Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 16 Issue 3(2024), Pages 45-61
https://doi.org/10.54671/BMAA-2024-3-4

# SOME COUPLED FIXED POINT THEOREMS ON ORTHOGONAL $b$-METRIC SPACES WITH APPLICATIONS

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Abstract. In this research, we establish certain well-known coupled fixed point theorems in complete orthogonal $b$-metric spaces. Our research yields as few conclusions as possible. We also provide an illustration to support the significance of our primary findings. Our results expand upon or generalize various findings in the literature. Some relevant applications are offered by us.

## 1. Introduction

The development of fixed point theory is based on the generalization of contraction conditions in one direction or/and generalization of ambient spaces of the operator under consideration on the other. Banach contraction principle plays an important role in solving nonlinear equations, and it is one of the most useful results in fixed point theory. As a generalization of metric space, Czerwik [13] created the idea of $b$-metric space or metric type space. Afterwards, many authors studied the existence of fixed points for a single-valued and multi-valued mappings in b-metric spaces under certain contraction conditions. For more details, we refer [3, 7, 6] and references therein. Guo and Lakshmikantham [17] introduced the notion of coupled fixed point in 1987. After that, Bhaskar and Lakshmikantham [9] established the idea of the mixed monotone property and demonstrated a few connected coupled fixed point theorems. These theorems have drawn the attention of numerous authors in a variety of metric spaces, including $b$-metric, bipolar, modular, partial, cone, and many more, due to their significant applicability in numerous mathematical domains. For more details, we refer [1, 2, 5, 10, 11, 12, 22, 20, 21, 24, 25].

On the other hand, Gordji et al. [16] introduced the new concept of an orthogonality in metric spaces and proved the fixed point result for contraction mappings in metric spaces endowed with this new type of orthogonality. Furthermore, they gave the application of this results for the existence and uniqueness of the solution of a first-order ordinary differential equation, while the Banach contraction mapping principle cannot be applied in this situation. This new concept of an orthogonal set

1991 *Mathematics Subject Classification.* 47H10, 54H25.
*Key words and phrases.* Coupled fixed points; orthogonal $b$-metric space; integral equation; functional equation.

Submitted May 29, 2024. Accepted September 22, 2024. Published September 30, 2024.
Communicated by H.K. Nashine.

has many applications, and there are also many types of the orthogonality. For find more details about $O$-sets and orthogonal metric spaces, the readers are referred to [4, 8, 19, 18].

We denote $\mathbb{N}$ and $\mathbb{R}$, respectively, as the collection of positive integers and real numbers.

**Definition 1.** [16] Let $\perp \subseteq X \times X$ be a binary relation defined on a nonempty set $X$. If the relation $\perp$ satisfies the following condition: there exists $x_0 \in X$ such that

$$[\text{for all } y,\ y \perp x_0] \text{ or } [\text{for all } y,\ x_0 \perp y]$$

then $X$ is called an orthogonal set (briefly, $O$-set) and $x_0$ is called an orthogonal element. We denote this $O$-set by $(X, \perp)$.

As an illustration, let us consider the following examples:

**Example 1.** [16] Let $X$ be the set of all peoples in the word. We define $x \perp y$ if $x$ can give blood to $y$. According to the following table, if $x_0$ is a person such that his / her blood type is O-, then we have $x_0 \perp y$ for all $y \in X$. This means that $(X, \perp)$ is an $O$-set. In this $O$-set, $x_0$ (in definition) is not unique. Note that, in this example, $x_0$ may be a person with blood type AB + . In this case, we have $y \perp x_0$ for all $y \in X$.

| Type | You can give blood to | You can receive blood from |
|---|---|---|
| A+ | A+ AB+ | A+ A O+ O |
| O+ | O+ A+ B+ AB+ | O+ O |
| B+ | B+ AB+ | B+ B O+ O |
| AB+ | AB+ | Everyone |
| A | A+ A AB+ AB | A O |
| O | Everyone | O |
| B | B+ B AB+ AB | B O |
| AB | AB+ AB | AB B O A |

**Remark.** Every continuous mapping is $\perp$-continuous and the converse is not true [16].

**Example 2.** [16] Let $X = \mathbb{Z}$. We define $m \perp n$ if there exists $k \in \mathbb{Z}$ such that $m = kn$. It is obvious that $0 \perp n$ for all $n \in \mathbb{Z}$. So, $(X, \perp)$ is an $O$-set.

**Example 3.** [16] Let $X = [0, \infty)$. We define $x \perp y$ if $xy \in \{x, y\}$. For orthogonal elements $x_0 = 0$ or $x_0 = 1, (X, \perp)$ is an $O$-set.

**Definition 2.** [16] Let $(X, \perp)$ be $O$-set. A sequence $\{x_i\}_{i \in \mathbb{N}}$ is called an orthogonal sequence ($O$-sequence) if

$$[\text{for all } i,\ x_i \perp x_{i+1}] \text{ or } [\text{for all } i,\ x_{i+1} \perp x_i].$$

**Definition 3.** [16] The triplet $(X, \perp, d)$ is called an orthogonal metric space if $(X, \perp)$ is an $O$-set and $(X, d)$ is a metric space.

**Remark.** [16] Every complete metric space is $O$-complete and the converse is not true.

**Definition 4.** [16] Let $(X, \perp, d)$ be an orthogonal metric space. Then, a mapping $f : X \to X$ is said to be orthogonally continuous (or $\perp$-continuous) in $x \in X$ if for each $O$-sequence $\{x_n\}$ in $X$ with $x_n \to x$ as $n \to \infty$, we have $fx_n \to fx$ as $n \to \infty$. Also, $f$ is said to be $\perp$-continuous on $X$ if $f$ is $\perp$-continuous in each $x \in X$.

**Definition 5.** [16] Let $(X, \perp, d)$ be an orthogonal metric space. Then, $X$ is said to be orthogonally complete (briefly, $O$-complete) if every Cauchy $O$-sequence is convergent.

**Definition 6.** [13] Let $X$ be a non-empty set and $s \geq 1$ is a given real number. A function $d : X \times X \to [0, \infty)$ is said to be a $b$-metric if the following conditions are satisfied: for any $x, y, z \in X$

(i) $0 \leq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair $(X, d)$ is called a $b$-metric space with coefficient $s$.

**Definition 7.** [14] The triplet $(X, \perp, d)$ is called an orthogonal $b$-metric space if $(X, \perp)$ is an $O$-set and $(X, d)$ is a $b$-metric space.

**Definition 8.** [15] Let $(X, \perp, d)$ be an orthogonal $b$-metric space. Then

(i) an orthogonal sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X$ that converges at a point $x$ if $\lim_{n\to\infty} d(x_n, x) = 0$;
(ii) an orthogonal sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X$ is said to be orthogonal $b$-Cauchy if $\overline{\lim_{n\to\infty}} d(x_n, x_m) = 0, m > n$.

**Definition 9.** [23] Let $(X, \perp)$ be an $O$-set. A mapping $f : X \times X \to X$ is said to be $\perp$-preserving if $x \perp \eta$ and $y \perp \nu$ implies $f(x, y) \perp f(\eta, \nu)$.

Recently, Özkan [23] established the following theorems in orthogonal complete metric spaces.

**Theorem 1.1.** [23] Let $(X, \perp, d)$ be an $O$-complete metric space (not necessarily complete metric space) and $f : X \times X \to X$ be $\perp$-preserving mapping. If the conditions are

$$d(f(x, y), f(\eta, \nu)) \leq kd(x, \eta) + ld(y, \nu) \tag{1.1}$$

$$d(f(x, y), f(\eta, \nu)) \leq kd(f(x, y), x) + ld(f(\eta, \nu), \eta) \tag{1.2}$$

$$d(f(x, y), f(\eta, \nu)) \leq kd(f(x, y), \eta) + ld(f(\eta, \nu), x) \tag{1.3}$$

hold for all $x, y, \eta, \nu \in X$ with $x \perp \eta$ and $y \perp \nu$ where $k, l \geq 0$ and $k + l < 1$, then $f$ has a unique coupled fixed point.

Motivated by the work of Özkan [23], we prove the existence and uniqueness of coupled fixed points in orthogonal complete $b$-metric spaces. We provide examples to support our results. Finally, we provide related applications.

## 2. Main Results

The subsequent lemma is helpful in demonstrating our findings.

