L_1, w_2, w_3, ..., w_n\|_0^+ \geq \frac{\varepsilon}{2}\right\}$$

and

$$B(\varepsilon) = \left\{(k,l) \in I_m \times J_r : \|x_{kl} - L_2, w_2, w_3, ..., w_n\|_0^+ \geq \frac{\varepsilon}{2}\right\}.$$

So that $\delta_\lambda^2(A(\varepsilon)) = 0$ and $\delta_\lambda^2(B(\varepsilon)) = 0$. It follows that there are $k \in I_m, l \in J_r$ such that $\|x_{kl} - L_1, w_2, w_3, ..., w_n\|_0^+ < \frac{\varepsilon}{2}$ and $\|x_{kl} - L_2, w_2, w_3, ..., w_n\|_0^+ < \frac{\varepsilon}{2}$. Further, for these k and l we have

$$\begin{aligned} 2\varepsilon &= \|L_2 - L_1, w_2, w_3, ..., w_n\|_0^+ \\ &\leq \|x_{kl} - L_2, w_2, w_3, ..., w_n\|_0^+ + \|x_{kl} - L_1, w_2, w_3, ..., w_n\|_0^+ < \varepsilon \end{aligned}$$

which is a contradiction. This completes the proof. □

The next theorem gives the algebraic characterization of λ^2 statistical convergence on fuzzy n normed spaces.

Theorem 3.4. Let (x_{kl}) and (y_{kl}) be sequences in fuzzy n normed space $(X, \|\cdot, ..., \cdot\|)$ such that $x_{kl} \overset{FnS_\lambda^2}{\to} L_1$ and $y_{kl} \overset{FnS_\lambda^2}{\to} L_2$ and $\lambda \in \Lambda_2$ where $L_1, L_2 \in X$. Then we have
i) $(x_{kl} + y_{kl}) \overset{FnS_\lambda^2}{\to} L_1 + L_2$,
ii) $tx_{kl} \overset{FnS_\lambda^2}{\to} tL_1$ $(t \in \mathbb{R})$,

Proof. i) Suppose that $FnS_\lambda^2 - \lim x_{kl} = L_1$ and $FnS_\lambda^2 - \lim y_{kl} = L_2$. We define the following sets as

$$A(\varepsilon) = \left\{(k,l) \in I_m \times J_r : \|x_{kl} - L_1, w_2, w_3, ..., w_n\|_0^+ \geq \frac{\varepsilon}{2}\right\}$$

and

$$B(\varepsilon) = \left\{(k,l) \in I_m \times J_r : \|y_{kl} - L_2, w_2, w_3, ..., w_n\|_0^+ \geq \frac{\varepsilon}{2}\right\}.$$

So that $\delta_\lambda^2(A(\varepsilon)) = 0$ and $\delta_\lambda^2(B(\varepsilon)) = 0$. It follows that there are $k \in I_m, l \in J_r$ such that $\|x_{kl} - L_1, w_2, w_3, ..., w_n\|_0^+ < \frac{\varepsilon}{2}$ and $\|y_{kl} - L_2, w_2, w_3, ..., w_n\|_0^+ < \frac{\varepsilon}{2}$. Further, for these k and l we have

$$\begin{aligned} & \|x_{kl} + y_{kl} - L_1 - L_2, w_2, w_3, ..., w_n\|_0^+ \\ \leq & \|x_{kl} - L_1, w_2, w_3, ..., w_n\|_0^+ + \|y_{kl} - L_2, w_2, w_3, ..., w_n\|_0^+ < \varepsilon. \end{aligned}$$

We define the following sets as

$$C(\varepsilon) = \left\{(k,l) \in I_m \times J_r : \|x_{kl} + y_{kl} - L_1 - L_2, w_2, w_3, ..., w_n\|_0^+ \geq \varepsilon\right\}.$$

$\delta_\lambda^2(C(\varepsilon)) = 0$ and so we get $FnS_\lambda^2 - \lim(x_{kl} + y_{kl}) = L_1 + L_2$

ii) It can be done in a similar way to the first proof. □

Theorem 3.5. Let $(X, \|\cdot, ..., \cdot\|)$ be a fuzzy n normed space and $\lambda \in \Lambda_2$. If a sequence $x = (x_{kl})$ is FS_2-convergent to L and $\liminf\limits_{m,r\to\infty} \left(\frac{\lambda_{mr}}{mr}\right) > 0$ then it is FnS_λ^2- convergent to L .

