Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 16 Issue 3(2024), Pages 13-27
https://doi.org/10.54671/BMAA-2024-3-2

Ψ-HILFER FRACTIONAL-ORDER LANGEVIN EQUATIONS: EXISTENCE AND UNIQUENESS REVISITED

SAVITA RATHEE, YOGEETA NARWAL

Abstract. This work discusses the existence and uniqueness findings for Langevin equations of fractional-order with mixed nonlocal boundary conditions. The existence findings are derived using Krasnoselskii's fixed point theorem and the nonlinear alternative of Leray-Schauder. Uniqueness is proved using the Banach contraction mapping principle. The fractional derivatives are described in the Ψ-Hilfer sense. There are numerous examples highlighting the key findings.

1. Introduction

Initial and boundary conditions for a range of *fractional differential equations* (FDEs) have been analysed by plenty of academics within the prior ten years. Since the advent of fractional calculus in 1695, there have been numerous definitions of integral and fractional derivatives (FDs), and these definitions have evolved [16]. For continuous functions, the most common is the Riemann-Liouville (R-L) and Caputo FD of order $\nu > 0$. Hilfer [8] proposed a generalized fractional derivative of order $\nu \in (0,1)$ and type $\beta \in [0,1]$, which may be reduced to the R-L FD for $\beta = 0$ and Caputo FD for $\beta = 1$. Several writers commonly refer to it as the *Hilfer fractional derivative* (HFD).

In 2018, Asawasamrit *et al.* [1] proposed a new class of the Hilfer boundary value problem (HBVP) and developed various existence and uniqueness criteria for their solution using nonlocal integral boundary conditions:

$$^{H}\mathcal{D}^{\nu,\beta}y(\mathfrak{t}) = \mathcal{G}(t, y(\mathfrak{t})), \quad \mathfrak{t} \in [a,b],\, 1 < \nu < 2,\, 0 \leq \beta \leq 1, \tag{1.1}$$

$$y(a) = 0,\, y(b) = \sum_{i=1}^{m} \lambda_i I_i^{\mu} y(\xi_i), \quad \mu_i > 0,\, \lambda_i \in \mathbb{R},\, \xi_i \in [a,b], \tag{1.2}$$

where $^{H}\mathcal{D}^{\nu,\beta}$ is the HFD of order ν, and parameter β, I^{μ_i} is the R-L fractional integral of order $\mu_i > 0$, $\xi_i \in [a,b]$, $a \geq 0$ and $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$.

2000 *Mathematics Subject Classification.* 26A33, 34A08, 34B15.

Key words and phrases. Ψ-Hilfer fractional derivative; Banach fixed point theorem; Krasnoselskii's fixed point theorem; Langevin equation.

Submitted April 2, 2024. Accepted August 7, 2024. Published September 11, 2024.
Communicated by Tongxing Li.

Corresponding Author: Yogeeta Narwal (yogeetawork@gmail.com).

As there are so many distinct definitions of integrals and fractional derivatives, a fractional derivative of one function with respect to another was required [13]. In 2018, Sousa *et al.* [16], introduced the Ψ-Hilfer fractional derivative (Ψ-HFD), which is the fractional derivative of one function with respect to another function in the Hilfer sense. In [15], Ntouyas and Vivek investigated the existence and uniqueness of solutions for a new type of sequential Ψ-Hilfer fractional differential equations with multi-point boundary conditions of the form

$$\left({}^{H}\mathcal{D}^{\nu,\beta;\Psi} + k^{H}\mathcal{D}^{\nu-1,\beta;\Psi}\right)y(\mathfrak{t}) = \mathcal{G}(t, y(\mathfrak{t})), \quad \mathfrak{t} \in [a,b],\ 1 < \nu < 2,\ 0 \leq \beta \leq 1, \tag{1.3}$$

$$y(a) = 0,\ y(b) = \sum_{i=1}^{m} \lambda_i y(\theta_i), \quad k, \lambda_i \in \mathbb{R},\ \theta_i \in [a,b], \tag{1.4}$$

where ${}^{H}\mathcal{D}^{\nu,\beta;\Psi}$ is the Ψ-HFD of order ν, $1 < \nu < 2$ and parameter β, $0 \leq \beta \leq 1$, $\mathcal{G} : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $i = 1, 2, \ldots, m$.

Nuchpong *et al.* [14] in the year 2021 studied the existence and uniqueness results for the following BVP of Langevin FDE with Ψ-Hilfer fractional derivative and nonlocal integral boundary conditions:

$$\mathcal{D}^{\chi_1,\beta_1;\Psi}(\mathcal{D}^{\chi_2,\beta_2;\Psi} + k)y(\mathfrak{t}) = \mathcal{G}(t, y(\mathfrak{t})), \quad t \in [a,b], \tag{1.5}$$

$$y(a) = 0,\ y(b) = \sum_{i=1}^{m} \lambda_i I^{\delta_i;\Psi} y(\theta_i), \quad \theta_i \in [a,b], \tag{1.6}$$

where $\mathcal{D}^{\chi_i,\beta_i;\Psi}$, $i = 1, 2$ is the Ψ-HFD of order χ_i, $0 < \chi_i < 1$ and type β_i, $0 \leq \beta_i \leq 1$, $i = 1, 2$; $1 \leq \chi_1 + \chi_2 \leq 2$, $k \in \mathbb{R}$, $a \geq 0$ and $\mathcal{G} : \Xi \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I^{\delta_i;\Psi}$ is Ψ-R-L fractional integral of order $\delta_i > 0$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$.

Similarly, Guida *et al.* [7] in 2021 studied the existence and uniqueness for sequential Ψ-Hilfer fractional pantograph differential equations with mixed nonlocal boundary conditions as follows:

$$\left({}^{H}\mathcal{D}^{\nu,\beta;\Psi} + p^{H}\mathcal{D}^{\nu-1,\beta;\Psi}\right)y(\mathfrak{t}) = \mathcal{G}(t, y(\mathfrak{t}), y(\sigma t)), \quad \mathfrak{t} \in [0,T],\ 0 < \sigma < 1, \tag{1.7}$$

$$y(0) = 0,\ \sum_{i=1}^{m} \delta_i y(\eta_i) + \sum_{j=1}^{n} \omega_j I^{\beta;\Psi} y(\theta_j) + \sum_{k=1}^{r} \lambda_k {}^{H}\mathcal{D}^{\mu;\Psi} y(\xi_k) = A, \tag{1.8}$$

where ${}^{H}\mathcal{D}^{\nu,\beta;\Psi}$ are the Ψ-Hilfer derivatives (HD) of order $u = \{\nu, \mu_k\}$, $1 < \mu_k < \nu \leq 2$, $0 \leq \beta \leq 1$, $I^{\beta;\Psi}$ are the Ψ-R-L fractional integrals of order β_j, with $\beta_j > 0$ for $1 \leq j \leq n$; p, A, δ_i, ω_j and $\lambda_k \in \mathbb{R}$ are given constants, the points η_i, θ_j, ξ_k are in J, for $1 \leq i \leq m$; $1 \leq j \leq n$; $1 \leq k \leq r$ and the function $\mathcal{G} : \Xi \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, $J = [0,T]$, $T > 0$.

Motivated by [1, 7, 14, 15], the existence and uniqueness of solutions for the following fractional-order langevin equation with nonlocal mixed multi-point integro-differential boundary conditions are investigated:

$$\mathcal{D}^{\nu_1,\beta_1;\Psi}(\mathcal{D}^{\nu_2,\beta_2;\Psi} + p)\varkappa(\mathfrak{t}) = \mathcal{F}(\mathfrak{t}, x(\mathfrak{t})),\ 0 < \nu_i < 1,\ 0 \leq \beta_i \leq 1,\ i = 1, 2, \mathfrak{t} \in [a,b], \tag{1.9}$$

$$\varkappa(a) = 0, \varkappa(b) = \sum_{i \in \mathcal{I}} \delta_i \varkappa(\eta_i) + \sum_{j \in \mathcal{J}} \omega_j I^{\tau_j;\Psi} \varkappa(\theta_j) + \sum_{k \in \mathcal{K}} \lambda_k {}^{H}\mathcal{D}^{\mu;\Psi} \varkappa(\xi_k), \tag{1.10}$$

where $\mathcal{D}^{\nu_i,\beta_i;\Psi}$, $i = 1, 2$ is the Ψ-HFD of order ν_i, $0 < \nu_i < 1$ and type β_i, $0 \le \beta_i \le 1$; $1 < \nu_1 + \nu_2 \le 2$, $k \in \mathbb{R}$, $a \ge 0$, $\mathcal{F} : \Xi \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I^{\tau;\Psi}$ are the Ψ-RL FI of order τ_j, with $\tau_j > 0$ for $j \in \mathcal{J}$; p, A, δ_i, ω_j and $\lambda_k \in \mathbb{R}$ are given constants. The points η_i, θ_j, ξ_k are in Ξ for $\mathcal{I} = \{i \in \mathcal{I} : i = 1, 2, \ldots, m\}$, $\mathcal{J} = \{j \in \mathcal{J} : j = 1, 2, \ldots, n\}$ and $\mathcal{K} = \{k \in \mathcal{K} : k = 1, 2, \ldots, r\}$.

Special Cases:

(i) For $\delta_i = 0$, $\forall\ i \in \mathcal{I}$; $\lambda_k = 0$, $\forall\ j \in \mathcal{J}$; the investigations of [14] regarding the hybrid differential equation of integer order are incorporated into the results of the current work

$$\mathcal{D}^{\nu_1,\beta_1;\Psi}(D^{\nu_2,\beta_2;\Psi} + p)\varkappa(\mathfrak{t}) = \mathcal{F}(\mathfrak{t}, x(\mathfrak{t})),$$
$$\varkappa(a) = 0,\ \varkappa(b) = \sum_{j\in\mathcal{J}} \omega_j I^{\tau_j;\Psi}\varkappa(\theta_j).$$

(ii) For $\nu_2 = 0$, $p = 0$, $\delta_i = 0$, $\forall\ i \in \mathcal{I}$; $\lambda_k = 0$, $\forall\ k \in \mathcal{K}$, we obtain the nonlocal BVP

$$\mathcal{D}^{\nu_1,\beta_1;\Psi}\varkappa(\mathfrak{t}) = \mathcal{F}(\mathfrak{t}, x(\mathfrak{t})),$$
$$\varkappa(a) = 0,\ \varkappa(b) = \sum_{j\in\mathcal{J}} \omega_j I^{\tau_j;\Psi}\varkappa(\theta_j),$$

the findings of existence and uniqueness for which are acquired in [1].