**Lemma 2.1.** *Let $(X, \perp, d)$ be an orthogonal $b$-metric space with parameter $\mathfrak{s} \geq 1$, and $\{x_n\}_n \in \mathbb{N}$, $\{y_n\}_n \in \mathbb{N}$ are orthogonal $b$-convergent to $x$, $y$ respectively, then we have*

$$\frac{1}{\mathfrak{s}^2} d(x, y) \leq \underline{\lim_{n\to\infty}} d(x_n, y_n) \leq \overline{\lim_{n\to\infty}} d(x_n, y_n) \leq \mathfrak{s}^2 d(x, y).$$

*For $x = y$, we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover for each $\eta \in X$ we have*

$$\frac{1}{\mathfrak{s}} d(x,\eta) \le \underline{\lim}_{n\to\infty} d(x_n,\eta) \le \overline{\lim}_{n\to\infty} d(x_n,\eta) \le \mathfrak{s} d(x,\eta).$$

**Theorem 2.2.** *Let $(X, \perp, d)$ be an $O$-complete $b$-metric space and $f : X \times X \to X$ be $\perp$-preserving map. If the condition*

$$\begin{aligned} d(f(x,y), f(\eta,\nu)) \le \kappa_1 d(x,\eta) + \kappa_2 d(y,\nu) + \kappa_3 d(f(x,y),x) + \kappa_4 d(f(\eta,\nu),\eta) \\ +\kappa_5 d(f(x,y),\eta) + \kappa_6 d(f(\eta,\nu),x) \end{aligned} \tag{2.1}$$

*holds for all $x, y, \eta, \nu \in X$ with $x \perp \eta$ and $y \perp \nu$ where $\kappa_i \ge 0, i = 1,2,3,4,5,6$ and $\kappa_1 + \kappa_2 + \mathfrak{s}\kappa_3 + \kappa_4 + \mathfrak{s}^2\kappa_5 + 2\mathfrak{s}\kappa_6 < 1$, then $f$ has a unique coupled fixed point.*

*Proof.* Since $X$ is $O$-set, there exist orthogonal elements $x_0, y_0 \in X$ such that

$$[x_0 \perp \varrho \text{ for all } \varrho \in X] \text{ or } [\varrho \perp x_0 \text{ for all } \varrho \in X] \text{ and}$$
$$[y_0 \perp \varrho \text{ for all } \varrho \in X] \text{ or } [\varrho \perp y_0 \text{ for all } \varrho \in X].$$

So that $x_0 \perp f(x_0,y_0)$ or $f(x_0,y_0) \perp x_0$ and $y_0 \perp f(y_0,x_0)$ or $f(y_0,x_0) \perp y_0$ for $x_0, y_0 \in X$.
We take

$$\begin{aligned} &x_1 = f(x_0,y_0) \text{ and } y_1 = f(y_0,x_0)\\ &x_2 = f(x_1,y_1) \text{ and } y_2 = f(y_1,x_1)\\ &\vdots\\ &x_i = f(x_{i-1},y_{i-1}) \text{ and } y_i = f(y_{i-1},x_{i-1}) \text{ for } i \in \mathbb{N}. \end{aligned}$$

Thus, we get

$$x_0 \perp f(x_0,y_0) = x_1 \text{ or } x_1 = f(x_0,y_0) \perp x_0 \text{ and } y_0 \perp f(y_0,x_0) = y_1 \text{ or } y_1 = f(y_0,x_0) \perp y_0.$$

As $f$ is $\perp$-preserving, we have

$$x_1 = f(x_0,y_0) \perp f(x_1,y_1) = x_2 \text{ or } x_2 = f(x_1,y_1) \perp f(x_0,y_0) = x_1$$

and

$$y_1 = f(y_0,x_0) \perp f(y_1,x_1) = y_2 \text{ or } y_2 = f(y_1,x_1) \perp f(y_0,x_0) = y_1.$$

We continue in this manner, we get

$$x_{i-1} \perp x_i \text{ or } x_i \perp x_{i-1} \text{ and } y_{i-1} \perp y_i \text{ or } y_i \perp y_{i-1} \text{ for all } i \in \mathbb{N}.$$

Therefore, $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ are $O$-sequences. We now see that $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ are orthogonal $b$-Cauchy sequences. From (2.1), we have

$$\begin{aligned} d(x_i, x_{i+1}) &= d(f(x_{i-1},y_{i-1}), f(x_i,y_i))\\ &\le \kappa_1 d(x_{i-1},x_i) + \kappa_2 d(y_{i-1},y_i) + \kappa_3 d(f(x_{i-1},y_{i-1}),x_{i-1}) + \kappa_4 d(f(x_i,y_i),x_i)\\ &\quad +\kappa_5 d(f(x_{i-1},y_{i-1}),x_i) + \kappa_6 d(f(x_i,y_i),x_{i-1})\\ &= \kappa_1 d(x_{i-1},x_i) + \kappa_2 d(y_{i-1},y_i) + \kappa_3 d(x_i,x_{i-1}) + \kappa_4 d(x_{i+1},x_i) + \kappa_5 d(x_i,x_i) + \kappa_6 d(x_{i+1},x_{i-1})\\ &\le \kappa_1 d(x_{i-1},x_i) + \kappa_3 d(x_i,x_{i-1}) + \kappa_4 d(x_{i+1},x_i) + \kappa_6 \mathfrak{s}[d(x_{i+1},x_i) + d(x_i,x_{i-1})] + \kappa_2 d(y_{i-1},y_i) \end{aligned}$$

which implies that

$$(1 - \kappa_4 - \mathfrak{s}\kappa_6) d(x_i, x_{i+1}) \le (\kappa_1 + \kappa_3 + \mathfrak{s}\kappa_6) d(x_{i-1}, x_i) + \kappa_2 d(y_{i-1}, y_i). \tag{2.2}$$

And similarly,

$$(1 - \kappa_4 - \mathfrak{s}\kappa_6) d(y_i, y_{i+1}) \le (\kappa_1 + \kappa_3 + \mathfrak{s}\kappa_6) d(y_{i-1}, y_i) + \kappa_2 d(x_{i-1}, x_i). \tag{2.3}$$

From (2.2) and (2.3), we get

$$\begin{aligned}(1-\kappa_4-\mathfrak{s}\kappa_6)(d(x_i,x_{i+1})+d(y_i,y_{i+1})) &\leq (\kappa_1+\kappa_3+\mathfrak{s}\kappa_6)d(x_{i-1},x_i)+\kappa_2 d(y_{i-1},y_i)\\ &\quad +(\kappa_1+\kappa_3+\mathfrak{s}\kappa_6)d(y_{i-1},y_i)+\kappa_2 d(x_{i-1},x_i)\\ \Rightarrow d(x_i,x_{i+1})+d(y_i,y_{i+1}) &\leq \left(\tfrac{\kappa_1+\kappa_2+\kappa_3+\mathfrak{s}\kappa_6}{1-\kappa_4-\mathfrak{s}\kappa_6}\right)(d(x_{i-1},x_i)+d(y_{i-1},y_i))\\ &= \tau\left(d(x_{i-1},x_i)+d(y_{i-1},y_i)\right),\end{aligned}$$

where

$$\tau=\left(\frac{\kappa_1+\kappa_2+\kappa_3+\mathfrak{s}\kappa_6}{1-\kappa_4-\mathfrak{s}\kappa_6}\right)<1.$$

We continue this argument, we get

$$\begin{aligned}d(x_i,x_{i+1})+d(y_i,y_{i+1}) &\leq \tau[d(x_{i-1},x_i)+d(y_{i-1},y_i)]\\ &\leq \tau^2[d(x_{i-2},x_{i-1})+d(y_{i-2},y_{i-1})]\\ &\ \vdots\\ &\leq \tau^i[d(x_0,x_1)+d(y_0,y_1)] \text{ for all } i\in\mathbb{N}.\end{aligned} \tag{2.4}$$

As $i\to\infty$ in (2.4), we get

$$\lim_{i\to\infty} d(x_i,x_{i+1})=0 \text{ and } \lim_{i\to\infty} d(y_i,y_{i+1})=0. \tag{2.5}$$

If $d(x_0,x_1)+d(y_0,y_1)=0$ then we get $d(x_0,x_1)=0\Rightarrow x_0=x_1=f(x_0,y_0)$ and $d(y_0,y_1)=0\Rightarrow y_0=y_1=f(y_0,x_0)$. Hence, $(x_0,y_0)$ is a coupled fixed point of $f$. Suppose $d(x_0,x_1)+d(y_0,y_1)>0$, then for any $i,j\in\mathbb{N}$ with $i\leq j$ we have

$$d(x_i,x_j)\leq \mathfrak{s}d(x_i,x_{i+1})+\mathfrak{s}^2 d(x_{i+1},x_{i+2})+\cdots+\mathfrak{s}^{j-i}d(x_{j-1},x_j) \tag{2.6}$$

and

$$d(y_i,y_j)\leq \mathfrak{s}d(y_i,y_{i+1})+\mathfrak{s}^2 d(y_{i+1},y_{i+2})+\cdots+\mathfrak{s}^{j-i}d(y_{j-1},y_j). \tag{2.7}$$

From (2.4), (2.6) and (2.7), we have

$$\begin{aligned}d(x_i,x_j)+d(y_i,y_j) &\leq \mathfrak{s}[d(x_i,x_{i+1})+d(y_i,y_{i+1})]+\mathfrak{s}^2[d(x_{i+1},x_{i+2})+d(y_{i+1},y_{i+2})]\\ &\quad +\cdots+\mathfrak{s}^{j-i}[d(x_{j-1},x_j)+d(y_{j-1},y_j)]\\ &\leq (\mathfrak{s}\tau^i+\mathfrak{s}^2\tau^{i+1}+\cdots+\mathfrak{s}^{j-i}\tau^{j-1})(d(x_0,x_1)+d(y_0,y_1))\\ &\leq (\mathfrak{s}\tau^i+\mathfrak{s}^2\tau^{i+1}+\cdots+\mathfrak{s}^{j-i}\tau^{j-1}+\cdots)(d(x_0,x_1)+d(y_0,y_1))\\ &= \mathfrak{s}\tau^i(1+\mathfrak{s}\tau+\mathfrak{s}^2\tau^2+\cdots)(d(x_0,x_1)+d(y_0,y_1))\\ &= \left(\tfrac{\mathfrak{s}\tau^i}{1-\mathfrak{s}\tau}\right)(d(x_0,x_1)+d(y_0,y_1))\to 0 \text{ as } i\to\infty.\end{aligned}$$