Proof. For given $\varepsilon > 0$, we have had

$$\begin{aligned} & \left\{(k,l) : k \leq m, l \leq r,\ D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \geq \varepsilon\right\} \\ \supset & \left\{(k,l) \in I_m \times J_r : D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \geq \varepsilon\right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{1}{mr}\left|\left\{(k,l) : k \leq m, l \leq r,\ D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \geq \varepsilon\right\}\right| \\ \geq & \frac{1}{mr}\left|\left\{(k,l) \in I_m \times J_r : D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \geq \varepsilon\right\}\right| \\ \geq & \frac{\lambda_{mr}}{mr} \cdot \frac{1}{\lambda_{mr}}\left|\left\{(k,l) \in I_m \times J_r : D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \geq \varepsilon\right\}\right|. \end{aligned}$$

Taking the limit as $m, r \to \infty$ and using $\liminf\limits_{m,r\to\infty} \left(\frac{\lambda_{mr}}{mr}\right) > 0$, we get $FnS_\lambda^2 - \lim x_{kl} = L$. □

Theorem 3.6. Let $(X, \|\cdot, ..., \cdot\|)$ be a fuzzy n normed space. If a double sequence $x = (x_{kl})$ is convergent to L with respect to fuzzy n norm on X then it is FnS_λ^2- convergent to L .

Proof. Let $x_{kl} \overset{FN}{\to} L$. Then for every $\varepsilon > 0$, there is a couple $(k_0, l_0) \in \mathbb{N} \times \mathbb{N}$ such that $D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \geq \varepsilon$ for all $k \geq k_0, l \geq l_0$. Hence the set

$$\left\{(k,l) : k \in I_m, l \in J_r,\ D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \geq \varepsilon\right\}$$

has natural density zero that is $FnS_\lambda^2 - \lim x_{kl} = L$. □

Theorem 3.7. Let $(X, \|\cdot, ..., \cdot\|)$ be a fuzzy n normed space and $\lambda = (\lambda_{mr}) \in \Lambda_2$. If $\lim\limits_{m,r\to\infty} \frac{\lambda_{mr}}{mr} = 1$ then $FnS_2(X) = FnS_\lambda^2(X)$.

Proof. Suppose that $x_{kl} \overset{FnS_\lambda^2}{\to} L$. Since $\lim_{m,r\to\infty} \frac{\lambda_{mr}}{mr} = 1$, then for $\varepsilon > 0$, we observe that

$$\begin{aligned}
& \frac{1}{mr}\left|\left\{(k,l) : k \le m, l \le r,\ D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \ge \varepsilon\right\}\right| \\
\le\ & \frac{1}{mr}\left|\left\{(k,l) : k \le m - \gamma_m, l \le r - \mu_r,\ D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \ge \varepsilon\right\}\right| \\
& + \frac{1}{mr}\left|\left\{(k,l) \in I_m \times J_r :\ D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \ge \varepsilon\right\}\right| \\
\le\ & \frac{(m-\gamma_m)(r-\mu_r)}{mr} \\
& + \frac{1}{mr}\left|\left\{(k,l) \in I_m \times J_r :\ D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \ge \varepsilon\right\}\right| \\
=\ & \frac{(m-\gamma_m)(r-\mu_r)}{mr} \\
& + \frac{\lambda_{mr}}{mr}\frac{1}{\lambda_{mr}}\left|\left\{(k,l) \in I_m \times J_r :\ D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \ge \varepsilon\right\}\right|.
\end{aligned}$$

This implies that $x_{kl} \overset{FnS_2}{\to} L$. Thus $FnS_\lambda^2(X) \subset FnS_2(X)$.

On the other hand, since $\lim_{m,r\to\infty} \frac{\lambda_{mr}}{mr} = 1$, implies that $\liminf_{m,r\to\infty}\left(\frac{\lambda_{mr}}{mr}\right) > 0$, then from Theorem 3.5, we have $FnS_2(X) \subset FnS_\lambda^2(X)$. Hence $FnS_\lambda^2(X) = FnS_2(X)$. □

Definition 3.8. Let $(X, \|\cdot, ..., \cdot\|)$ be a fuzzy n normed space. A sequence (x_{kl}) in X is $\lambda-$statistically Cauchy with respect to the fuzzy n norm on X provided that for every $\varepsilon > 0$, there exist a positive integers t and v such that for all $k, p \ge t$ and $l, q \ge v$, such that

$$\delta_\lambda^2\left\{(k,l) \in I_m \times J_r : \|x_{kl} - x_{pq}, w_2, w_3, ..., w_n\|_0^+ \ge \varepsilon\right\} = 0.$$

In the sequel, (x_{kl}) is FnS_λ^2-Cauchy means that (x_{kl}) is λ^2-statistically Cauchy with respect to the fuzzy n norm on X.