(iii) For $\omega_j = 0$, $\forall\ j \in \mathcal{I}$; $\lambda_k = 0$, $\forall\ k \in \mathcal{J}$; we obtain a nonlocal BVP for Ψ-Hilfer fractional order Langevin equations with multi-point boundary conditions

$$\mathcal{D}^{\nu_1,\beta_1;\Psi}(\mathcal{D}^{\nu_2,\beta_2;\Psi} + p)\varkappa(\mathfrak{t}) = \mathcal{F}(\mathfrak{t}, x(\mathfrak{t})),$$
$$\varkappa(a) = 0,\ \varkappa(b) = \sum_{i\in\mathcal{I}} \delta_i\varkappa(\eta_i).$$

(iv) For $\beta_1,\ \beta_2 = 0$, $\Psi(\mathfrak{t}) = \mathfrak{t}$; we get nonlinear FDEs with multi-point fractional integro-differential boundary conditions

$$^{RL}\mathcal{D}^{\nu_1}(^{RL}\mathcal{D}^{\nu_2} + p)\varkappa(\mathfrak{t}) = \mathcal{F}(\mathfrak{t}, x(\mathfrak{t})),$$
$$\varkappa(a) = 0,\ \varkappa(b) = \sum_{i\in\mathcal{I}} \delta_i\varkappa(\eta_i) + \sum_{j\in\mathcal{J}} \omega_j I^{\tau_j;\Psi}\varkappa(\theta_j) + \sum_{k\in\mathcal{K}} \lambda_k\, {}^{H}\mathcal{D}^{\mu;\Psi}\varkappa(\xi_k).$$

Organization of paper is as follows: Section 2 reviews some necessary preliminaries from fractional calculus. Section 3 focuses on the existence and uniqueness of solutions to the BVP (1.9)-(1.10) using FPT. To show the important conclusions, examples are built. The findings are summarised in Section 4.

2. Auxiliary Results

For all $\mathfrak{t} \in \Xi = [a, b]$ and $\Psi'(\mathfrak{t}) \ne 0$, consider an increasing function $\Psi \in \mathcal{C}^1(\Xi, \mathbb{R})$.

Definition 2.1 ([11]). *Let $\nu > 0$, $\nu \in \mathbb{R}$, and $g \in L^1([a, b], \mathbb{R})$. The Ψ-R-L fractional integral of a function g with respect to Ψ is defined by*

$$I^{\nu;\Psi} = \frac{1}{\Gamma(\nu)} \int_a^t \Psi'(s)(\Psi(\mathfrak{t}) - \Psi(s))^{\nu-1} g(s)ds.$$

Definition 2.2 ([16])**.** *Let $n-1<\nu<n$, $n\in\mathbb{N}$ and $g\in C^n([a,b],\mathbb{R})$. The Ψ-HFD ${}^H\mathcal{D}^{\nu,\beta;\Psi}(\cdot)$ of a function g of order ν and type $0\le\beta\le 1$ is defined by*

$$ {}^H\mathcal{D}^{\nu,\beta;\Psi}g(\mathfrak{t}) = I^{\beta(n-\nu);\Psi}\left(\frac{1}{\Psi'(s)}\frac{d}{dt}\right)^n I^{(1-\beta)(n-\nu);\Psi}g(\mathfrak{t}). $$

Lemma 2.3 ([16])**.** *Let $\nu>0$ and $\delta>0$. Then*

(i) $I^{\nu;\Psi}I^{\nu;\Psi}h(\mathfrak{t}) = I^{\nu+\nu;\Psi}h(\mathfrak{t})$;
(ii) $I^{\nu;\Psi}(\Psi(\mathfrak{t})-\Psi(s))^{\delta-1} = \frac{\Gamma(\nu)}{\Gamma(\nu+\delta)}(\Psi(\mathfrak{t})-\Psi(s))^{\nu+\delta-1}$.

We note also that ${}^H\mathcal{D}^{(\nu,\beta;\Psi)}(\Psi(\mathfrak{t})-\Psi(s))^{(\gamma-1)} = 0$.

Lemma 2.4 ([16])**.** *Let $f\in L(a,b)$, $n-1<\nu\le n$, $n\in\mathbb{N}$, $0\le\beta\le 1$, $\gamma=\nu+\beta(1-\nu)$, $I^{(1-\beta)(n-\nu)}f\in AC^k[a,b]$. Then*

$$ \begin{aligned} &\left(I^{\nu;\Psi\,H}\mathcal{D}^{\nu,\beta;\Psi}f\right)(\mathfrak{t}) \\ &\quad = f(\mathfrak{t}) - \sum_{k=1}^{n}\frac{(\Psi(\mathfrak{t})-\Psi(s))^{\gamma-k}}{\Gamma(\gamma-k+1)}\left(\frac{1}{\Psi'(s)}\frac{d}{dt}\right)^n \lim_{t\to a^+}(I^{(1-\beta)(n-\nu)}f)(\mathfrak{t}). \end{aligned} $$

Here we mention some of the FPT used in this paper for the convenience of the readers.

Lemma 2.5 ([3], Banach fixed point theorem)**.** *Let $\varkappa$ be a Banach space, $D\subset X$ closed, and $F:D\to D$ a strict contraction, i.e., $|F\varkappa - Fy|\le k|\varkappa - y|$ for some $k\in(0,1)$ and all $\varkappa, y\in D$. Then F has a fixed point in D.*

Lemma 2.6 ([12], Krasnoselskii's fixed point theorem)**.** *Let M be a closed, bounded, convex and nonempty subset of a Banach space $\varkappa$. Let A, B be the operators such that*

(i) *$A\varkappa + By\in M$ whenever $\varkappa, y\in M$;*
(ii) *A is compact and continuous;*
(iii) *B is a contraction mapping.*

Then there exists $z = Az + Bz$.

Lemma 2.7 ([6], Nonlinear alternative for single valued maps)**.** *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0\in U$. Suppose that $A:U\to[C]$ is a continuous and compact map such that*

(i) *A has a fixed point in U;*
(ii) *there is an $\varkappa\in\partial U$ (the boundary of U in C) and $\lambda\in(0,1)$ with $\varkappa=\lambda A(\varkappa)$*

Then, there exist $z = Az + Bz$.

Remark. *To simplify the notation and the proof of some results, we will introduce the following notation $\mathcal{Q}_{\Psi}^{\sigma-1}(\mathfrak{t},a) = (\Psi(\mathfrak{t})-\Psi(a))^{\sigma-1}$ and $\nu_1+\nu_2=\mathfrak{z}$.*

3. Main Results

We start by proving an auxiliary lemma for the BVP (1.9)-(1.10).

Lemma 3.1. *The function $\varkappa$ is a solution of the following BVP*

$$ \mathcal{D}^{\nu_1,\beta_1;\Psi}(\mathcal{D}^{\nu_2,\beta_2;\Psi}+p)\varkappa(\mathfrak{t}) = h(\mathfrak{t});\quad 0\le\nu_i\le 1,\ h\in\mathcal{C}(\Xi,\mathbb{R}), \tag{3.1} $$

$$ \varkappa(a)=0,\ \varkappa(b)=\sum_{i\in\mathcal{I}}\delta_i\varkappa(\eta_i) + \sum_{j\in\mathcal{J}}\omega_j I^{\tau_j;\Psi}\varkappa(\theta_j) + \sum_{k\in\mathcal{K}}\lambda_k\,{}^H\mathcal{D}^{\mu_k;\Psi}\varkappa(\xi_k),\ \text{where } a\ge 0, \tag{3.2} $$

if and only if

$$\begin{aligned}\varkappa(\mathfrak{t}) = I^{\mathfrak{z};\Psi}h(\mathfrak{t}) - pI^{\nu_2;\Psi}\varkappa(\mathfrak{t}) + \frac{\mathcal{Q}_\Psi^{\gamma_1+\nu_2-1}(\mathfrak{t},a)}{\Lambda\Gamma(\gamma_1+\nu_2)} \\ \cdot\Big[\varkappa(b) + p\Big(\sum_{i\in\mathcal{I}}\delta_i I^{\nu_2;\Psi}\varkappa(\eta_i) + \sum_{j\in\mathcal{J}}\omega_j I^{\nu_2+\tau_j;\Psi}\varkappa(\theta_j) + \sum_{k\in\mathcal{K}}\lambda_k I^{\nu_2-\mu_k;\Psi}\varkappa(\xi_k)\Big) \\ -\Big(\sum_{i\in\mathcal{I}}\delta_i I^{\mathfrak{z};\Psi}h(\eta_i) + \sum_{j\in\mathcal{J}}\omega_j I^{\mathfrak{z}+\tau_j;\Psi}h(\theta_j) + \sum_{k\in\mathcal{K}}\lambda_k I^{\mathfrak{z}-\mu_k;\Psi}h(\xi_k)\Big)\Big],\ \mathfrak{t}\in\Xi,\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}\Lambda = \sum_{i\in\mathcal{I}}\delta_i\frac{\mathcal{Q}_\Psi^{\gamma_1+\nu_2-1}(\eta_i,a)}{\Gamma(\gamma_1+\nu_2)} + \sum_{j\in\mathcal{J}}\omega_j\frac{\mathcal{Q}_\Psi^{\gamma_1+\nu_2+\tau_j-1}(\theta_j,a)}{\Gamma(\gamma_1+\nu_2+\tau_j)} \\ + \sum_{k\in\mathcal{K}}\lambda_k\frac{\mathcal{Q}_\Psi^{\gamma_1+\nu_2-\mu_k-1}(\xi_k,a)}{\Gamma(\gamma_1+\nu_2-\mu_k)} \neq 0\end{aligned} \tag{3.4}$$

and $\gamma_i = \nu_i + \beta_i(1-\nu_i)$*,* $i = 1,2$*;* $1 \le \mathfrak{z} \le 2$*.*

Proof. Applying Ψ-R-L FI of order ν_1 and Lemma 2.4 on (3.1), we obtain

$$\mathcal{D}^{\nu_2,\beta_2;\Psi}\varkappa(\mathfrak{t}) + p\varkappa(\mathfrak{t}) = I^{\nu_1;\Psi}h(\mathfrak{t}) + \frac{c_0}{\Gamma(\gamma_1)}\mathcal{Q}_\Psi^{\gamma_1-1}(\mathfrak{t},a). \tag{3.5}$$