Therefore, $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ are orthogonal $b$-Cauchy sequences in $X$. Since $X$ is an $O$-complete $b$-metric space, there exist $u,v\in X$ such that $x_i\to u$ and $y_i\to v$. By choice of $u$ and $v$, we have $u\perp x_i$ or $x_i\perp u$ and $v\perp y_i$ or $y_i\perp v$.
From (2.1), we have

$$\begin{aligned}d(f(u,v),u) &\leq \mathfrak{s}[d(f(u,v),x_{i+1})+d(x_{i+1},u)]\\ &= \mathfrak{s}d(f(u,v),f(x_i,y_i))+\mathfrak{s}d(x_{i+1},u)\\ &\leq \mathfrak{s}[\kappa_1 d(u,x_i)+\kappa_2 d(v,y_i)+\kappa_3 d(f(u,v),u)+\kappa_4 d(f(x_i,y_i),x_i)\\ &\quad +\kappa_5 d(f(u,v),x_i)+\kappa_6 d(f(x_i,y_i),u)]+\mathfrak{s}d(x_{i+1},u).\end{aligned} \tag{2.8}$$

Letting upper limit as $i\to\infty$ in (2.8) and from (2.5), using Lemma 2.1, we get

$$d(f(u,v),u)\leq \kappa_3 d(f(u,v),u)+\mathfrak{s}^2\kappa_5 d(f(u,v),u)\Rightarrow (1-\kappa_3-\mathfrak{s}^2\kappa_5)d(f(u,v),u)\leq 0.$$

Hence $d(f(u,v),u)=0$ i.e., $f(u,v)=u$.
Similarly, we can easily see that $f(v,u)=v$.

Therefore, $f$ has a coupled fixed point $(u,v)$.
Let $(u^*,v^*)\neq(u,v)$ be a coupled fixed point of $f$,
i.e., $f(u^*,v^*)=u^*$ and $f(v^*,u^*)=v^*$. Therefore,
$d(f(u,v),f(u^*,v^*))=d(u,u^*)>0$ and $d(f(v,u),f(v^*,u^*))=d(v,v^*)>0$.
Since $f$ is $\perp$-preserving, we get

$$u\perp u^* \text{ or } u^*\perp u$$

and

$$v\perp v^* \text{ or } v^*\perp v.$$

From (2.1), we obtain

$$\begin{aligned} d(u,u^*)=d(f(u,v),f(u^*,v^*)) &\leq \kappa_1 d(u,u^*)+\kappa_2 d(v,v^*)+\kappa_3 d(f(u,v),u)\\ &\quad+\kappa_4 d(f(u^*,v^*),u^*)+\kappa_5 d(f(u,v),u^*)+\kappa_6 d(f(u^*,v^*),u)\\ &=(\kappa_1+\kappa_4+\kappa_5)d(u,u^*)+\kappa_2 d(v,v^*) \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} d(v,v^*)=d(f(v,u),f(v^*,u^*)) &\leq \kappa_1 d(v,v^*)+\kappa_2 d(u,u^*)+\kappa_3 d(f(v,u),v)\\ &\quad+\kappa_4 d(f(v^*,u^*),v^*)+\kappa_5 d(f(v,u),v^*)+\kappa_6 d(f(v^*,u^*),v)\\ &=(\kappa_1+\kappa_4+\kappa_5)d(v,v^*)+\kappa_2 d(u,u^*). \end{aligned} \tag{2.10}$$

From (2.9) and (2.10), we get

$$d(u,u^*)+d(v,v^*)\leq(\kappa_1+\kappa_2+\kappa_4+\kappa_5)(d(u,u^*)+d(v,v^*)).$$

As $(\kappa_1+\kappa_2+\kappa_4+\kappa_5)<1$, we obtain $d(u,u^*)=d(v,v^*)=0$
implies that $u=u^*, v=v^*$. Therefore, $(u,v)=(u^*,v^*)$.
Thus, $f$ has a unique coupled fixed point.

□

**Corollary 2.3.** *Let $(X,\perp,d)$ be an $O$-complete $b$-metric space and $f:X\times X\to X$ be $\perp$-preserving map such that*

$$\begin{aligned} d(f(x,y),f(\eta,\nu)) &\leq \kappa_1 d(x,\eta)+\kappa_2 d(f(x,y),x)+\kappa_3 d(f(\eta,\nu),\eta)+\kappa_4 d(f(x,y),\eta)\\ &\quad+\kappa_5 d(f(\eta,\nu),x), \end{aligned}$$

*for all $x,y,\eta,\nu\in X$ with $x\perp\eta$ and $y\perp\nu$ where $\kappa_i\geq 0, i=1,2,3,4,5$ and $\kappa_1+\mathfrak{s}\kappa_2+\kappa_3+\mathfrak{s}^2\kappa_4+2\mathfrak{s}\kappa_5<1$, then $f$ has a unique coupled fixed point.*

**Corollary 2.4.** *Let $(X,\perp,d)$ be an $O$-complete $b$-metric space and $f:X\times X\to X$ be $\perp$-preserving map. If the condition*

$$d(f(x,y),f(\eta,\nu))\leq\kappa_1 d(x,\eta)+\kappa_2 d(y,\nu)$$

*holds, for all $x,y,\eta,\nu\in X$ with $x\perp\eta$ and $y\perp\nu$ where $\kappa_1,\kappa_2\geq 0$ and $\kappa_1+\kappa_2<1$, then $f$ has only one coupled fixed point.*

**Corollary 2.5.** *Let $(X,\perp,d)$ be an $O$-complete $b$-metric space and $f:X\times X\to X$ be $\perp$-preserving map. If the condition*

$$d(f(x,y),f(\eta,\nu))\leq\kappa_1 d(f(x,y),x)+\kappa_2 d(f(\eta,\nu),\eta)$$

*holds, for all $x,y,\eta,\nu\in X$ with $x\perp\eta$ and $y\perp\nu$ where $\kappa_1,\kappa_2\geq 0$ and $\mathfrak{s}\kappa_1+\kappa_2<1$, then $f$ has one and only one coupled fixed point.*

**Corollary 2.6.** *Let $(X, \perp, d)$ be an $O$-complete $b$-metric space and $f : X \times X \to X$ be $\perp$-preserving map. If the condition*

$$d(f(x, y), f(\eta, \nu)) \leq \kappa_5 d(f(x, y), \eta) + \kappa_6 d(f(\eta, \nu), x)$$

*holds, for all $x, y, \eta, \nu \in X$ with $x \perp \eta$ and $y \perp \nu$ where $\kappa_1, \kappa_2 \geq 0$ and $\mathfrak{s}^2\kappa_1 + 2\mathfrak{s}\kappa_2 < 1$, then $f$ has a unique coupled fixed point.*

**Remark.** Corollary 2.4, Corollary 2.5 and Corollary 2.6 are extend and generalize (1.1), (1.2) and (1.3) of Theorem 1.1 to orthogonal $b$-metric spaces respectively.

**Theorem 2.7.** *Let $(X, \perp, d)$ be an $O$-complete $b$-metric space and $f : X \times X \to X$ be $\perp$-preserving map. If the condition*

$$\begin{aligned} d(f(x, y), f(\eta, \nu)) \leq{}& \kappa_1[d(x, \eta) + d(y, \nu)] + \kappa_2[d(x, f(x, y) + d(\eta, f(\eta, \nu))] \\ &+ \kappa_3[d(x, f(\eta, \nu)) + d(\eta, f(x, y))] + \kappa_4[\Delta(x, y, \eta, \nu) + h\delta(x, y, \eta, \nu)] \end{aligned} \tag{2.11}$$

*holds, for all $x, y, \eta, \nu \in X$ with $x \perp \eta$ and $y \perp \nu$ where $\Delta(x, y, \eta, \nu) = \max\{d(x, f(\eta, \nu), d(\eta, f(x, y)\}, \delta(x, y, \eta, \nu) = \min\{d(x, f(\eta, \nu), d(\eta, f(x, y)\}$, $\kappa_i \geq 0, i = 1, 2, 3, 4$ and $\kappa_1 + \mathfrak{s}\kappa_2 + \mathfrak{s}^2\kappa_3 + \mathfrak{s}^2\kappa_4 < \frac{1}{2}$, then $f$ has a unique coupled fixed point.*

*Proof.* Consider $O$-sequences $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ as in the proof of Theorem 2.2. Then, we have $x_{i+1} = f(x_i, y_i), y_{i+1} = f(y_i, x_i)$ and

$$\begin{aligned} &x_i \perp x_{i+1} \text{ or } x_{i+1} \perp x_i, \\ &y_i \perp y_{i+1} \text{ or } y_{i+1} \perp y_i \text{ for all } i \in \mathbb{N} \end{aligned}$$