Theorem 3.9. Let $(X, \|\cdot, ..., \cdot\|)$ be a fuzzy n normed space and (x_{kl}) be a double sequence in X. In $(X, \|\cdot, ..., \cdot\|)$, Every FnS_λ^2-convergent sequence is also an FnS_λ^2-Cauchy with respect to the fuzzy n norm on X.

Proof. Let $x_{kl} \overset{FnS_\lambda^2}{\to} L$ and $\varepsilon > 0$. Then we have $\|x_{kl} - L, w_2, w_3, ..., w_n\|_0^+ < \varepsilon/2$ for a.a.k and l. Choose a positive integers $t \le p$ and $v \le q$ such that $\|x_{pq} - L, w_2, w_3, ..., w_n\|_0^+ < \varepsilon/2$. Now, $\|\cdot, ..., \cdot\|_0^+$ being a norm in the usual sense, we get

$$\begin{aligned}
& \|x_{kl} - x_{pq}, w_2, w_3, ..., w_n\|_0^+ \\
=\ & \|(x_{kl} - L, w_2, w_3, ..., w_n) + (L - x_{pq}, w_2, w_3, ..., w_n)\|_0^+ \\
\le\ & \|x_{kl} - L, w_2, w_3, ..., w_n\|_0^+ + \|x_{pq} - L, w_2, w_3, ..., w_n\|_0^+ \\
<\ & \varepsilon/2 + \varepsilon/2 = \varepsilon
\end{aligned}$$

for all $k, p \ge t$ and $l, q \ge v$. This shows that (x_{kl}) is FnS_λ^2- Cauchy. □

4. λ^2-SUMMABILITY

In this section, we have introduced and studied the concepts of λ^2- summability with respect to fuzzy n norm on X and found its relation with λ^2- statistically convergent with respect to fuzzy n norm on X. Before giving the promised relations we have given definitions of λ^2-summability with respect to fuzzy n norm on X.

Definition 4.1. Let $(X, \|\cdot, ..., \cdot\|)$ be a fuzzy n normed space and let $\gamma = (\gamma_m)$ and $\mu = (\mu_r)$ be two non-decreasing sequences of positive real numbers each tending to ∞ and such that

$$\begin{aligned} \gamma_{m+1} &\leq \gamma_m + 1, \ \gamma_1 = 1 \text{ and} \\ \mu_{r+1} &\leq \mu_r + 1, \ \mu_1 = 1. \end{aligned}$$

and $x = (x_{kl})$ be a double sequence in X. The sequence x is said to be λ^2-*summable* with respect to fuzzy n norm on X if there is a $L \in X$ such that

$$\lim_{m,r\to\infty} \frac{1}{\lambda_{mr}} \sum_{k\in I_m} \sum_{l\in J_r} D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) = 0$$

where $I_m = [m - \gamma_m + 1, m]$, $J_r = [r - \mu_r + 1, r]$ and $\lambda_{mr} = \gamma_m \mu_r$.

In this case, x is said to be $\left(V, \lambda^2\right)_{FnN}$-summable to L . If $\lambda_{mr} = mr$, then $\left(V, \lambda^2\right)_{FnN}$-summability reduces to $(C, 1-1)$-summability with respect to fuzzy n norm on X defined as follows:

$$\lim_{m,r\to\infty} \frac{1}{mr} \sum_{k=1}^{m} \sum_{l=1}^{r} D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) = 0$$

So, we have written for some L

$$\left[V, \lambda^2\right]_{FnN}(X) = \left\{ x = (x_{kl}) : \lim_{m,r\to\infty} \tfrac{1}{\lambda_{mr}} \sum_{k\in I_m, l\in J_r} D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) = 0 \right\}$$

$$[C, 1-1]_{FnN}(X) = \left\{ x = (x_{kl}) : \lim_{m,r\to\infty} \tfrac{1}{mr} \sum_{k,l=1,1}^{m,r} D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) = 0 \right\}$$

Theorem 4.2. Let $(X, \|\cdot, ..., \cdot\|)$ be a fuzzy n normed space and (x_{kl}) be a double sequence in X. If a sequence $x = (x_{kl})$ is $\left[V, \lambda^2\right]_{FnN}$-summable to L , then it is FnS_λ^2- convergent to L .