Applying Ψ-R-L FI of order ν_2 and Lemma 2.4 on the above equation, we get

$$\varkappa(\mathfrak{t}) = I^{\mathfrak{z};\Psi}h(\mathfrak{t}) - pI^{\nu_2;\Psi}\varkappa(\mathfrak{t}) + \frac{c_0}{\Gamma(\gamma_1+\nu_2)}\mathcal{Q}_\Psi^{\gamma_1+\nu_2-1}(\mathfrak{t},a) + \frac{c_1}{\Gamma(\gamma_2)}\mathcal{Q}_\Psi^{\gamma_2-1}(\mathfrak{t},a). \tag{3.6}$$

Using $\varkappa(a) = 0$, we get $c_1 = 0$. Thus,

$$\varkappa(\mathfrak{t}) = \frac{c_0}{\Gamma(\gamma_1+\nu_2)}\mathcal{Q}_\Psi^{\gamma_1+\nu_2-1}(\mathfrak{t},a) + I^{\mathfrak{z};\Psi}h(\mathfrak{t}) - pI^{\nu_2;\Psi}\varkappa(\mathfrak{t}). \tag{3.7}$$

Applying operators ${}^H\mathcal{D}^{\mu_k,\epsilon;\Psi}$ and $I^{\tau_j;\Psi}$ to the above equation

$$\begin{aligned}{}^H\mathcal{D}^{\mu_k,\epsilon;\Psi}\varkappa(\mathfrak{t}) &= \frac{c_0}{\Gamma(\gamma_1+\nu_2-\mu_k)}\mathcal{Q}_\Psi^{\gamma_1+\nu_2-\mu_k-1}(\mathfrak{t},a) - pI^{\nu_2-\mu_k;\Psi}\varkappa(\mathfrak{t}) + I^{\mathfrak{z}-\mu_k;\Psi}h(\mathfrak{t}), \\ I^{\tau_j;\Psi}\varkappa(\mathfrak{t}) &= \frac{c_0}{\Gamma(\gamma_1+\nu_2+\tau_j)}\mathcal{Q}_\Psi^{\gamma_1+\nu_2+\tau_j-1}(\mathfrak{t},a) - pI^{\nu_2+\tau_j;\Psi}\varkappa(\mathfrak{t}) + I^{\mathfrak{z}+\tau_j;\Psi}h(\mathfrak{t}).\end{aligned}$$

Now, using the second boundary condition, we get

$$\begin{aligned}c_0\Big[\sum_{i\in\mathcal{I}}\delta_i\frac{\mathcal{Q}_\Psi^{\gamma_1+\nu_2-1}(\eta_i,a)}{\Gamma(\gamma_1+\nu_2)} + \sum_{j\in\mathcal{J}}\omega_j\frac{\mathcal{Q}_\Psi^{\gamma_1+\nu_2+\tau_j-1}(\theta_j,a)}{\Gamma(\gamma_1+\nu_2+\tau_j)} + \sum_{k\in\mathcal{K}}\lambda_k\frac{\mathcal{Q}_\Psi^{\gamma_1+\nu_2-\mu_k-1}(\xi_k,a)}{\Gamma(\gamma_1+\nu_2-\mu_k)}\Big] \\ - p\Big[\sum_{i\in\mathcal{I}}\delta_i I^{\nu_2;\Psi}\varkappa(\eta_i) + \sum_{j\in\mathcal{J}}\omega_j I^{\nu_2+\tau_j;\Psi}\varkappa(\theta_j) + \sum_{k\in\mathcal{K}}\lambda_k I^{\nu_2-\mu_k;\Psi}\varkappa(\xi_k)\Big] \\ + \sum_{i\in\mathcal{I}}\delta_i I^{\mathfrak{z};\Psi}h(\eta_i) + \sum_{j\in\mathcal{J}}\omega_j I^{\mathfrak{z}+\tau_j;\Psi}h(\theta_j) + \sum_{k\in\mathcal{K}}\lambda_k I^{\mathfrak{z}-\mu_k;\Psi}h(\xi_k) = \varkappa(b).\end{aligned}$$

This implies

$$c_0 = \frac{1}{\Lambda}\Big[\varkappa(b) + p\Big(\sum_{i\in\mathcal{I}}\delta_i I^{\nu_2;\Psi}\varkappa(\eta_i) + \sum_{j\in\mathcal{J}}\omega_j I^{\nu_2+\tau_j;\Psi}\varkappa(\theta_j) + \sum_{k\in\mathcal{K}}\lambda_k I^{\nu_2-\mu_k;\Psi}\varkappa(\xi_k)\Big)$$

$$-\Bigg(\sum_{i\in\mathcal{I}}\delta_i I^{\mathfrak{z};\Psi}h(\eta_i)+\sum_{j\in\mathcal{J}}\omega_j I^{\mathfrak{z}+\tau_j;\Psi}h(\theta_j)+\sum_{k\in\mathcal{K}}\lambda_k I^{\mathfrak{z}-\mu_k;\Psi}h(\xi_k)\Bigg)\Bigg] \tag{3.8}$$

Substituting c_0 in (3.6) we get (3.3). Direct computation using the specification of Ψ-HFD of order ν and type β and Lemma 2.3 yields the converse. □

Consider the Banach space $\mathcal{C}=\mathcal{C}(\Xi,\mathbb{R})$ of all functions $\Phi:\Xi\to\mathbb{R}$ which are continuous and endowed with the norm

$$\|\varkappa\|=\sup_{\mathfrak{t}\in[a,b]}|\varkappa(\mathfrak{t})|$$

In light of Lemma **??**, we set $\zeta:\mathcal{C}\to\mathcal{C}$ as follows:

$$\begin{aligned}(\zeta\varkappa)(\mathfrak{t}) &= I^{\mathfrak{z};\Psi}f(\mathfrak{t},\varkappa(\mathfrak{t}))-pI^{\nu_2;\Psi}\varkappa(\mathfrak{t})+\frac{\mathcal{Q}_\Psi^{\gamma_1+\nu_2-1}(b,a)}{\Lambda\Gamma(\gamma_1+\nu_2)}\\ &\quad\cdot\Bigg[\varkappa(b)+p\Bigg(\sum_{i\in\mathcal{I}}\delta_i I^{\nu_2;\Psi}\varkappa(\eta_i)+\sum_{j\in\mathcal{J}}\omega_j I^{\nu_2+\tau_j;\Psi}\varkappa(\theta_j)+\sum_{k\in\mathcal{K}}\lambda_k I^{\nu_2-\mu_k;\Psi}\varkappa(\xi_k)\Bigg)\\ &\quad-\Bigg(\sum_{i\in\mathcal{I}}\delta_i I^{\mathfrak{z};\Psi}f(\eta_i,\varkappa(\eta_i))+\sum_{j\in\mathcal{J}}\omega_j I^{\mathfrak{z}+\tau_j;\Psi}f(\theta_j,\varkappa(\theta_j))\\ &\quad+\sum_{k\in\mathcal{K}}\lambda_k I^{\mathfrak{z}-\mu_k;\Psi}f(\xi_k,\varkappa(\xi_k))\Bigg)\Bigg].\end{aligned} \tag{3.9}$$

It is worth noting that the sequential BVP (1.9)-(1.10) has a solution if and only if ζ has a fixed point. For the sake of brevity, we set:

$$\begin{aligned}\Omega_1 &= \frac{\mathcal{Q}_\Psi^{\mathfrak{z}}(b,a)}{\Gamma(\mathfrak{z}+1)}+\frac{\mathcal{Q}_\Psi^{\gamma_1+\nu_2-1}(b,a)}{|\Lambda|\Gamma(\gamma_1+\nu_2)}\\ &\quad\cdot\Bigg[\sum_{i\in\mathcal{I}}|\delta_i|\frac{\mathcal{Q}_\Psi^{\mathfrak{z}}(\eta_i,a)}{\Gamma(\mathfrak{z}+1)}+\sum_{j\in\mathcal{J}}|\omega_j|\frac{\mathcal{Q}_\Psi^{\mathfrak{z}+\tau_j}(\theta_j,a)}{\Gamma(\mathfrak{z}+\tau_j+1)}+\sum_{k\in\mathcal{K}}|\lambda_k|\frac{\mathcal{Q}_\Psi^{\mathfrak{z}+\mu_k}(\xi_k,a)}{\Gamma(\mathfrak{z}-\mu_k+1)}\Bigg],\end{aligned} \tag{3.10}$$

$$\begin{aligned}\Omega_2 &= |p|\Bigg\{\frac{\mathcal{Q}_\Psi^{\nu_2}(b,a)}{\Gamma(\nu_2+1)}+\frac{\mathcal{Q}_\Psi^{\gamma_1+\nu_2-1}(b,a)}{|\Lambda|\Gamma(\gamma_1+\nu_2)}\\ &\quad\cdot\Bigg[\sum_{i\in\mathcal{I}}|\delta_i|\frac{\mathcal{Q}_\Psi^{\nu_2}(\eta_i,a)}{\Gamma(\nu_2+1)}+\sum_{j\in\mathcal{J}}|\omega_j|\frac{\mathcal{Q}_\Psi^{\nu_2+\tau_j}(\theta_j,a)}{\Gamma(\nu_2+\tau_j)}+\sum_{k\in\mathcal{K}}|\lambda_k|\frac{\mathcal{Q}_\Psi^{\nu_2-\mu_k}(\xi_k,a)}{\Gamma(\nu_2-\mu_k+1)}\Bigg]\Bigg\}.\end{aligned} \tag{3.11}$$

3.1. **Existence results.** We use Krasnoselskii's FPT to demonstrate our inaugural result for the existence of solution for the BVP (1.9)-(1.10).

Theorem 3.2. *Assume that:*

(H1) $|\mathcal{F}(\mathfrak{t},\varkappa(\mathfrak{t}))|\leq\phi(\mathfrak{t})$ *is satisfied for a continuous function* $\mathcal{F}:\Xi\times\mathbb{R}\to\mathbb{R}$, *and* $\forall\ (\mathfrak{t},\varkappa)\in\Xi\times\mathbb{R}$ *with* $\phi\in\mathcal{C}(\Xi,\mathbb{R})$.

(H2) $\Omega_2<1$, *where* Ω_2 *is given by* (3.11).

Then, (1.9)-(1.10) *has at least one solution on* Ξ.