We now see that $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ are $b$-Cauchy $O$-sequences.
From (2.11), we have

$$\begin{aligned} d(x_i, x_{i+1}) &= d(f(x_{i-1}, y_{i-1}), f(x_i, y_i)) \\ &\leq \kappa_1[d(x_{i-1}, x_i) + d(y_{i-1}, y_i)] + \kappa_2[d(x_{i-1}, f(x_{i-1}, y_{i-1})) + d(x_i, f(x_i, y_i))] \\ &\qquad + \kappa_3[d(x_i, f(x_{i-1}, y_{i-1}) + d(x_{i-1}, f(x_i, y_i))] \\ &\qquad + \kappa_4[\Delta(x_{i-1}, y_{i-1}, x_i, y_i) + h\delta(x_{i-1}, y_{i-1}, x_i, y_i)] \end{aligned} \tag{2.12}$$

where
$\Delta(x_{i-1}, y_{i-1}, x_i, y_i) = \max\{d(x_i, f(x_{i-1}, y_{i-1}), d(x_{i-1}, f(x_i, y_i))\} = d(x_{i-1}, x_{i+1})$ and $\delta(x_{i-1}, y_{i-1}, x_i, y_i) = \min\{d(x_i, f(x_{i-1}, y_{i-1}), d(x_{i-1}, f(x_i, y_i))\} = 0$.
From (2.12), we have

$$\begin{aligned} d(x_i, x_{i+1}) &\leq \kappa_1[d(x_{i-1}, x_i) + d(y_{i-1}, y_i)] + \kappa_2[d(x_{i-1}, x_i) + d(x_i, x_{i+1})] \\ &\qquad + \kappa_3 d(x_{i-1}, x_{i+1}) + \kappa_4 d(x_{i-1}, x_{i+1}) \\ &\leq \kappa_1[d(x_{i-1}, x_i) + d(y_{i-1}, y_i)] + \kappa_2[d(x_{i-1}, x_i) + d(x_i, x_{i+1})] \\ &\qquad + \mathfrak{s}\kappa_3[d(x_{i-1}, x_i) + d(x_i, x_{i+1})] + \mathfrak{s}\kappa_4[d(x_{i-1}, x_i) + d(x_i, x_{i+1})] \end{aligned}$$

which implies that

$$(1 - \kappa_2 - \mathfrak{s}\kappa_3 - \mathfrak{s}\kappa_4)d(x_i, x_{i+1}) \leq (\kappa_1 + \kappa_2 + \mathfrak{s}\kappa_3 + \mathfrak{s}\kappa_4)d(x_{i-1}, x_i) + \kappa_1 d(y_{i-1}, y_i) \tag{2.13}$$

and similarly

$$(1 - \kappa_2 - \mathfrak{s}\kappa_3 - \mathfrak{s}\kappa_4)d(y_i, y_{i+1}) \leq (\kappa_1 + \kappa_2 + \mathfrak{s}\kappa_3 + \mathfrak{s}\kappa_4)d(y_{i-1}, y_i) + \kappa_1 d(x_{i-1}, x_i). \tag{2.14}$$

From (2.13) and (2.14), we get

$$\begin{aligned}
(1-\kappa_2-\mathfrak{s}\kappa_3-\mathfrak{s}\kappa_4)(d(x_i,x_{i+1})+d(y_i,y_{i+1})) &\leq (\kappa_1+\kappa_2+\mathfrak{s}\kappa_3+\mathfrak{s}\kappa_4)d(x_{i-1},x_i)\\
&\quad +\kappa_1 d(y_{i-1},y_i)+(\kappa_1+\kappa_2+\mathfrak{s}\kappa_3+\mathfrak{s}\kappa_4)d(y_{i-1},y_i)+\kappa_1 d(x_{i-1},x_i)\\
\Rightarrow d(x_i,x_{i+1})+d(y_i,y_{i+1}) &\leq \left(\tfrac{2\kappa_1+\kappa_2+\mathfrak{s}\kappa_3+\mathfrak{s}\kappa_4}{1-\kappa_2-\mathfrak{s}\kappa_3-\mathfrak{s}\kappa_4}\right)(d(x_{i-1},x_i)+d(y_{i-1},y_i))\\
&= \iota\left(d(x_{i-1},x_i)+d(y_{i-1},y_i)\right),
\end{aligned}$$

where

$$\iota=\left(\frac{2\kappa_1+\kappa_2+\mathfrak{s}\kappa_3+\mathfrak{s}\kappa_4}{1-\kappa_2-\mathfrak{s}\kappa_3-\mathfrak{s}\kappa_4}\right)<1.$$

Proceeding similar to Theorem 2.2, we get that $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ are orthogonal $b$-Cauchy sequences in $X$.

Since $X$ is an $O$-complete $b$-metric space, there exist $u,v\in X$ such that $x_i\to u$ and $y_i\to v$. By choice of $u$ and $v$, we have $u\perp x_i$ or $x_i\perp u$ and $v\perp y_i$ or $y_i\perp v$.

From (2.11), we have

$$\begin{aligned}
d(f(u,v),u) &\leq \mathfrak{s}[d(f(u,v),x_{i+1})+d(x_{i+1},u)]\\
&= \mathfrak{s}d(f(u,v),f(x_i,y_i))+\mathfrak{s}d(x_{i+1},u)\\
&\leq \mathfrak{s}[\kappa_1[d(u,x_i)+d(v,y_i)]+\kappa_2[d(u,f(u,v)),d(x_i,f(x_i,y_i))]\\
&+\kappa_3[d(x_i,f(u,v))+d(u,f(x_i,y_i))]+\kappa_4[\Delta(u,v,x_i,y_i)+\delta(u,v,x_i,y_i)]]+\mathfrak{s}d(x_{i+1},u)
\end{aligned} \tag{2.15}$$

where

$\Delta(u,v,x_i,y_i)=\max\{d(x_i,f(u,v)),d(u,f(x_i,y_i))\}$ and

$\delta(u,v,x_i,y_i)=\min\{d(x_i,f(u,v)),d(u,f(x_i,y_i))\}$.

On taking upper limit as $i\to\infty$, we get

$\limsup\limits_{i\to\infty}\Delta(u,v,x_i,y_i)\leq \mathfrak{s}d(f(u,v),u)$ and $\limsup\limits_{i\to\infty}\delta(u,v,x_i,y_i)=0$.

Letting limit superior as $i\to\infty$ in (2.15), and using Lemma 2.1, we get

$$\begin{aligned}
&d(f(u,v),u)\leq \mathfrak{s}\kappa_2 d(f(u,v),u)+\mathfrak{s}^2\kappa_3 d(f(u,v),u)+\mathfrak{s}^2\kappa_4 d(f(u,v),u)\\
&\Rightarrow (1-\mathfrak{s}\kappa_2-\mathfrak{s}^2\kappa_3-\mathfrak{s}^2\kappa_4)d(f(u,v),u)\leq 0.
\end{aligned}$$

Thus, $d(f(u,v),u)=0$ and hence $f(u,v)=u$. Similarly, we obtain that $f(v,u)=v$.

Therefore, $f$ has a coupled fixed point $(u,v)$.

Let $(u^*,v^*)\neq(u,v)$ be a coupled fixed point of $f$.

i.e., $f(u^*,v^*)=u^*$ and $f(v^*,u^*)=v^*$.

Therefore,

$d(f(u,v),f(u^*,v^*))=d(u,u^*)>0$ and $d(f(v,u),f(v^*,u^*))=d(v,v^*)>0$.

Since $f$ is $\perp$-preserving, we get

$$u\perp u^* \text{ or } u^*\perp u$$

and

$$v\perp v^* \text{ or } v^*\perp v.$$

From (2.11), we obtain

$$\begin{aligned}
d(u,u^*) &= d(f(u,v),f(u^*,v^*))\\
&\leq \kappa_1[d(u,u^*)+d(v,v^*)]+\kappa_2[d(u,f(u,v))+d(u^*,f(u^*,v^*))]\\
&+\kappa_3[d(u,f(u^*,v^*))+d(u^*,f(u,v))]+\kappa_4[\Delta(u,v,u^*,v^*)+h\delta(u,v,u^*,v^*)]
\end{aligned}$$

where $\Delta(u,v,u^*,v^*)=\max\{d(u,f(u^*,v^*)),d(u^*,f(u,v))\}=d(u,u^*)$,

$\delta(u,v,u^*,v^*)=\min\{d(u,f(u^*,v^*)),d(u^*,f(u,v))\}=d(u,u^*)$.