Proof. Let $\varepsilon > 0$. Since

$$\begin{aligned} &\sum_{k\in I_m, l\in J_r} D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \\ \geq &\sum_{\substack{k\in I_m, l\in J_r \\ D(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}) \geq \varepsilon}} D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \\ \geq &\ \varepsilon. \left| \left\{ (k,l) \in I_m \times J_r : D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \geq \varepsilon \right\} \right| . \end{aligned}$$

This implies that if $\left[V, \lambda^2\right]_{FnN} -$ *summable* to L, then x is FnS_λ^2- convergent to L . □

Theorem 4.3. Let $(X, \|\cdot, ..., \cdot\|)$ be a fuzzy n normed space and (x_{kl}) be a double sequence in X. If a bounded $x = (x_{kl})$ is FnS_λ^2- convergent to L, then it is $\left[V, \lambda^2\right]_{FnN}$-summable to L , and hence x is $[C, 1-1]_{FnN}$-summable to L, provided $x = (x_{kl})$ is not eventually constant.

Proof. Suppose that $x = (x_{kl})$ is bounded and FnS_λ^2- convergent to L. Since x is bounded, there exists a $M > 0$ such that we have written

$D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) < M$ for all k, l.

Given $\varepsilon > 0$, we have had

$$\begin{aligned}
&\tfrac{1}{\lambda_{mr}} \sum_{k\in I_m, l\in J_r} D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \\
&= \tfrac{1}{\lambda_{mr}} \sum_{\substack{k\in I_m, l\in J_r \\ D(\|x_{kl}-L,w_2,w_3,...,w_n\|,\tilde{0})\geq\varepsilon}} D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \\
&+ \tfrac{1}{\lambda_n} \sum_{\substack{k\in I_m, l\in J_r \\ D(\|x_{kl}-L,w_2,w_3,...,w_n\|,\tilde{0})<\varepsilon}} D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \\
&\leq \tfrac{M}{\lambda_{mr}} \left|\left\{(k, l) \in I_m \times J_r : D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \geq \varepsilon\right\}\right| + \varepsilon
\end{aligned}$$

which implies that x is $\left[V, \lambda^2\right]_{FnN}$-summable to L.
Further, we have had

$$\begin{aligned}
&\tfrac{1}{mr} \sum_{k,l=1}^{m,r} D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \\
&= \tfrac{1}{mr} \sum_{k,l=1}^{m-\gamma_m, r-\mu_r} D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \\
&+ \tfrac{1}{mr} \sum_{k\in I_m, l\in J_r} D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \\
&\leq \tfrac{1}{\lambda_{mr}} \sum_{k,l=1,1}^{m-\gamma_m, r-\mu_r} D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \\
&+ \tfrac{1}{\lambda_{mr}} \sum_{k\in I_m, l\in J_r} D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right) \\
&\leq \tfrac{2}{\lambda_{mr}} \sum_{k\in I_m, l\in J_r} D\left(\|x_{kl} - L, w_2, w_3, ..., w_n\|, \tilde{0}\right).
\end{aligned}$$

Hence, x is $[C, 1-1]_{FnN}$-summable to L since x is $\left[V, \lambda^2\right]_{FnN}$- summable to L. □

By combining Theorems 4.2 and 4.3, we obtain the following result.

Theorem 4.4. Let $(X, \|\cdot, ..., \cdot\|)$ be a fuzzy n normed space and (x_{kl}) be a double sequence in X. In this case, $FnS_\lambda^2(X) \cap l_\infty^2(X) = \left[V, \lambda^2\right]_{FnN}(X) \cap l_\infty^2(X)$.

If we let $\lambda_{mr} = mr$ in Theorem 4.2,4.3 and 4.4 then we have the following corollary.

Corollary 4.5. Let $(X, \|\cdot, ..., \cdot\|)$ be a fuzzy n normed space and (x_{kl}) be a double sequence in X. Then

(i) $(x_{kl}) \to L\left([C, 1-1]_{FnN}\right) \Rightarrow (x_{kl}) \to L\left(FnS_2\right)$,
(ii) $(x_{kl}) \in l_\infty^2$ and $(x_{kl}) \to L\left(FnS_2\right)$ then $(x_{kl}) \to L\left([C, 1-1]_{FnN}\right)$
(iii) $FnS_2(X) \cap l_\infty^2(X) = [C, 1-1]_{FnN} \cap l_\infty^2(X)$.

5. Conclusion

In this study, we define λ^2-statistical convergent sequences, λ^2- statistically Cauchy sequence, λ^2-summability, and Cesaro summability for double sequences in fuzzy n normed spaces. Besides we have studied the relation between λ^2-statistically convergence and λ^2-summability in fuzzy n normed spaces. Finally, for bounded double sequences, we saw that

$$\begin{aligned} FnS_{\lambda}^{2}(X) &= \left[V,\lambda^{2}\right]_{FnN}(X), \\ FnS_{2}(X) &= [C,1-1]_{FnN}. \end{aligned}$$

Acknowledgments. The author would like to thank the anonymous referee for his/her comments that helped us improve this article.

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