Proof. We shall prove that ζ satisfies the prerequisites of Krasnoselskii's FPT. On the closed ball $B_\epsilon=\left\{\varkappa\in\mathcal{C}:\|\varkappa\|\leq\epsilon \text{ with } \epsilon\geq\dfrac{\|\phi\|\Omega_1}{1-\Omega_2}\right\}$, we divide the operator ζ

into the sum of two operators ζ_1 and ζ_2, where

$$\sup_{\mathfrak{t}\in\Xi}\phi(\mathfrak{t}) = \|\phi\|,$$

$$\begin{aligned}(\zeta_1\varkappa)(\mathfrak{t}) &= I^{\mathfrak{z};\Psi}\mathcal{F}(\mathfrak{t},\varkappa(\mathfrak{t})) - \frac{\mathcal{Q}_\Psi^{\gamma_1+\nu_2-1}(\mathfrak{t},a)}{\Lambda\Gamma(\gamma_1+\nu_2)}\Bigg[\sum_{i\in\mathcal{I}}\delta_i I^{\mathfrak{z};\Psi}\mathcal{F}(\eta_i,\varkappa(\eta_i))\\ &\quad+\sum_{j\in\mathcal{J}}\omega_j I^{\mathfrak{z}+\tau_j;\Psi}\mathcal{F}(\theta_j,\varkappa(\theta_j)) + \sum_{k\in\mathcal{K}}\lambda_k I^{\mathfrak{z}-\mu_k;\Psi}\mathcal{F}(\xi_k,\varkappa(\xi_k))\Bigg] \qquad (3.12)\end{aligned}$$

and

$$\begin{aligned}(\zeta_2\varkappa)(\mathfrak{t}) &= -pI^{\nu_2;\Psi}\varkappa(\mathfrak{t}) + \frac{\mathcal{Q}_\Psi^{\gamma_1+\nu_2-1}(\mathfrak{t},a)}{\Lambda\Gamma(\gamma_1+\nu_2)}\\ &\quad\cdot\Bigg[\varkappa(b) + k\Bigg(\sum_{i\in\mathcal{I}}\delta_i I^{\nu_2;\Psi}\varkappa(\eta_i) + \sum_{j\in\mathcal{J}}\omega_j I^{\nu_2+\tau_j;\Psi}\varkappa(\theta_j)\\ &\quad+\sum_{k\in\mathcal{K}}\lambda_k I^{\nu_2-\mu_k;\Psi}\varkappa(\xi_k)\Bigg)\Bigg]. \qquad (3.13)\end{aligned}$$

For any $\varkappa, y \in B_\epsilon$, we have

$$\begin{aligned}|(\zeta_1\varkappa)(\mathfrak{t}) + (\zeta_2 y)(\mathfrak{t})| &\le \sup_{\mathfrak{t}\in\Xi}\Bigg\{I^{\mathfrak{z};\Psi}|\mathcal{F}(\mathfrak{t},\varkappa(\mathfrak{t}))| + |p|I^{\nu_2;\Psi}|y(\mathfrak{t})| + \frac{\mathcal{Q}_\Psi^{\gamma_1+\nu_2-1}(\mathfrak{t},a)}{|\lambda|\Gamma(\gamma_1+\nu_2)}\\ &\quad\cdot\Bigg[|y(b)| + |p|\Bigg(\sum_{i\in\mathcal{I}}|\delta_i|I^{\nu_2;\Psi}|y(\eta_i)| + \sum_{j\in\mathcal{J}}|\omega_j|I^{\nu_2+\tau_j;\Psi}|y(\theta_j)|\\ &\quad+\sum_{k\in\mathcal{K}}|\lambda_k|I^{\nu_2-\mu_k;\Psi}|y(\xi_k)|\Bigg) + \sum_{i\in\mathcal{I}}|\delta_i|I^{\mathfrak{z};\Psi}|\mathcal{F}(\eta_i,\varkappa(\eta_i))|\\ &\quad+\sum_{j\in\mathcal{J}}|\omega_j|I^{\mathfrak{z}+\tau_j;\Psi}|\mathcal{F}(\theta_j,\varkappa(\theta_j))|\\ &\quad+\sum_{k\in\mathcal{K}}|\lambda_k|I^{\mathfrak{z}-\mu_k;\Psi}|\mathcal{F}(\xi_k,\varkappa(\xi_k))|\Bigg]\Bigg\}\\ &\le \|\phi\|\Bigg\{\frac{\mathcal{Q}_\Psi^{\mathfrak{z}}(b,a)}{\Gamma(\mathfrak{z}+1)} + \frac{\mathcal{Q}_\Psi^{\gamma_1+\nu_2-1}(b,a)}{|\Lambda|\Gamma(\gamma_1+\nu_2)}\\ &\quad\cdot\Bigg[\sum_{i\in\mathcal{I}}|\delta_i|\frac{\mathcal{Q}_\Psi^{\mathfrak{z}}(\eta_i,a)}{\Gamma(\mathfrak{z}+1)} + \sum_{j\in\mathcal{J}}|\omega_j|\frac{\mathcal{Q}_\Psi^{\mathfrak{z}+\tau_j}(\theta_j,a)}{\Gamma(\mathfrak{z}+\tau_j+1)}\\ &\quad+\sum_{k\in\mathcal{K}}|\lambda_k|\frac{\mathcal{Q}_\Psi^{\mathfrak{z}+\mu_k}(\xi_k,a)}{\Gamma(\mathfrak{z}-\mu_k+1)}\Bigg]\Bigg\}\\ &\quad+\|y\||p|\Bigg\{\frac{\mathcal{Q}_\Psi^{\nu_2}(b,a)}{\Gamma(\nu_2+1)} + \frac{\mathcal{Q}_\Psi^{\gamma_1+\nu_2-1}(b,a)}{|\Lambda|\Gamma(\gamma_1+\nu_2)}\Bigg[\sum_{i\in\mathcal{I}}|\delta_i|\frac{\mathcal{Q}_\Psi^{\nu_2}(\eta_i,a)}{\Gamma(\nu_2+1)}\\ &\quad+\sum_{j\in\mathcal{J}}|\omega_j|\frac{\mathcal{Q}_\Psi^{\nu_2+\tau_j}(\theta_j,a)}{\Gamma(\nu_2+\tau_j)} + \sum_{k\in\mathcal{K}}|\lambda_k|\frac{\mathcal{Q}_\Psi^{\nu_2-\mu_k}(\xi_k,a)}{\Gamma(\nu_2-\mu_k+1)}\Bigg]\Bigg\}\\ &\le \|\phi\|\Omega_1 + \epsilon\Omega_2\\ &\le \epsilon\end{aligned}$$

and hence $\|\zeta_1\varkappa + \zeta_2 y\| \leq \epsilon$ which implies that $\zeta_1\varkappa + \zeta_2 y \in B_\epsilon$. ζ_2 is a contraction mapping on using (H2). Since, $\mathcal{F}$ is continuous. As a result, ζ_1 is a continuous operator. Furthermore, on $B\epsilon$; ζ_1 is uniformly bounded as

$$\|\zeta_1\varkappa\| \leq \Omega_1\|\phi\| \tag{3.14}$$

We now show that the operator ζ_1 is compact.
Let

$$\sup_{(\mathfrak{t},\varkappa)\in J\times B_\epsilon} |\mathcal{F}(\mathfrak{t},\varkappa)| = F < \infty$$

and thus, we have

$$\begin{aligned}
|(\zeta_1\varkappa_n)(\mathfrak{t}) - (\zeta_1\varkappa)(\mathfrak{t})| \leq I^{\mathfrak{z};\Psi}|(\mathcal{F}_{\varkappa_n})(\mathfrak{t}) - (\mathcal{F}_{\varkappa})(\mathfrak{t})| &+ \frac{\mathcal{Q}_\Psi^{\gamma_1+\nu_2-1}(\mathfrak{t},a)}{|\Lambda|\Gamma(\gamma_1+\nu_2)} \\
&\cdot \Bigg[\sum_{i\in\mathcal{I}} |\delta_i| I^{\mathfrak{z};\Psi}|(\mathcal{F}_{\varkappa_n})(\eta_i) - (\mathcal{F}_{\varkappa})(\eta_i)| \\
&+ \sum_{j\in\mathcal{J}} |\omega_j| I^{\mathfrak{z}+\tau_j;\Psi}|(\mathcal{F}_{\varkappa_n})(\theta_j) - (\mathcal{F}_{\varkappa})(\theta_j)| \\
&+ \sum_{k\in\mathcal{K}} |\lambda_k| I^{\mathfrak{z}-\mu_k;\Psi}|(\mathcal{F}_{\varkappa_n})(\xi_k) - (\mathcal{F}_{\varkappa})(\xi_k)| \Bigg] \\
\leq \frac{\mathcal{Q}_\Psi^{\mathfrak{z}}(\mathfrak{t},a)}{\Gamma(\mathfrak{z})}\|\mathcal{F}_{\varkappa_n} - \mathcal{F}_{\varkappa}\| &+ \frac{\mathcal{Q}_\Psi^{\mathfrak{z}-1}(\mathfrak{t},a)}{|\Lambda|\Gamma(\gamma_1+\nu_2)} \\
&\cdot \Bigg(\sum_{i\in\mathcal{I}} |\delta_i| \frac{\mathcal{Q}_\Psi^{\mathfrak{z}}(\eta_i,a)}{\Gamma(\mathfrak{z})} + \sum_{j\in\mathcal{J}} |\omega_j| \frac{\mathcal{Q}_\Psi^{\mathfrak{z}+\tau_j}(\theta_j,a)}{\Gamma(\mathfrak{z}+\tau_j)} \\
&+ \sum_{k\in\mathcal{K}} |\lambda_k| \frac{\mathcal{Q}_\Psi^{\mathfrak{z}+\mu_k}(\xi_k,a)}{\Gamma(\mathfrak{z}-\mu_k)} \Bigg) \|\mathcal{F}_{\varkappa_n} - f_{\varkappa}\|.
\end{aligned}$$

Since $\mathcal{F}$ is continuous, this implies $\|\mathcal{F}_{\varkappa_n} - \mathcal{F}_{\varkappa}\| \to 0$ as $n \to \infty$. ζ_1 is thus equicontinuous. So, on B_ϵ, ζ_1 is relatively compact. Arzela-Ascoli theorem implies the compactness of ζ_1 on B_ϵ. Thus, by applying Krasnoselskii's FPT, (1.9)-(1.10) has at least one solution on Ξ. □