Therefore,

$$d(u,u^*)\leq(2\kappa_1+2\kappa_3+\kappa_4(1+h))d(u,u^*). \tag{2.16}$$

Similarly, we obtain that

$$d(v,v^*) \le (2\kappa_1 + 2\kappa_3 + \kappa_4(1+h))d(v,v^*). \tag{2.17}$$

From (2.16) and (2.17), we get

$$d(u,u^*) + d(v,v^*) \le (2\kappa_1 + 2\kappa_3 + \kappa_4(1+h))(d(u,u^*) + d(v,v^*)).$$

As $(2\kappa_1 + 2\kappa_3 + (1+h)\kappa_4) < 1$, we obtain
$d(u,u^*) + d(v,v^*) = 0$ and that $u = u^*, v = v^*$.
Hence $(u,v) = (u^*,v^*)$.
Thus, $f$ has only one coupled fixed point. □

**Theorem 2.8.** *Let $(X,\perp,d)$ be an $O$-complete $b$-metric space and $f : X \times X \to X$ be $\perp$-preserving map. If the condition*

$$\begin{aligned}\mathfrak{s}^3 d(f(x,y), f(\eta,\nu)) \le \kappa \max\{&d(x,\eta), d(y,\nu), d(x,f(x,y)), d(\eta,f(\eta,\nu)), d(y,f(y,x)),\\ &d(\nu,f(\nu,\eta)), \tfrac{d(x,f(\eta,\nu))+d(\eta,f(x,y))}{2\mathfrak{s}}, \tfrac{d(y,f(\nu,\eta))+d(\nu,f(y,x))}{2\mathfrak{s}}\}\end{aligned} \tag{2.18}$$

*holds for all $x,y,\eta,\nu \in X$ with $x \perp \eta$ and $y \perp \nu$, where $\kappa \in [0,1)$. Then, there is only one coupled fixed point for $f$.*

*Proof.* We consider $O$-sequences $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ as in the proof of Theorem 2.2.
Then we have $x_{i+1} = f(x_i,y_i), y_{i+1} = f(y_i,x_i)$ and

$$\begin{aligned}&x_i \perp x_{i+1} \text{ or } x_{i+1} \perp x_i,\\ &y_i \perp y_{i+1} \text{ or } y_{i+1} \perp y_i \text{ for all } i \in \mathbb{N}.\end{aligned}$$

We now see that $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ are $b$-Cauchy $O$-sequences. From (2.18), we have

$$\begin{aligned}d(x_i,x_{i+1}) &\le \mathfrak{s}^3 d(x_i,x_{i+1})\\ &= \mathfrak{s}^3 d(f(x_{i-1},y_{i-1}), f(x_i,y_i))\\ &\le \kappa \max\{d(x_{i-1},x_i), d(y_{i-1},y_i), d(x_{i-1},f(x_{i-1},y_{i-1})), d(x_i,f(x_i,y_i)),\\ &\quad d(y_{i-1},f(y_{i-1},x_{i-1})), d(y_i,f(y_i,x_i)), \tfrac{d(x_{i-1},f(x_i,y_i))+d(x_i,f(x_{i-1},y_{i-1}))}{2\mathfrak{s}},\\ &\quad \tfrac{d(y_{i-1},f(y_i,x_i))+d(y_i,f(y_{i-1},x_{i-1}))}{2\mathfrak{s}}\}\\ &= \kappa \max\{d(x_{i-1},x_i), d(y_{i-1},y_i), d(x_{i-1},x_i), d(x_i,x_{i+1}), d(y_{i-1},y_i),\\ &\quad d(y_i,y_{i+1}), \tfrac{d(x_{i-1},x_{i+1})+d(x_i,x_i)}{2\mathfrak{s}}, \tfrac{d(y_{i-1},y_{i+1})+d(y_i,y_i)}{2\mathfrak{s}}\}\\ &\le \kappa \max\{d(x_{i-1},x_i), d(y_{i-1},y_i), d(x_i,x_{i+1}), d(y_i,y_{i+1}),\\ &\quad \tfrac{d(x_{i-1},x_i)+d(x_i,x_{i+1})}{2}, \tfrac{d(y_{i-1},y_i)+d(y_i,y_{i+1})}{2}\}\\ &\le \kappa \max\{d(x_{i-1},x_i), d(y_{i-1},y_i), d(x_i,x_{i+1}), d(y_i,y_{i+1})\}.\end{aligned} \tag{2.19}$$

Similarly, we obtain

$$
\begin{aligned}
d(y_i, y_{i+1}) \leq\ & \mathfrak{s}^3 d(y_i, y_{i+1}) \\
&= \mathfrak{s}^3 d(f(y_{i-1}, x_{i-1}), f(y_i, x_i)) \\
&\leq \kappa \max\{d(y_{i-1}, y_i), d(x_{i-1}, x_i), d(y_{i-1}, f(y_{i-1}, x_{i-1})), d(y_i, f(y_i, x_i)), \\
&\qquad d(x_{i-1}, f(x_{i-1}, y_{i-1})), d(x_i, f(x_i, y_i)), \\
&\qquad \tfrac{d(y_{i-1}, f(y_i,x_i)) + d(y_i, f(y_{i-1}, x_{i-1}))}{2\mathfrak{s}}, \tfrac{d(x_{i-1}, f(x_i,y_i)) + d(x_i, f(x_{i-1}, y_{i-1}))}{2\mathfrak{s}}\} \\
&= \kappa \max\{d(y_{i-1}, y_i), d(x_{i-1}, x_i), d(y_{i-1}, y_i), d(y_i, y_{i+1}), d(x_{i-1}, x_i), \\
&\qquad d(x_i, x_{i+1}), \tfrac{d(y_{i-1}, y_{i+1}) + d(y_i, y_i)}{2\mathfrak{s}}, \tfrac{d(x_{i-1}, x_{i+1}) + d(x_i, x_i)}{2\mathfrak{s}}\} \\
&\leq \kappa \max\{d(x_{i-1}, x_i), d(y_{i-1}, y_i), d(x_i, x_{i+1}), d(y_i, y_{i+1}), \\
&\qquad \tfrac{d(x_{i-1}, x_i) + d(x_i, x_{i+1})}{2}, \tfrac{d(y_{i-1}, y_i) + d(y_i, y_{i+1})}{2}\} \\
&\leq \kappa \max\{d(x_{i-1}, x_i), d(y_{i-1}, y_i), d(x_i, x_{i+1}), d(y_i, y_{i+1})\}.
\end{aligned} \tag{2.20}
$$

If $\max\{d(x_{i-1}, x_i), d(y_{i-1}, y_i), d(x_i, x_{i+1}), d(y_i, y_{i+1})\} = d(x_i, x_{i+1})$ then from (2.19), we get that

$$\mathfrak{s}^3 d(x_i, x_{i+1}) \leq \kappa d(x_i, x_{i+1}) < d(x_i, x_{i+1}),$$

which is a contradiction.
Suppose, $\max\{d(x_{i-1}, x_i), d(y_{i-1}, y_i), d(x_i, x_{i+1}), d(y_i, y_{i+1})\} = d(y_i, y_{i+1})$ then from (2.20), we get that

$$\mathfrak{s}^3 d(y_i, y_{i+1}) \leq \kappa d(y_i, y_{i+1}) < d(y_i, y_{i+1}),$$

a contradiction. And from (2.19) and (2.20), we get

$$\mathfrak{s}^3[d(x_i, x_{i+1}) + d(y_i, y_{i+1})] \leq \kappa[d(x_i, x_{i+1}) + d(y_i, y_{i+1})] < d(x_i, x_{i+1}) + d(y_i, y_{i+1}),$$

it is a contradiction.
If $\max\{d(x_{i-1}, x_i), d(y_{i-1}, y_i), d(x_i, x_{i+1}), d(y_i, y_{i+1})\} = d(x_{i-1}, x_i)$ then from (2.19) and (2.20), we have

$$\mathfrak{s}^3[d(x_i, x_{i+1}) + d(y_i, y_{i+1})] \leq 2\kappa d(x_{i-1}, x_i). \tag{2.21}$$

Similarly, if $\max\{d(x_{i-1}, x_i), d(y_{i-1}, y_i), d(x_i, x_{i+1}), d(y_i, y_{i+1})\} = d(y_{i-1}, y_i)$ then from (2.19) and (2.20), we have

$$\mathfrak{s}^3[d(x_i, x_{i+1}) + d(y_i, y_{i+1})] \leq 2\kappa d(y_{i-1}, y_i) \tag{2.22}$$

From (2.21) and (2.22)

$$
\begin{aligned}
&\mathfrak{s}^3[d(x_i, x_{i+1}) + d(y_i, y_{i+1})] \leq \kappa[d(y_{i-1}, y_i) + d(x_{i-1}, x_i)] \\
&\Rightarrow d(x_i, x_{i+1}) + d(y_i, y_{i+1}) \leq \tfrac{\kappa}{\mathfrak{s}^3}[d(y_{i-1}, y_i) + d(x_{i-1}, x_i)] \\
&\qquad\qquad = a[d(y_{i-1}, y_i) + d(x_{i-1}, x_i)], a = \tfrac{\kappa}{\mathfrak{s}^3} < 1.
\end{aligned}
$$

As in Theorem 2.2, it can be shown that $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ are orthogonal $b$-Cauchy sequences in $X$.
Since $X$ is an $O$-complete $b$-metric space, there exist $u, v \in X$ such that $x_i \to u$ and $y_i \to v$. By choice of $u$ and $v$, we have $u \perp x_i$ or $x_i \perp u$ and $v \perp y_i$ or $y_i \perp v$.