Remark. *Some special cases of the above theorem are given by:*
For $\beta_1 = 0$, we get $\gamma_1 = \nu_1$ by setting the constant Ω_2^0 as

$$\begin{aligned}
\Omega_2^0 = |p| \Bigg\{ &\frac{\mathcal{Q}_\Psi^{\nu_2}(b,a)}{\Gamma(\nu_2+1)} + \frac{\mathcal{Q}_\Psi^{\mathfrak{z}-1}(b,a)}{|\Lambda_0|\Gamma(\mathfrak{z})} \\
&\cdot \Bigg[\sum_{i\in\mathcal{I}} |\delta_i| \frac{\mathcal{Q}_\Psi^{\nu_2}(\eta_i,a)}{\Gamma(\nu_2+1)} + \sum_{j\in\mathcal{J}} |\omega_j| \frac{\mathcal{Q}_\Psi^{\nu_2+\tau_j}(\theta_j,a)}{\Gamma(\nu_2+\tau_j)} + \sum_{k\in\mathcal{K}} |\lambda_k| \frac{\mathcal{Q}_\Psi^{\nu_2-\mu_k}(\xi_k,a)}{\Gamma(\nu_2-\mu_k+1)} \Bigg] \Bigg\},
\end{aligned} \tag{3.15}$$

where

$$\Lambda_0 = \sum_{i\in\mathcal{I}} \delta_i \frac{\mathcal{Q}_\Psi^{\mathfrak{z}-1}(\eta_i,a)}{\Gamma(\mathfrak{z})} + \sum_{j\in\mathcal{J}} \omega_j \frac{\mathcal{Q}_\Psi^{\mathfrak{z}+\tau_j-1}(\theta_j,a)}{\Gamma(\mathfrak{z}+\tau_j)} + \sum_{k\in\mathcal{K}} \lambda_k \frac{\mathcal{Q}_\Psi^{\mathfrak{z}-\mu_k-1}(\xi_k,a)}{\Gamma(\mathfrak{z}-\mu_k)}. \tag{3.16}$$

Corollary 3.3. *Let* (H1) *hold along with* $\Omega_2^0 < 1$. *Then* (1.9)-(1.10) *has a unique solution on* Ξ, *where* Ω_2^0 *is defined by* (3.15).
For $\beta_1 = 1$, *we get* $\gamma_1 = 1$ *by setting the constant* Ω_2^1 *as*

$$\Omega_2^1 = |p|\Bigg\{\frac{\mathcal{Q}_\Psi^{\nu_2}(b,a)}{\Gamma(\nu_2+1)} + \frac{\mathcal{Q}_\Psi^{\nu_2}(b,a)}{|\Lambda_1|\Gamma(\nu_2+1)} \cdot \Bigg[\sum_{i\in\mathcal{I}}|\delta_i|\frac{\mathcal{Q}_\Psi^{\nu_2}(\eta_i,a)}{\Gamma(\nu_2+1)} + \sum_{j\in\mathcal{J}}|\omega_j|\frac{\mathcal{Q}_\Psi^{\nu_2+\tau_j}(\theta_j,a)}{\Gamma(\nu_2+\tau_j+1)} + \sum_{k\in\mathcal{K}}|\lambda_k|\frac{\mathcal{Q}_\Psi^{\nu_2-\mu_k}(\xi_k,a)}{\Gamma(\nu_2-\mu_k+1)}\Bigg]\Bigg\}, \tag{3.17}$$

where

$$\Lambda_1 = \sum_{i\in\mathcal{I}}\delta_i\frac{\mathcal{Q}_\Psi^{\nu_2}(\eta_i,a)}{\Gamma(\nu_2+1)} + \sum_{j\in\mathcal{J}}\omega_j\frac{\mathcal{Q}_\Psi^{\nu_2+\tau_j}(\theta_j,a)}{\Gamma(\nu_2+\tau_j+1)} + \sum_{k\in\mathcal{K}}\lambda_k\frac{\mathcal{Q}_\Psi^{\nu_2-\mu_k}(\xi_k,a)}{\Gamma(\nu_2-\mu_k+1)}. \tag{3.18}$$

Corollary 3.4. *Assume* (H1) *hold along with* $\Omega_2^1 < 1$. *Then* (1.9)-(1.10) *has a unique solution on* Ξ, *for* Ω_2^1 *defined by* (3.17).

Example 3.5. *Taking the BVP of a fractional-order Langevin equation with a nonlocal mixed multi-point boundary condition and a* Ψ*-Hilfer fractional derivative:*

$$\mathcal{D}^{\frac{5}{4},\frac{3}{4};t}\left(\mathcal{D}^{\frac{1}{5},\frac{3}{5};t} + \frac{1}{12}\right)\varkappa(\mathfrak{t}) = \frac{1}{5\sqrt{\pi}}\left(\sin\mathfrak{t}\tan^{-1}\varkappa + \frac{\pi}{2}\right), \quad \mathfrak{t}\in[0,1], \tag{3.19}$$

$$\varkappa(0) = 0,\ \varkappa(1) = \frac{1}{2}\varkappa\left(\frac{1}{3}\right) + \frac{1}{4}I^{\frac{1}{2};t}\varkappa\left(\frac{2}{3}\right) + \frac{1}{8}{}^H\mathcal{D}^{\frac{1}{3};t}\varkappa\left(\frac{1}{4}\right). \tag{3.20}$$

Here, $\nu_1 = \frac{5}{4}$, $\nu_2 = \frac{1}{5}$, $\beta_1 = \frac{3}{4}$, $\beta_2 = \frac{3}{5}$, $p = \frac{1}{12}$, $a = 0$, $b = 3$, $m = 1$, $n = 1$, $r = 1$, $\delta_1 = \frac{1}{2}$, $\omega_1 = \frac{1}{4}$, $\lambda_1 = \frac{1}{8}$, $\eta_1 = \frac{1}{3}$, $\theta_1 = \frac{2}{3}$, $\xi_1 = \frac{1}{4}$, $\tau_1 = \frac{1}{2}$, $\mu_1 = \frac{1}{3}$, $\Psi(\mathfrak{t}) = \mathfrak{t}$.
We see that $|\mathcal{F}(\mathfrak{t},\varkappa)| \leq \frac{\sqrt{\pi}}{5}$. *Using the given data, we get* $\gamma_1 = \nu_1 + \beta_1(1-\nu_1) = \frac{17}{16}$, $|\Lambda| \approx 1.5514$.
Thus, $\Omega_1 \approx 0.8952$. *All the requirements of Theorem* 3.2 *have been met. As a result, the given BVP has at least one solution on* $[0,1]$.

In this context, we apply the Leray-Schauder's Nonlinear Alternative to derive the second existence result.

Theorem 3.6. *Let* (H2) *hold and*

(H3) $\exists$ *a function* $q \in \mathcal{C}(\Xi,\mathbb{R}^+)$ *and a function* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *which is continuous and non-decreasing such that*

$$|\mathcal{F}(\mathfrak{t},\varkappa)| \leq q(\mathfrak{t})\phi(|\varkappa|), \quad \forall\ (\mathfrak{t},\varkappa)\in\Xi\times\mathbb{R};$$

(H4) $\exists$ *a constant* $\mathcal{R} > 0$ *such that*

$$\frac{(1-\Omega_2)\mathcal{R}}{\phi(\mathcal{R})\|q\|\Omega_1} > 1\,.$$

Then, (1.9)-(1.10) *has at least one solution on* Ξ, *for* Ω_1, Ω_2 *provided by* (3.10) *and* (3.11), *respectively.*

Proof. Let ζ be defined by (3.9).
Step 1. To show that in $\mathcal{C}(\Xi,\mathbb{R})$, ζ maps bounded sets into bounded sets. Consider a bounded set $B_\epsilon = \{\varkappa\in\mathcal{C}(\Xi,\mathbb{R}) : \|\varkappa\| \leq \epsilon\}$ in $\mathcal{C}(\Xi,\mathbb{R})$ for a number $\epsilon > 0$. Then,

for $\mathfrak{t} \in \Xi$

$$
\begin{aligned}
|(\zeta\varkappa)(\mathfrak{t})| \leq \sup_{\mathfrak{t}\in\Xi} \Bigg\{ & I^{\mathfrak{z};\Psi}|\mathcal{F}(\mathfrak{t},\varkappa(\mathfrak{t}))| + |p| I^{\nu_2;\Psi}|\varkappa(\mathfrak{t})| + \frac{\mathcal{Q}_{\Psi}^{\gamma_1+\nu_2-1}(\mathfrak{t},a)}{|\Lambda|\Gamma(\gamma_1+\nu_2)} \\
& \cdot \Bigg[\sum_{i\in\mathcal{I}} |\delta_i| I^{\mathfrak{z};\Psi}|\mathcal{F}(\eta_i,\varkappa(\eta_i))| + \sum_{j\in\mathcal{J}} |\omega_j| I^{\mathfrak{z}+\tau_j;\Psi}|\mathcal{F}(\theta_j,\varkappa(\theta_j))| \\
& + \sum_{k\in\mathcal{K}} |\lambda_k| I^{\mathfrak{z}-\mu_k;\Psi}(|\mathcal{F}(\xi_k,\varkappa(\xi_k))|\Bigg] + |p| \Bigg(\sum_{i\in\mathcal{I}} |\delta_i| I^{\nu_2;\Psi}|\varkappa(\eta_i)| \\
& + \sum_{j\in\mathcal{J}} |\omega_j| I^{\nu_2+\tau_j;\Psi}|\varkappa(\theta_j)| + \sum_{k\in\mathcal{K}} |\lambda_k| I^{\nu_2-\mu_k;\Psi}|\varkappa(\xi_k)| + |\varkappa(b)| \Bigg) \Bigg] \Bigg\} \\
\leq \|q\|\phi(\|\varkappa\|) \Bigg\{ & \frac{\mathcal{Q}_{\Psi}^{\mathfrak{z}}(b,a)}{\Gamma(\mathfrak{z}+1)} + \frac{\mathcal{Q}_{\Psi}^{\gamma_1+\nu_2-1}(b,a)}{\Lambda\Gamma(\gamma_1+\nu_2)} \\
& \cdot \Bigg[\sum_{i\in\mathcal{I}} |\delta_i| \frac{\mathcal{Q}_{\Psi}^{\mathfrak{z}}(\eta_i,a)}{\Gamma(\mathfrak{z}+1)} + \sum_{j\in\mathcal{J}} |\omega_j| \frac{\mathcal{Q}_{\Psi}^{\mathfrak{z}+\tau_j}(\theta_j,a)}{\Gamma(\mathfrak{z}+\tau_j+1)} \\
& + \sum_{k\in\mathcal{K}} |\lambda_k| \frac{\mathcal{Q}_{\Psi}^{\mathfrak{z}+\mu_k}(\xi_k,a)}{\Gamma(\mathfrak{z}-\mu_k+1)} \Bigg] \Bigg\} + \|\varkappa\||p| \Bigg\{ \frac{\mathcal{Q}_{\Psi}^{\nu_2}(b,a)}{\Gamma(\nu_2+1)} \\
& + \frac{\mathcal{Q}_{\Psi}^{\gamma_1+\nu_2-1}(b,a)}{\Lambda\Gamma(\gamma_1+\nu_2)} \Bigg[\sum_{i\in\mathcal{I}} |\delta_i| \frac{\mathcal{Q}_{\Psi}^{\nu_2}(\eta_i,a)}{\Gamma(\nu_2+1)} \\
& + \sum_{j\in\mathcal{J}} |\omega_j| \frac{\mathcal{Q}_{\Psi}^{\nu_2+\tau_j}(\theta_j,a)}{\Gamma(\nu_2+\tau_j+1)} + \sum_{k\in\mathcal{K}} |\lambda_k| \frac{\mathcal{Q}_{\Psi}^{\nu_2-\mu_k}(\xi_k,a)}{\Gamma(\nu_2-\mu_k+1)} + |\varkappa(b)| \Bigg] \Bigg\} \\
= \|q\|\phi(\|\varkappa\|)\Omega_1 & + \|\varkappa\|\Omega_2
\end{aligned}
$$

$$\implies \quad \|\zeta\varkappa\| \leq \|q\|\phi(\epsilon)\Omega_1 + \Omega_2\epsilon$$