From (2.18), we have

$$\begin{aligned} d(f(u,v),u) &\le \mathfrak{s}[d(f(u,v),x_{i+1}) + d(x_{i+1},u)] \\ &= \mathfrak{s}d(f(u,v),f(x_i,y_i)) + \mathfrak{s}d(x_{i+1},u) \\ &\le \mathfrak{s}\kappa \max\{d(u,x_i), d(v,y_i), d(u,f(u,v)), d(x_i,f(x_i,y_i)), d(v,f(v,u)), \\ &\qquad d(y_i,f(y_i,x_i)), \tfrac{d(u,f(x_i,y_i))+d(x_i,f(u,v))}{2\mathfrak{s}}, \tfrac{d(v,f(y_i,x_i))+d(y_i,f(v,u))}{2\mathfrak{s}}\} \\ &\qquad +\mathfrak{s}d(x_{i+1},u) \\ &= \mathfrak{s}\kappa \max\{d(u,x_i), d(v,y_i), d(u,f(u,v)), d(x_i,x_{i+1}), d(v,f(v,u)), \\ &\qquad d(y_i,y_{i+1}), \tfrac{d(u,x_{i+1})+d(x_i,f(u,v))}{2\mathfrak{s}}, \tfrac{d(v,y_{i+1})+d(y_i,f(v,u))}{2\mathfrak{s}}\} \\ &\qquad +\mathfrak{s}d(x_{i+1},u). \end{aligned} \tag{2.23}$$

As $i \to \infty$ in (2.23), and using Lemma 2.1, we get

$$d(f(u,v),u) \le \mathfrak{s}\kappa \max\{d(f(u,v),u), d(f(v,u),v)\}. \tag{2.24}$$

If $\max\{d(f(u,v),u), d(f(v,u),v)\} = d(f(u,v),u)$ then we get $d(f(u,v),u) = 0$ and that $f(u,v) = u$.
Suppose that $\max\{d(f(u,v),u), d(f(v,u),v)\} = d(f(v,u),v)$ then from (2.24), we have

$$d(f(u,v),u) \le \mathfrak{s}\kappa d(f(v,u),v). \tag{2.25}$$

Similarly, we can easily see that $f(v,u) = v$ and

$$d(f(v,u),v) \le \mathfrak{s}\kappa d(f(u,v),u). \tag{2.26}$$

Hence from (2.25) and (2.26), we have

$$d(f(u,v),u) + d(f(v,u),v) \le \mathfrak{s}\kappa[d(f(u,v),u) + d(f(v,u),v)]$$

which implies that $d(f(u,v),u) + d(f(v,u),v) = 0$ and that $f(u,v) = u$ and $f(v,u) = v$.
Therefore $f$ has a coupled fixed point $(u,v)$.

Uniqueness of coupled fixed point follows from the inequality (2.18). □

The following serves as an illustration for Theorem 2.2.

**Example 4.** Let $X = [0,\infty)$ and define $x \perp y$ if $x < y$. So, $(X,\perp)$ is an $O$-set.
We define $d : X \times X \to \mathbb{R}^+$ by $d(x,y) = \begin{cases} (x+y)^2 & \text{if } x \ne y, \\ 0 & \text{if } x = y \end{cases}$.
Then $(X,\perp,d)$ is an $O$-complete $b$-metric space with $\mathfrak{s} = 2$. Let $f : X \times X \to X$ be a mapping defined by

$$f(x,y) = \begin{cases} \frac{x+y}{1+x^2+y^2} & \text{if } x < y, \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } x,y \in X.$$

Let $x \perp \eta$ and $y \perp \nu$. Then $x < \eta$ and $y < \nu$.
Now, $\frac{x+y}{1+x^2+y^2} < \frac{\eta+\nu}{1+\eta^2+\nu^2} \Rightarrow f(x,y) \perp f(\eta,\nu)$.
Thus, $f$ is $\perp$-preserving on $X$. Let $x \perp \eta$ and $y \perp \nu$. Now
Case (i): If $x < y$ and $\eta < \nu$, then $f(x,y) = \frac{x+y}{1+x^2+y^2}$ and $f(\eta,\nu) = \frac{\eta+\nu}{1+\eta^2+\nu^2}$ for all $x,y,\eta,\nu \in X$.
Case (ii): If $x < y$ and $\eta \ge \nu$, then $f(x,y) = \frac{x+y}{1+x^2+y^2}$ and $f(\eta,\nu) = 0$ for all $x,y,\eta,\nu \in X$.
Case (iii): If $x \ge y$ and $\eta < \nu$, then $f(x,y) = 0$ and $f(\eta,\nu) = \frac{\eta+\nu}{1+\eta^2+\nu^2}$ for all $x,y,\eta,\nu \in X$.
Case (iv): If $x \ge y$ and $\eta \ge \nu$, then $f(x,y) = 0$ and $f(\eta,\nu) = 0$ for all $x,y,\eta,\nu \in X$.

Take $\kappa_1=\kappa_2=\kappa_4=\frac{1}{8},\kappa_3=\frac{1}{16},\kappa_5=\kappa_6=\frac{1}{32}$.
For all the above cases, the condition (2.1)

$$\begin{aligned}
d(f(x,y),f(\eta,\nu)) &= (f(x,y)+f(\eta,\nu))^2\\
&= (\tfrac{x+y}{1+x^2+y^2}+\tfrac{\eta+\nu}{1+\eta^2+\nu^2})^2\\
&\leq \tfrac{1}{8}(x+\eta)^2+\tfrac{1}{8}(y+\nu)^2\\
&\leq \tfrac{1}{8}(x+\eta)^2+\tfrac{1}{8}(y+\nu)^2+\tfrac{1}{16}(x+\tfrac{x+y}{x^2+y^2})^2+\tfrac{1}{8}(\eta+\tfrac{\eta+\nu}{\eta^2+\nu^2})^2\\
&\quad +\tfrac{1}{32}(\eta+\tfrac{x+y}{x^2+y^2})^2+\tfrac{1}{32}(x+\tfrac{\eta+\nu}{\eta^2+\nu^2})^2\\
&= \kappa_1 d(x,\eta)+\kappa_2 d(y,\nu)+\kappa_3 d(f(x,y),x)+\kappa_4 d(f(\eta,\nu),\eta)\\
&\quad +\kappa_5 d(f(x,y),\eta)+\kappa_6 d(f(\eta,\nu),x)
\end{aligned}$$

is satisfied with $\kappa_1+\kappa_2+\mathfrak{s}\kappa_3+\kappa_4+\mathfrak{s}^2\kappa_5+2\mathfrak{s}\kappa_6<1$. As a result, Theorem 2.2 is met in its entirety, and $(0,0)$ is the only coupled fixed point of $f$.

**Example 5.** Let $X=[0,1]$ and define $x\perp y$ if $x<y$. So, $(X,\perp)$ is an $O$-set. We define $d:X\times X\to\mathbb{R}^+$ by $d(x,y)=\begin{cases}(x+y)^2 & \text{if } x\neq y,\\ 0 & \text{if } x=y\end{cases}$.
Then $(X,\perp,d)$ is an $O$-complete $b$-metric space with $\mathfrak{s}=2$. Let $f:X\times X\to X$ be a mapping defined by

$$f(x,y)=\begin{cases}\frac{\log(1+x^2+y^2)}{3} & \text{if } x<y,\\ 0 & \text{elsewhere}\end{cases}$$

for all $x,y\in X$. It is obvious that $f$ is $\perp$-preserving on $X$. Let $x\perp\eta$ and $y\perp\nu$. We have
Case (i): If $x<y$ and $\eta<\nu$, then $f(x,y)=\frac{\log(1+x^2+y^2)}{3}$ and $f(\eta,\nu)=\frac{\log(1+\eta^2+\nu^2)}{3}$ for all $x,y,\eta,\nu\in X$.
Case (ii): If $x<y$ and $\eta\geq\nu$, then $f(x,y)=\frac{\log(1+x^2+y^2)}{3}$ and $f(\eta,\nu)=0$ for all $x,y,\eta,\nu\in X$.
Case (iii): If $x\geq y$ and $\eta<\nu$, then $f(x,y)=0$ and $f(\eta,\nu)=\frac{\log(1+\eta^2+\nu^2)}{3}$ for all $x,y,\eta,\nu\in X$.
Case (iv): If $x\geq y$ and $\eta\geq\nu$, then $f(x,y)=0$ and $f(\eta,\nu)=0$ for all $x,y,\eta,\nu\in X$.
Take $\kappa=\frac{9}{10}$. For all the above cases, we verify the condition (2.18)

$$\begin{aligned}
\mathfrak{s}^3 d(f(x,y),f(\eta,\nu)) &= 8(f(x,y)+f(\eta,\nu))^2\\
&= 8(\tfrac{\log(1+x^2+y^2)}{3}+\tfrac{\log(1+\eta^2+\nu^2)}{3})^2\\
&= \tfrac{8}{9}(\log(1+x^2+y^2)+\log(1+\eta^2+\nu^2))^2\\
&\leq \tfrac{9}{10}(x+\eta)^2\\
&= \kappa d(x,\eta)\\
&\leq \kappa\max\{d(x,\eta),d(y,\nu),d(x,f(x,y)),d(\eta,f(\eta,\nu)),\\
&\quad d(y,f(y,x)),d(\nu,f(\nu,\eta)),\tfrac{d(x,f(\eta,\nu))+d(\eta,f(x,y))}{2\mathfrak{s}},\tfrac{d(y,f(\nu,\eta))+d(\nu,f(y,x))}{2\mathfrak{s}}\}.
\end{aligned}$$

As a result, Theorem 2.8 is satisfied in its entirety, and $(0,0)$ is the only coupled fixed point of $f$.

## 3. Application to nonlinear integral equations

Using Theorem 2.2, we demonstrate in this section that the following system of integral equations has a unique solution

$$\begin{aligned} x(t) &= \int_{a_l}^{b_u} \mathscr{F}(t, x(s), y(s))ds \\ y(t) &= \int_{a_l}^{b_u} \mathscr{F}(t, y(s), x(s))ds \end{aligned} \tag{3.1}$$

where $t \in [a_l, b_u], 0 \leq a_l < b_u$ and $\mathscr{F} : [a_l, b_u] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The class of real-valued continuous functions on the interval $[a_l, b_u]$ is denoted by $\mathscr{C}([a_l, b_u], \mathbb{R})$.