Step 2. For $\varkappa \in B_\epsilon$ and $\mathfrak{t}_1, \mathfrak{t}_2 \in \Xi$ with $\mathfrak{t}_1 < \mathfrak{t}_2$, we shall establish that ζ maps bounded sets into equicontinuous sets of $\mathcal{C}(\Xi,\mathbb{R})$.

$$
\begin{aligned}
& |(\zeta\varkappa)(\mathfrak{t}_2) - (\zeta\varkappa)(\mathfrak{t}_1)| \\
& = \frac{\|q\|\Psi(\epsilon)}{\Gamma(\mathfrak{z}+1)} [2\mathcal{Q}_{\Psi}^{\mathfrak{z}}(\mathfrak{t}_2,\mathfrak{t}_1) + |\mathcal{Q}_{\Psi}^{\mathfrak{z}}(\mathfrak{t}_2,a) - \mathcal{Q}_{\Psi}^{\mathfrak{z}}(\mathfrak{t}_1,a)|] + \frac{|p|\epsilon}{\gamma(\nu_2+1)} \mathcal{Q}_{\Psi}^{\nu_2}(\mathfrak{t}_2,\mathfrak{t}_1) \\
& + \frac{\mathcal{Q}_{\Psi}^{\gamma_1+\nu_2-1}(\mathfrak{t}_2,a) - \mathcal{Q}_{\Psi}^{\gamma_1+\nu_2-1}(\mathfrak{t}_1,a)}{|\Lambda|\Gamma(\gamma_1+\nu_2)} \Bigg[\sum_{i\in\mathcal{I}} |\delta_i| \|q\|\Psi(\epsilon) \frac{\mathcal{Q}_{\Psi}^{\mathfrak{z}}(\eta_i,a)}{\Gamma(\mathfrak{z}+1)} \\
& + \sum_{j\in\mathcal{J}} |\omega_j| \|q\|\Psi(\epsilon) \frac{\mathcal{Q}_{\Psi}^{\mathfrak{z}+\tau_j}(\theta_j,a)}{\Gamma(\mathfrak{z}+\tau_j+1)} + \sum_{k\in\mathcal{K}} |\lambda_k| \|q\|\Psi(\epsilon) \frac{\mathcal{Q}_{\Psi}^{\mathfrak{z}+\mu_k}(\xi_k,a)}{\Gamma(\mathfrak{z}-\mu_k+1)} \Bigg] \\
& + \frac{\mathcal{Q}_{\Psi}^{\gamma_1+\nu_2-1}(\mathfrak{t}_2,a) - \mathcal{Q}_{\Psi}^{\gamma_1+\nu_2-1}(\mathfrak{t}_1,a)}{|\Lambda|\Gamma(\gamma_1+\nu_2)} \Bigg[\sum_{i\in\mathcal{I}} |\delta_i| \frac{\mathcal{Q}_{\Psi}^{\nu_2}(\eta_i,a)}{\Gamma(\nu_2+1)} \\
& + \sum_{j\in\mathcal{J}} |\omega_j| \frac{\mathcal{Q}_{\Psi}^{\nu_2+\tau_j}(\theta_j,a)}{\Gamma(\nu_2+\tau_j+1)} + \sum_{k\in\mathcal{K}} |\lambda_k| \frac{\mathcal{Q}_{\Psi}^{\nu_2-\mu_k}(\xi_k,a)}{\Gamma(\nu_2+\mu_k+1)} \Bigg].
\end{aligned}
$$

Now, $\mathfrak{t}_2 - \mathfrak{t}_1 \to 0$ implies R.H.S. tends to 0, independently of $\varkappa \in B\epsilon$. The set ζB_ϵ is relatively compact and equicontinuous. Arzela-Ascoli theorem implies complete continuity of ζ.

Step 3. To show that $\Theta = \{\varkappa : \varkappa = \lambda\zeta\varkappa, \forall\, \lambda \in (0,1)\}$ is bounded. If $\varkappa$ is a solution to this equation, then we have for $\mathfrak{t} \in \Xi$ and almost the exact computations as in *Step* 1; we have

$$|\varkappa(\mathfrak{t})| \leq \phi(\|\varkappa\|)\|q\|\Omega_1 + \|\varkappa\|\Omega_2$$

which gives

$$\frac{(1-\Omega_2)\|\varkappa\|}{\phi(\|\varkappa\|)\|q\|\Omega_1} \leq 1\,.$$

In view of (H3), $\exists\, \mathcal{R}$ such that $\|\varkappa\| \neq \mathcal{R}$. Considering $\mathfrak{U} = \{\varkappa \in \mathcal{C}(\Xi,\mathbb{R}) : \|\varkappa\| < \mathcal{R}\}$, ζ is a continuous and completely continuous mapping from $\overline{\mathfrak{U}} \to \mathcal{C}(\Xi,\mathbb{R})$. By the choice of $\mathfrak{U}$ we could not find any $\varkappa \in \partial\mathfrak{U}$ such that $\varkappa \in \Theta$. From Lemma 2.7, the fixed point $\varkappa \in \overline{\mathfrak{U}}$ of ζ is also a solution of (1.9)-(1.10). □

Example 3.7. *Given the multi-point BVP with Ψ-HFD*

$$\mathcal{D}^{\frac{2}{3},\frac{4}{5};\mathfrak{t}}\left(\mathcal{D}^{\frac{4}{5},\frac{2}{3};\mathfrak{t}} + \frac{1}{14}\right)\varkappa(\mathfrak{t}) = \frac{1}{10}\left(\frac{1}{6}|\varkappa| + \frac{1}{8}\cos\varkappa + \frac{|\varkappa|}{4(1+|\varkappa|)} + \frac{1}{16}\right),\ \mathfrak{t} \in [0,1], \tag{3.21}$$

$$\varkappa(0) = 0,\ \varkappa(1) = 0.2\varkappa\left(\frac{1}{3}\right) + 0.25 I^{0.5;\mathfrak{t}}\varkappa\left(\frac{2}{3}\right) + 0.6\mathcal{D}^{1.5;\mathfrak{t}}\varkappa\left(\frac{1}{4}\right). \tag{3.22}$$

Here, $\nu_1 = \frac{2}{3}$, $\nu_2 = \frac{4}{5}\beta_1 = \frac{4}{5}$, $\beta_2 = \frac{2}{3}$, $\Psi(\mathfrak{t}) = \mathfrak{t}$, $p = \frac{1}{14}$, $\delta_1 = 0.2$, $\omega_1 = 0.25$, $\lambda_1 = 0.6$, $\eta_1 = \frac{1}{3}$, $\theta_1 = \frac{2}{3}$, $\xi_1 = \frac{1}{4}$, $\tau_1 = 0.5$, $\mu_1 = 1.5$. *We observe that*

$$|\mathcal{F}(\mathfrak{t},\varkappa)| \leq \frac{1}{10}\left(\frac{1}{6}|\varkappa| + \frac{7}{16}\right)$$

and for $\phi(|\varkappa|) = \frac{1}{10}\left(\frac{1}{6}|\varkappa| + \frac{7}{16}\right)$, $\|q\| = \frac{1}{10}$ (H3) *is satisfied.*
Also, $\Lambda \approx 0.25713$, $\Omega_1 \approx 0.9782$ *and* $\Omega_2 = 0.30266$. *Now,*

$$\frac{(1-\Omega_2)\mathcal{R}}{\phi(\mathcal{R})\|q\|\Omega_1} > 1$$

for $\mathcal{R} > 0.16746$. *Thus, by Theorem 3.6, problem* (3.21)-(3.22) *must have at least one solution on* $[0,1]$.

3.2. **Uniqueness results.** We have the following uniqueness result based on Banach FPT.

Theorem 3.8. *Assume that*

(H1) $\mathcal{F}$ *satisfies Lipschitz condition for second variable with constant* $\mathcal{L}$,

and $\mathcal{L}\Omega_1 + \Omega_2 < 1$, *for* Ω_1 *and* Ω_2 *defined by* (3.10) *and* (3.11), *respectively. Then, the BVP* (1.9)-(1.10) *has a unique solution on* Ξ.