We define $f : X \times X \to X$ by $f(x, y)(t) = \int_{a_l}^{b_u} \mathscr{F}(t, x(s), y(s))ds, t \in [a_l, b_u], \theta, \rho \in X$.

**Theorem 3.1.** *Suppose $\mathscr{F} : [a_l, b_u] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a mapping. We assume that the following circumstances exist:*

*($\mathscr{I}\mathscr{E}_1$) $f$ is a continuous mapping,*

*($\mathscr{I}\mathscr{E}_2$) there exists $\kappa_i \geq 0, i = 0, 1, 2, 3, 4, 5, 6$ with $\kappa_1 + \kappa_2 + \mathfrak{s}\kappa_3 + \kappa_4 + \mathfrak{s}^2\kappa_5 + 2\mathfrak{s}\kappa_6 < 1$ such that*

$$0 \leq | \mathscr{F}(t, \eta, \nu) - \mathscr{F}(t, x, y) |^{\wp} \leq \frac{1}{(b_u - a_l)^{\wp}} \nabla(x, y, \eta, \nu)$$

*where*

$$\nabla(x, y, \eta, \nu) = \kappa_1|\eta - x|^{\wp} + \kappa_2|\nu, y|^{\wp} + \kappa_3|x - f(x, y)|^{\wp} + \kappa_4|\eta - f(\eta, \nu)|^{\wp} + \kappa_5|f(x, y) - \eta|^{\wp} + \kappa_6|x - f(\eta, \nu)|^{\wp}$$

*for all $x, y, \eta, \nu \in \mathbb{R}, x, y, \eta, \nu \geq 0$ and for all $t \in [a_l, b_u]$. Then, there is only one solution to the system of integral equations* (3.1).

*Proof.* $X = \{x \in \mathscr{C}([a_l, b_u], \mathbb{R}) : x(t) \geq 0, \text{ for all } t \in [a_l, b_u]\}$.
We consider the orthogonality relationship in $X$ by $x \perp y \iff y(t) \geq x(t)$, for all $t \in [a_l, b_u]$.
We take an arbitrary $t$ and define

$$d(x, y) = \max_{t \in [a_l, b_u]} | x(t) - y(t) |^{\wp} \text{ for all } x, y \in X.$$

We can easily see that $(X, d)$ is a $b$-metric space with $\mathfrak{s} = 2^{\wp-1}, \wp > 1$ a real number. We consider a Cauchy $O$-sequence $\{x_i\}_{i\in\mathbb{N}} \subseteq X$. It is easily say that $\{x_i\}_{i\in\mathbb{N}}$ is convergent to a point $u \in \mathscr{C}([a_l, b_u], \mathbb{R})$. We take arbitrary $t \in [a_l, b_u]$. We have $x_i \perp x_{i+1}$ for each $i$. Since $x_i \geq 0$ for all $i \in \mathbb{N}$, this sequence converges to $u(t)$. This means $u(t) \geq 0$ and that $u \in X$.
We now show that $f$ is $\perp$-preserving.
For all $x, y, \eta, \nu \in X$ with $x \perp \eta, y \perp \nu$ and $t \in [a_l, b_u]$, from ($\mathscr{I}\mathscr{E}_2$), we get
$0 \leq \mathscr{F}(t, \eta(s), \nu(s)) - \mathscr{F}(t, x(s), y(s)) \Rightarrow \mathscr{F}(t, x(s), y(s)) \leq \mathscr{F}(t, \eta(s), \nu(s))$.
So, we get

$$\begin{aligned} f(x, y)(t) &= \int_{a_l}^{b_u} \mathscr{F}(t, x(s), y(s))ds \\ &\leq \int_{a_l}^{b_u} \mathscr{F}(t, \eta(s), \nu(s))ds \\ &= f(\eta, \nu)(t). \end{aligned}$$

It follows that $f(\eta,\nu)(t) \geq f(x,y)(t)$ and that $f(x,y) \perp f(\eta,\nu)$.
Take $q \in \mathbb{R}$ with $\frac{1}{\wp} + \frac{1}{q} = 1$ using the Holder inequality and $x, y, \eta, \nu \in X$ with $x \perp \eta, y \perp \nu$ and $t \in [a_l, b_u]$, we have

$$\begin{aligned}
d(f(x,y), f(\eta,\nu)) &= d(f(\eta,\nu), f(x,y)) \\
&= \max_{t\in[a_l,b_u]} |f(\eta,\nu)(t) - f(x,y)(t)|^{\wp} \\
&= \max_{t\in[a_l,b_u]} \left| \int_{a_l}^{b_u} \mathscr{F}(t,\eta(s),\nu(s))ds - \int_{a_l}^{b_u} \mathscr{F}(t,x(s),y(s))ds \right|^{\wp} \\
&= \max_{t\in[a_l,b_u]} \left| \int_{a_l}^{b_u} (\mathscr{F}(t,\eta(s),\nu(s)) - \mathscr{F}(t,x(s),y(s)))ds \right|^{\wp} \\
&\leq \left[ \max_{t\in[a_l,b_u]} \left( \int_{a_l}^{b_u} 1^{\wp} ds \right)^{\frac{1}{q}} \left( \int_{a_l}^{b_u} |(\mathscr{F}(t,\eta(s),\nu(s)) - \mathscr{F}(t,x(s),y(s)))|^{\wp} ds \right)^{\frac{1}{\wp}} \right]^{\wp} \\
&= (b_u - a_l)^{\frac{\wp}{q}} \max_{t\in[a_l,b_u]} \left( \int_{a_l}^{b_u} |(\mathscr{F}(t,\eta(s),\nu(s)) - \mathscr{F}(t,x(s),y(s)))|^{\wp} ds \right) \\
&= (b_u - a_l)^{\wp-1} \max_{t\in[a_l,b_u]} \left( \int_{a_l}^{b_u} |(\mathscr{F}(t,\eta(s),\nu(s)) - \mathscr{F}(t,x(s),y(s)))|^{\wp} ds \right) \\
&\leq \max_{t\in[a_l,b_u]} (\kappa_1|\eta - x|^{\wp} + \kappa_2|\nu, y|^{\wp} + \kappa_3|x - f(x,y)|^{\wp} \\
&\qquad + \kappa_4|\eta - f(\eta,\nu)|^{\wp} + \kappa_5|f(x,y) - \eta|^{\wp} + \kappa_6|x - f(\eta,\nu)|^{\wp})
\end{aligned}$$

which implies that

$$\begin{aligned}
d(f(x,y), f(\eta,\nu)) \leq{}& \kappa_1 d(x,\eta) + \kappa_2 d(y,\nu) + \kappa_3 d(f(x,y),x) + \kappa_4 d(f(\eta,\nu),\eta) \\
&+ \kappa_5 d(f(x,y),\eta) + \kappa_6 d(f(\eta,\nu),x).
\end{aligned}$$

Therefore, from Theorem 2.2, (3.1) has a unique solution. □

## 4. Application to dynamic programming

In the section that follows, we go through whether functional equations that arises in dynamic programming have a bounded solution. Let $\tilde{\Theta}_1$ and $\tilde{\Theta}_2$ be two Banach spaces; $\tilde{\mathcal{D}} \subseteq \tilde{\Theta}_1$ is the decision space; $\tilde{\mathcal{S}} \subseteq \tilde{\Theta}_2$ is the state space; $\mho(\tilde{\mathcal{S}})$, the set of all bounded real valued functions on $\tilde{\mathcal{S}}$ with $b$-metric is defined by:

$$d(p_x, p_y) = \sup_{t\in\tilde{\mathcal{S}}} |p_x(t) - p_y(t)|^r, \text{ for all } p_x, p_y \in \mho(\tilde{\mathcal{S}}) \text{ with parameter } s = 2^{r-1}.$$

and for all $t \in \tilde{\mathcal{S}}, \|p_x(t)\|^r = \sup_{t\in\tilde{\mathcal{S}}} |p_x|^r$.
We consider the following functional equations:

$$\begin{cases} \xi(v_s) = \sup_{v_d\in\tilde{\mathcal{D}}} \{F(v_s,v_d) + \mathcal{C}(v_s,v_d,\xi(\omega(v_s,v_d)),\eta(\omega(v_s,v_d)))\} \text{ for all } v_s \in \tilde{\mathcal{S}}, \\ \zeta(v_s) = \sup_{v_d\in\tilde{\mathcal{D}}} \{F(v_s,v_d) + \mathcal{C}(v_s,v_d,\zeta(\omega(v_s,v_d)),\xi(\omega(v_s,v_d)))\} \text{ for all } v_s \in \tilde{\mathcal{S}}, \end{cases} \tag{4.1}$$

appearing in the study of dynamic programming, where $v_d$ is a decision vector, $v_s$ is a state vector, $\omega : \tilde{\mathcal{S}} \times \tilde{\mathcal{D}} \to \tilde{\mathcal{S}}, F : \tilde{\mathcal{S}} \times \tilde{\mathcal{D}} \to \mathbb{R}, \mathcal{C} : \tilde{\mathcal{S}} \times \tilde{\mathcal{D}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

and $\xi(v_s), \zeta(v_s)$ indicates the optimal return functions. Let $\aleph : \mho(\tilde{\mathcal{S}}) \to \mho(\tilde{\mathcal{S}})$ be a mapping defined by:

$$\aleph(\xi,\zeta)(v_s) = \sup_{v_d \in \tilde{\mathcal{D}}} \{\digamma(v_s, v_d) + \mathcal{C}(v_s, v_d, \xi(\omega(v_s, v_d))), \zeta(\omega(v_s, v_d)))\} \text{ for all } v_s \in \tilde{\mathcal{S}}. \tag{4.2}$$

**Theorem 4.1.** *Let $\mathcal{C} : \tilde{\mathcal{S}} \times \tilde{\mathcal{D}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a mapping and $\aleph$ be a mapping defined by* (4.2)*. We suppose the following circumstances are true:*

*($\mathscr{D}\mathscr{P}_1$) $\mathcal{C}$ is a continuous mapping,*
*($\mathscr{D}\mathscr{P}_2$) there exist $\kappa_i \geq 0, i = 1, 2, 3, 4$ with $\kappa_1 + \mathfrak{s}\kappa_2 + \mathfrak{s}^2\kappa_3 + \mathfrak{s}^2\kappa_4 < \frac{1}{2}$ such that*

$$\begin{aligned}0 &\leq \mathcal{C}(v_s, v_d, \eta(\omega(v_s, v_d)), \nu(\omega(v_s, v_d))) - \mathcal{C}(v_s, v_d, \xi(\omega(v_s, v_d)), \zeta(\omega(v_s, v_d)))\\ &\leq (\Delta_s(\xi, \zeta, \eta, \nu))^{\frac{1}{r}},\end{aligned}$$

*where*

$$\begin{aligned}\Delta_s(\xi,\zeta,\eta,\nu) &= \kappa_1[|\xi-\eta|^r + |\zeta-\nu|^r] + \kappa_2[|\xi-\aleph(\xi,\zeta)|^r + |\eta-\aleph(\eta,\nu)|^r]\\ &\quad + \kappa_3[|\xi-\aleph(\eta,\nu)|^r + |\eta-\aleph(\xi,\zeta)|^r]\\ &\quad + \kappa_4[\max\{|\xi-\aleph(\eta,\nu)|^r, |\eta-\aleph(\xi,\zeta)|^r\} + h\min\{|\xi-\aleph(\eta,\nu)|^r, |\eta-\aleph(\xi,\zeta)|^r\}].\end{aligned}$$

*Then, there is only one unique bounded solution to the system of functional equations defined by* (4.1)*.*

*Proof.* $\mathscr{M} = \{\xi \in \mho(\tilde{\mathcal{S}}) : \xi(v_s) \geq 0, \text{ for all } v_s \in \tilde{\mathcal{S}}\}$.
We consider the orthogonality relationship in $\mathscr{M}$ by $\xi \perp \zeta \iff \zeta(v_s) \geq \xi(v_s)$, for all $v_s \in \tilde{\mathcal{S}}$.
Clearly, $(\mathscr{M}, d)$ is a $b$-metric space with $\mathfrak{s} = 2^{r-1}, r > 1$ a real number. We consider a Cauchy $O$-sequence $\{\xi_i\}_{i\in\mathbb{N}} \subseteq \mathscr{M}$. It is easily say that $\{\xi_i\}_{i\in\mathbb{N}}$ is convergent to a point $u \in \mho(\tilde{\mathcal{S}})$. We take arbitrary $v_s \in \tilde{\mathcal{S}}$. We have $\xi_i \perp \xi_{i+1}$ for each $i$. Since $\xi_i \geq 0$ for all $i \in \mathbb{N}$, this sequence converges to $u(v_s)$. This means $u(v_s) \geq 0$ and that $u \in \mathscr{M}$.
We now show that $\aleph$ is $\perp$-preserving.
For all $\xi, \zeta, \eta, \nu \in \mathscr{M}$ with $\xi \perp \eta, \zeta \perp \nu$ and $v_s \in \tilde{\mathcal{S}}$, from ($\mathscr{D}\mathscr{P}_2$), we get

$$\begin{aligned}\aleph(\eta,\nu)(v_s) &= \sup_{v_d \in \tilde{\mathcal{D}}} \{\digamma(v_s, v_d) + \mathcal{C}(v_s, v_d, \eta(\omega(v_s, v_d))), \nu(\omega(v_s, v_d)))\}\\ &\geq \sup_{v_d \in \tilde{\mathcal{D}}} \{\digamma(v_s, v_d) + \mathcal{C}(v_s, v_d, \xi(\omega(v_s, v_d))), \zeta(\omega(v_s, v_d)))\}\\ &= \aleph(\xi,\zeta)(v_s).\end{aligned}$$

It follows that, $\aleph(\xi,\zeta) \leq \aleph(\eta,\nu)$ and that $\aleph(\xi,\zeta) \perp \aleph(\eta,\nu)$.
Let $\xi, \zeta, \eta, \nu \in \mathscr{M}$ with $\xi \perp \eta, \zeta \perp \nu$ and $v_s \in \tilde{\mathcal{S}}$, we have

$$|\aleph(\xi,\zeta)(v_s)-\aleph(\eta,\nu)(v_s)| = |\sup_{v_d\in\tilde{\mathcal{D}}}\{F(v_s,v_d)+\mathcal{C}(v_s,v_d,\xi(\omega(v_s,v_d))),\zeta(\omega(v_s,v_d)))\}$$
$$-\sup_{v_d\in\tilde{\mathcal{D}}}\{F(v_s,v_d)+\mathcal{C}(v_s,v_d,\eta(\omega(v_s,v_d))),\nu(\omega(v_s,v_d)))\}|$$
$$=\sup_{v_d\in\tilde{\mathcal{D}}}|\mathcal{C}(v_s,v_d,\xi(\omega(v_s,v_d))),\zeta(\omega(v_s,v_d)))-\mathcal{C}(v_s,v_d,\eta(\omega(v_s,v_d))),\nu(\omega(v_s,v_d)))|$$
$$\leq \sup_{v_d\in\tilde{\mathcal{D}}}(\Delta_s(\xi,\zeta,\eta,\nu))^{\frac{1}{r}}$$
$$\Rightarrow |\aleph(\xi,\zeta)(v_s)-\aleph(\eta,\nu)(v_s)|^r \leq \sup_{v_d\in\tilde{\mathcal{D}}}[\kappa_1[|\xi-\eta|^r+|\zeta-\nu|^r]$$
$$+\kappa_2[|\xi-\aleph(\xi,\zeta)|^r+|\eta-\aleph(\eta,\nu)|^r]+\kappa_3[|\xi-\aleph(\eta,\nu)|^r+|\eta-\aleph(\xi,\zeta)|^r]$$
$$+\kappa_4[\max\{|\xi-\aleph(\eta,\nu)|^r,|\eta-\aleph(\xi,\zeta)|^r\}+h\min\{|\xi-\aleph(\eta,\nu)|^r,|\eta-\aleph(\xi,\zeta)|^r\}]]$$
$$\leq \kappa_1[\|\xi-\eta\|^r+\|\zeta-\nu\|^r]+\kappa_2[\|\xi-\aleph(\xi,\zeta)\|^r+\|\eta-\aleph(\eta,\nu)\|^r]$$
$$+\kappa_3[\|\xi-\aleph(\eta,\nu)\|^r+\|\eta-\aleph(\xi,\zeta)\|^r]$$
$$+\kappa_4[\max\{\|\xi-\aleph(\eta,\nu)\|^r,\|\eta-\aleph(\xi,\zeta)\|^r\}+h\min\{\|\xi-\aleph(\eta,\nu)\|^r,\|\eta-\aleph(\xi,\zeta)\|^r\}]$$

which implies that

$$\sup_{v_s\in\tilde{\mathcal{S}}}|\aleph(\xi,\zeta)(v_s)-\aleph(\eta,\nu)(v_s)|^r \leq \sup_{v_s\in\tilde{\mathcal{S}}}\kappa_1[\|\xi-\eta\|^r+\|\zeta-\nu\|^r]$$
$$+\kappa_2[\|\xi-\aleph(\xi,\zeta)\|^r+\|\eta-\aleph(\eta,\nu)\|^r]+\kappa_3[\|\xi-\aleph(\eta,\nu)\|^r+\|\eta-\aleph(\xi,\zeta)\|^r]$$
$$+\kappa_4[\max\{\|\xi-\aleph(\eta,\nu)\|^r,\|\eta-\aleph(\xi,\zeta)\|^r\}+h\min\{\|\xi-\aleph(\eta,\nu)\|^r,\|\eta-\aleph(\xi,\zeta)\|^r\}].$$

Thus,

$$d(f(\xi,\zeta),f(\eta,\nu))\leq\kappa_1[d(\xi,\eta)+d(\zeta,\nu)]+\kappa_2[d(\xi,f(\xi,\zeta)+d(\eta,f(\eta,\nu))]$$
$$+\kappa_3[d(\xi,f(\eta,\nu))+d(\eta,f(\xi,\zeta))]$$
$$+\kappa_4[\max\{d(\xi,\aleph(\eta,\nu)),d(\eta,\aleph(\xi,\zeta))\}+h\min\{d(\xi,\aleph(\eta,\nu)),d(\eta,\aleph(\xi,\zeta))\}].$$

It is clear that Theorem 4.1 satisfies all the hypotheses Theorem 2.7. According to Theorem 2.7, the functional equations that are defined in (4.1) has a unique bounded solution. □

**Acknowledgments.** The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

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