Proof. To show equation (3.9) has a fixed point which is then a solution of the BVP (1.9)-(1.10). Set

$$\sup_{\mathfrak{t}\in\Xi} |\mathcal{F}(\mathfrak{t},0)| = \mathcal{R} < \infty$$

and choose

$$\epsilon \geq \frac{\mathcal{R}\Omega_1}{1-\mathcal{L}\Omega_1-\Omega_2}\,.$$

Now, we show that $\zeta B_\epsilon \subset B_\epsilon$, where $B_\epsilon = \{\varkappa \in \mathcal{C}(\Xi, \mathbb{R}); \|\varkappa\| \leq \epsilon\}$. For any $\varkappa \in B_\epsilon$, we have

$$\begin{aligned}
|(\zeta\varkappa)(\mathfrak{t})| \leq \sup_{\mathfrak{t}\in\Xi} \Bigg\{ & I^{\mathfrak{z};\Psi}(|\mathcal{F}(\mathfrak{t},\varkappa(\mathfrak{t})) - \mathcal{F}(\mathfrak{t},0)| + |\mathcal{F}(\mathfrak{t},0)|) + |p| I^{\nu_2;\Psi}|\varkappa(\mathfrak{t})| \\
& + \frac{\mathcal{Q}_\Psi^{\gamma_1+\nu_2-1}(\mathfrak{t},a)}{|\Lambda|\Gamma(\gamma_1+\nu_2)} \Bigg[\sum_{i\in\mathcal{I}} |\delta_i| I^{\mathfrak{z};\Psi}(|\mathcal{F}(\eta_i,\varkappa(\eta_i)) - \mathcal{F}(\eta_i,0)| + |\mathcal{F}(\eta_i,0)|) \\
& + \sum_{j\in\mathcal{J}} |\omega_j| I^{\mathfrak{z}+\tau_j;\Psi}(|\mathcal{F}(\theta_j,\varkappa(\theta_j)) - \mathcal{F}(\theta_j,0)| + |\mathcal{F}(\theta_j,0)|) \\
& + \sum_{k\in\mathcal{K}} |\lambda_k| I^{\mathfrak{z}-\mu_k;\Psi}(|\mathcal{F}(\xi_k,\varkappa(\xi_k)) - \mathcal{F}(\xi_k,0)| + |\mathcal{F}(\xi_k,0)|) \\
& + \varkappa(b) + |p| \Bigg(\sum_{i\in\mathcal{I}} |\delta_i| I^{\nu_2;\Psi}|\varkappa(\eta_i)| + \sum_{j\in\mathcal{J}} |\omega_j| I^{\nu_2+\tau_j;\Psi}|\varkappa(\theta_j)| \\
& + \sum_{k\in\mathcal{K}} |\lambda_k| I^{\nu_2-\mu_k;\Psi}|\varkappa(\xi_k)| \Bigg) \Bigg] \Bigg\} \\
\leq (\mathcal{L}\|\varkappa\| + \mathcal{R}) \Bigg\{ & \frac{\mathcal{Q}_\Psi^{\mathfrak{z}}(b,a)}{\Gamma(\mathfrak{z}+1)} + \frac{\mathcal{Q}_\Psi^{\gamma_1+\nu_2-1}(b,a)}{|\Lambda|\Gamma(\gamma_1+\nu_2)} \Bigg[\sum_{i\in\mathcal{I}} |\delta_i| \frac{\mathcal{Q}_\Psi^{\mathfrak{z}}(\eta_i,a)}{\Gamma(\mathfrak{z}+1)} \\
& + \sum_{j\in\mathcal{J}} |\omega_j| \frac{\mathcal{Q}_\Psi^{\mathfrak{z}+\tau_j}(\theta_j,a)}{\Gamma(\mathfrak{z}+\tau_j+1)} + \sum_{k\in\mathcal{K}} |\lambda_k| \frac{\mathcal{Q}_\Psi^{\mathfrak{z}+\mu_k}(\xi_k,a)}{\Gamma(\mathfrak{z}-\mu_k+1)} \Bigg] \Bigg\} \\
& + \|\varkappa\| \Bigg(|p| \Bigg\{ \frac{\mathcal{Q}_\Psi^{\nu_2}(b,a)}{\Gamma(\nu_2+1)} + \frac{\mathcal{Q}_\Psi^{\gamma_1+\nu_2-1}(\mathfrak{t},a)}{|\Lambda|\Gamma(\nu_2+1)} \Bigg[\sum_{i\in\mathcal{I}} |\delta_i| \frac{\mathcal{Q}_\Psi^{\nu_2}(\eta_i,a)}{\Gamma(\nu_2+1)} \\
& + \sum_{j\in\mathcal{J}} |\omega_j| \frac{\mathcal{Q}_\Psi^{\nu_2+\tau_j}(\theta_j,a)}{\Gamma(\nu_2+\tau_j+1)} + \sum_{ik=1}^{r} |\lambda_k| \frac{\mathcal{Q}_\Psi^{\nu_2-\mu_k}(\xi_k,a)}{\Gamma(\nu_2-\mu_k+1)} \Bigg\} \Bigg] \\
\leq (\mathcal{L}\|\varkappa\| + \mathcal{R})\Omega_1 & + \|\varkappa\| \\
\leq (\mathcal{L}\epsilon + \mathcal{R})\Omega_1 & + \epsilon\Omega_2 \\
\leq \epsilon\,. &
\end{aligned}$$

Therefore, $\|\zeta\varkappa\| \leq \epsilon$ which implies $\zeta B_\epsilon \subset B_\epsilon$. Next, let $\varkappa, y \in \mathcal{C}(\Xi, \mathbb{R})$. Then, for $\mathfrak{t} \in \Xi$, we have

$$\begin{aligned}
|(\zeta\varkappa)(\mathfrak{t}) - (\zeta y)(\mathfrak{t})| \leq \Bigg\{ & \frac{\mathcal{Q}_\Psi^{\mathfrak{z}}(b,a)}{\Gamma(\mathfrak{z}+1)} + \frac{\mathcal{Q}_\Psi^{\gamma_1+\nu_2-1}(b,a)}{|\Lambda|(\Gamma(\gamma_1+\nu_2)} \Bigg[\sum_{i\in\mathcal{I}} |\delta_i| \frac{\mathcal{Q}_\Psi^{\mathfrak{z}}(\eta_i,a)}{\Gamma(\mathfrak{z}+1)} \\
& + \sum_{j\in\mathcal{J}} |\omega_j| \frac{\mathcal{Q}_\Psi^{\mathfrak{z}+\tau_j}(\theta_j,a)}{\Gamma(\mathfrak{z}+\tau_j+1)} + \sum_{k\in\mathcal{K}} |\lambda_k| \frac{\mathcal{Q}_\Psi^{\mathfrak{z}+\mu_k}(\xi_k,a)}{\Gamma(\mathfrak{z}-\mu_k+1)} \Bigg] \Bigg\} \\
& + |p| \Bigg(\frac{\mathcal{Q}_\Psi^{\nu_2}(b,a)}{\Gamma(\nu_2+1)} + \frac{\mathcal{Q}_\Psi^{\gamma_1+\nu_2-1}(b,a)}{\Lambda|\Gamma(\gamma_1+\nu_2)} \\
& \cdot \Bigg[\sum_{i\in\mathcal{I}} |\delta_i| \frac{\mathcal{Q}_\Psi^{\nu_2}(\eta_i,a)}{\Gamma(\nu_2+1)} + \sum_{j\in\mathcal{J}} |\omega_j| \frac{\mathcal{Q}_\Psi^{\nu_2+\tau_j}(\theta_j,a)}{\Gamma(\nu_2+\tau_j+1)} \\
& + \sum_{k\in\mathcal{K}} |\lambda_k| \frac{\mathcal{Q}_\Psi^{\nu_2-\mu_k}(\xi_k,a)}{\Gamma(\nu_2-\mu_k+1)} \Bigg] \Bigg\} \|\varkappa - y\|
\end{aligned}$$

$$= (\mathcal{L}\Omega_1 + \Omega_2)\|\varkappa - y\|$$
$$\implies \quad \|\zeta\varkappa - \zeta y\| \le (\mathcal{L}\Omega_1 + \Omega_2)\|\varkappa - y\|.$$

Thus, ζ is a contraction mapping provided $(\mathcal{L}\Omega_1 + \Omega_2) < 1$, and owing to that Banach contraction mapping principle implies that the fixed point of ζ is nothing but the unique solution of the BVP (1.9)-(1.10). □

Remark. *In case, $\mathcal{L}\Omega_1+\Omega_2 \not< 1$, that is, the mapping is not a contraction mapping, then we can use Bielecki's renorming method* [2] *instead of the Banach contraction mapping principle.*

Remark. *Some special cases of the above theorem are given by:*

For $\beta_1 = 0$, we get $\gamma_1 = \nu_1$ by setting constants Ω_1^0 and Ω_2^0 as

$$\Omega_1^0 = \frac{\mathcal{Q}_\Psi^{\mathfrak{z}}(b,a)}{\Gamma(\mathfrak{z}+1)} + \frac{\mathcal{Q}_\Psi^{\mathfrak{z}-1}(b,a)}{|\Lambda_0|\Gamma(\mathfrak{z})} \cdot \Bigg[\sum_{i\in\mathcal{I}} |\delta_i| \frac{\mathcal{Q}_\Psi^{\mathfrak{z}}(\eta_i,a)}{\Gamma(\mathfrak{z}+1)} + \sum_{j\in\mathcal{J}} |\omega_j| \frac{\mathcal{Q}_\Psi^{\mathfrak{z}+\tau_j}(\theta_j,a)}{\Gamma(\mathfrak{z}+\tau_j+1)} + \sum_{k\in\mathcal{K}} |\lambda_k| \frac{\mathcal{Q}_\Psi^{\mathfrak{z}+\mu_k}(\xi_k,a)}{\Gamma(\mathfrak{z}-\mu_k+1)}\Bigg], \tag{3.23}$$

$$\Omega_2^0 = |p|\Bigg\{\frac{\mathcal{Q}_\Psi^{\nu_2}(b,a)}{\Gamma(\nu_2+1)} + \frac{\mathcal{Q}_\Psi^{\mathfrak{z}-1}(b,a)}{|\Lambda_1|\Gamma(\mathfrak{z})} \cdot \Bigg[\sum_{i\in\mathcal{I}} |\delta_i| \frac{\mathcal{Q}_\Psi^{\nu_2}(\eta_i,a)}{\Gamma(\nu_2+1)} + \sum_{j\in\mathcal{J}} |\omega_j| \frac{\mathcal{Q}_\Psi^{\nu_2+\tau_j}(\theta_j,a)}{\Gamma(\nu_2+\tau_j)} + \sum_{k\in\mathcal{K}} |\lambda_k| \frac{\mathcal{Q}_\Psi^{\nu_2-\mu_k}(\xi_k,a)}{\Gamma(\nu_2-\mu_k+1)}\Bigg]\Bigg\}, \tag{3.24}$$

where

$$\Lambda_0 = \sum_{i\in\mathcal{I}} \delta_i \frac{\mathcal{Q}_\Psi^{\mathfrak{z}-1}(\eta_i,a)}{\Gamma(\mathfrak{z})} + \sum_{j\in\mathcal{J}} \omega_j \frac{\mathcal{Q}_\Psi^{\mathfrak{z}+\tau_j-1}(\theta_j,a)}{\Gamma(\mathfrak{z}+\tau_j)} + \sum_{k\in\mathcal{K}} \lambda_k \frac{\mathcal{Q}_\Psi^{\mathfrak{z}-\mu_k-1}(\xi_k,a)}{\Gamma(\mathfrak{z}-\mu_k)}. \tag{3.25}$$

Corollary 3.9. *For $\mathcal{L}\Omega_1^0 + \Omega_2^0 < 1$ and* (H5) *being satisfied,* (1.9)-(1.10) *has a unique solution on Ξ, Ω_1^0 and Ω_2^0 specified by* (3.23) *and* (3.15)*.*

For $\beta_1 = 1$, we get $\gamma_1 = 1$ by setting constants Ω_1^1 and Ω_2^1 as

$$\Omega_1^1 = \frac{\mathcal{Q}_\Psi^{\mathfrak{z}}(b,a)}{\Gamma(\mathfrak{z}+1)} + \frac{\mathcal{Q}_\Psi^{\nu_2}(b,a)}{|\Lambda|\Gamma(\nu_2+1)} \cdot \Bigg[\sum_{i\in\mathcal{I}} |\delta_i| \frac{\mathcal{Q}_\Psi^{\mathfrak{z}}(\eta_i,a)}{\Gamma(\mathfrak{z}+1)} + \sum_{j\in\mathcal{J}} |\omega_j| \frac{\mathcal{Q}_\Psi^{\mathfrak{z}+\tau_j}(\theta_j,a)}{\Gamma(\mathfrak{z}+\tau_j+1)} + \sum_{k\in\mathcal{K}} |\lambda_k| \frac{\mathcal{Q}_\Psi^{\mathfrak{z}+\mu_k}(\xi_k,a)}{\Gamma(\mathfrak{z}-\mu_k+1)}\Bigg], \tag{3.26}$$

$$\Omega_2^1 = |p|\Bigg\{\frac{\mathcal{Q}_\Psi^{\nu_2}(b,a)}{\Gamma(\nu_2+1)} + \frac{\mathcal{Q}_\Psi^{\nu_2}(b,a)}{|\Lambda|\Gamma(\nu_2+1)} \cdot \Bigg[\sum_{i\in\mathcal{I}} |\delta_i| \frac{\mathcal{Q}_\Psi^{\nu_2}(\eta_i,a)}{\Gamma(\nu_2+1)} + \sum_{j\in\mathcal{J}} |\omega_j| \frac{\mathcal{Q}_\Psi^{\nu_2+\tau_j}(\theta_j,a)}{\Gamma(\nu_2+\tau_j+1)} + \sum_{k\in\mathcal{K}} |\lambda_k| \frac{\mathcal{Q}_\Psi^{\nu_2-\mu_k}(\xi_k,a)}{\Gamma(\nu_2-\mu_k+1)}\Bigg]\Bigg\} \tag{3.27}$$

where

$$\Lambda_1 = \sum_{i\in\mathcal{I}} \delta_i \frac{\mathcal{Q}_\Psi^{\nu_2}(\eta_i,a)}{\Gamma(\nu_2+1)} + \sum_{j\in\mathcal{J}} \omega_j \frac{\mathcal{Q}_\Psi^{\nu_2+\tau_j}(\theta_j,a)}{\Gamma(\nu_2+\tau_j+1)} + \sum_{k\in\mathcal{K}} \lambda_k \frac{\mathcal{Q}_\Psi^{\nu_2-\mu_k}(\xi_k,a)}{\Gamma(\nu_2-\mu_k+1)}. \tag{3.28}$$

Corollary 3.10. *If* $\mathcal{L}\Omega_1^1 + \Omega_2^1 < 1$ *and* (H5) *is satisfied, then the problem* (1.9)-(1.10) *has a unique solution on* Ξ, *for* Ω_1^1 *and* Ω_2^1 *defined by* (3.27) *and* (3.28).

Example 3.11. *Given the BVP of a fractional-order Langevin equation with a nonlocal mixed multi-point boundary condition and a* Ψ*-Hilfer fractional derivative:*

$$\mathcal{D}^{\frac{5}{3},\frac{2}{3};\mathfrak{t}}\left(\mathcal{D}^{\frac{3}{2},\frac{1}{2};\mathfrak{t}} + \frac{1}{15}\right)\varkappa(\mathfrak{t}) = \frac{1}{8(\mathfrak{t}+1)^2}\frac{|\varkappa(\mathfrak{t})|}{1+|\varkappa(\mathfrak{t})|}, \quad t \in [1,3], \tag{3.29}$$

$$\varkappa(1) = 0,\ \varkappa(3) = \frac{1}{8}\varkappa\left(\frac{3}{2}\right) + \frac{1}{7}I^{\frac{1}{2}}\varkappa\left(\frac{4}{3}\right) + \frac{1}{12}\mathcal{D}^{\frac{2}{3}}\varkappa\left(\frac{5}{4}\right). \tag{3.30}$$

Here, $\nu_1 = \frac{5}{3}$, $\nu_2 = \frac{3}{2}$, $\beta_1 = \frac{2}{3}$, $\beta_2 = \frac{1}{2}$, $\delta_1 = \frac{1}{8}$, $\omega_1 = \frac{1}{7}$, $\lambda_1 = \frac{1}{12}$, $\eta_1 = \frac{3}{2}$, $\theta_1 = \frac{4}{3}$, $\xi_1 = \frac{5}{4}$, $\tau_1 = \frac{1}{2}$, $\mu_1 = \frac{2}{3}$, $\Psi(\mathfrak{t}) = \mathfrak{t}$ *and* $\mathcal{F}(\mathfrak{t},\varkappa) = \frac{1}{8(\mathfrak{t}+1)^2}\frac{|\varkappa(\mathfrak{t})|}{1+|\varkappa(\mathfrak{t})|}$, $\mathfrak{t} \in [1,2]$.

For any $\varkappa, y \in \mathbb{R}$ *and* $\mathfrak{t} \in [1,3]$,

$$|\mathcal{F}(\mathfrak{t},\varkappa) - \mathcal{F}(\mathfrak{t},y)| \leq \frac{1}{32}|\varkappa - y|.$$

Here, $\gamma_1 = \nu_1 + \beta_1(1-\nu_1) = 1.22$. *From the given data, we observe that* $\gamma_1 + \nu_2 = 2.72$, $\gamma_1 + \nu_2 + \tau_1 = 3.22$, $\gamma_1 + \nu_2 - \mu_1 = 3.22$, $\mathfrak{z} = 3.16$, $\mathfrak{z} + \tau_1 = 3.66$, $\mathfrak{z} - \mu_1 = 1.72$. *Hence, condition* (H1) *is satisfied with* $\mathcal{L} = \frac{1}{32}$. *We discover using the data provided,* $\Lambda \approx 0.0477$, $\Omega_1 \approx 1.30023$ *and* $\Omega_2 = 0.3439$. *This implies that* $\mathcal{L}\Omega_1 + \Omega_2 \approx 0.384532 < 1$. *It follows from Theorem* 3.8 *that the problem* (3.29)-(3.30) *has a unique solution.*

4. Conclusion

The existence and uniqueness of fractional-order Langevin differential equations with multi-point integral and differential boundary conditions were investigated in this study. We first transformed the problem into an equivalent fixed point problem, and then demonstrated its existence using the Krasnoselskii FPT and then the Leray-Schauder Nonlinear Alternative. The Banach FPT was used to prove uniqueness. Also, our results are generalizations of fractional differential equations with multi-point and integral boundary conditions. Finally, we provided instances with each of the theories in order to buttress our theoretical conclusions.

References

[1] S. Asawasamrit, A. Kijjathanakorn, S.K Ntouyas, J. Tariboon, *Nonlocal boundary value problems for Hilfer fractional differential equations*, Bulletin of the Korean Mathematical Society **55**(6) (2018), 1639–1657.

[2] M. Bessenyei, Z. Ples, *Applications of the Bielecki renorming technique*, Journal of Fixed Point Theory and Applications **23**(2) (2021), 1–23.

[3] K. Deimling, *Nonlinear Functional Analysis*, SpringerVerlag, New York (1985).

[4] K. Diethelm, N.J. Ford, *Analysis of fractional differential equations*, Journal of Mathematical Analysis and Applications **265**(2) (2002), 229–248.

[5] K.M. Furati, M.D. Kassim, *Existence and uniqueness for a problem involving Hilfer fractional derivative*, Computers and Mathematics with Applications **64**(6) (2012), 1616–1626.

[6] A. Granas, J. Dugundji, Elementary fixed point theorems, In: *Fixed Point Theory* (pp. 9–84), Springer, New York, NY (2003).

[7] K. Guida, L. Ibnelazyz, K. Hilal, S. Melliani, *Existence and uniqueness results for sequential* Ψ*-Hilfer fractional pantograph differential equations with mixed nonlocal boundary conditions*, AIMS Mathematics **6**(8) (2021), 8239–8255.

[8] R. Hilfer (ed.), *Applications of Fractional Calculus in Physics*, World Scientific (2000).

[9] R. Hilfer, *Experimental evidence for fractional time evolution in glass forming materials*, Chemical Physics **284**(1-2) (2002), 399–408.

[10] R. Hilfer, Y. Luchko, Z. Tomovski, *Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives*, Fract. Calc. Appl. Anal. **12**(3) (2009), 299–318.
[11] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Vol. 204, Elsevier (2006).
[12] M.A. Krasnoselskii, *Two remarks on the method of successive approximations*, Uspekhi Mat. Nauk **10** (1955), 123–127.
[13] K.D. Kucche, A.D. Mali, J.V.D.C. Sousa, *On the nonlinear Ψ-Hilfer fractional differential equations*, Computational and Applied Mathematics **38**(2) (2019), 1–25.
[14] C. Nuchpong, S.K. Ntouyas, D. Vivek, J. Tariboon, *Nonlocal boundary value problems for Ψ-Hilfer fractional-order Langevin equations*, Boundary Value Problems **2021**(1) (2021), 1–12.
[15] S.K. Ntouyas, D. Vivek, *Existence and uniqueness results for sequential Ψ-Hilfer fractional differential equations with multi-point boundary conditions*, Acta Mathematica Universitatis Comenianae **90**(2) (2021), 171–185.
[16] J. V. D. C. Sousa, E. C. De Oliveira, *On the Ψ-Hilfer fractional derivative*, Communications in Nonlinear Science and Numerical Simulation **60** (2018), 72–91.

Savita Rathee
Department of Mathematics, Maharshi Dayanand University, Rohtak, Haryana 124001, India
E-mail address: savitarathee.math@mdurohtak.ac.in

Yogeeta Narwal
Government College, Baund Kalan, Charkhi Dadri, Haryana 127025, India
E-mail address: yogeetawork@gmail.com