Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 16 Issue 4(2024), Pages 17-34
https://doi.org/10.54671/BMAA-2024-4-2

COMPARISON OF ORTHOGONAL STABILITY OF CUBIC FUNCTIONAL EQUATION IN DIFFERENT SPACES

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Abstract. In this study, our focus is examining the orthogonal stability of the newly formulated following three-dimensional cubic functional equation using direct and fixed point methods within both modular and random normed space. To validate the stability conclusions, we present experimental outcomes. Additionally, we offer a comparative evaluation of the findings derived from this investigation:

$$\begin{aligned}
&f(2\upsilon_1+3\upsilon_2+4\upsilon_3) = 3f(\upsilon_1+3\upsilon_2+4\upsilon_3) + f(-\upsilon_1+3\upsilon_2+4\upsilon_3)\\
&+2f(\upsilon_1+3\upsilon_2) + 2f(\upsilon_1+4\upsilon_3) - 6f(\upsilon_1-3\upsilon_2) - 6f(\upsilon_1-4\upsilon_3)\\
&-3f(3\upsilon_2+4\upsilon_3) + 16[f(\upsilon_1-\frac{3}{2}\upsilon_2) + f(\upsilon_1-2\upsilon_3)] - 18f(\upsilon_1)\\
&-6f(3\upsilon_2) - 6f(4\upsilon_3).
\end{aligned}$$

where, $\upsilon_1, \upsilon_2, \upsilon_3$ are mutually orthogonal.

1. Introduction

In the present era, the domain of functional equations represents a continuously expanding area within mathematics, holding significant implications across various applications. The relatively recent emergence of functional equation theory has led to the creation of potent tools within modern mathematics. Functional equations encompass a traditional mathematical discipline that encompasses diverse avenues for algebraic, analytic, order-theoretic, and topological exploration. Concurrently, numerous mathematical concepts from various disciplines have become fundamental to the underpinnings of functional equations. This framework is progressively being employed to scrutinize challenges in unrelated fields like mathematical analysis, combinatorics, biology, behavioral and social sciences, as well as engineering.

In the field of functional equations, different sectors cover a range of research domains, one of which involves exploring the stability of functional equations. In 1940, S.M. Ulam [32] initiated research into the stability of functional equations,

1991 *Mathematics Subject Classification.* 39B82,46A80, 46B03.

Key words and phrases. Functional equation, Orthogonal Stability, Modular Space, Random normed space.

Submitted January 4, 2024. Accepted October 10, 2024. Published November 3, 2024.

Communicated by Janusz Brzdek.

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introducing the question: "Under what conditions is it valid that a solution to a slightly altered equation closely approximates the solution of the original equation?" Subsequent investigations have built upon this inquiry. D. H. Hyers [12] provided an affirmative answer to Ulam's stability problem for Banach spaces in 1941. Additionally, T. Aoki [3] delved deeper into the study of additive mappings in 1950. Numerous mathematicians[17, 22, 21, 24, 26, 31, 33, 34] have researched stability problems over the past few decades. See [3, 4, 6, 7, 13, 31, 14, 27, 30, 25, 8] for more information on the various stability problems with functional equations in various spaces.

Nakano [23] laid the groundwork for modular theory on linear spaces, shaping the perspective of modular linear spaces. Shimogaki, Koshi, and others from his mathematical school played a pivotal role in advancing this theory. Presently, the exploration of diverse Orlicz spaces and interpolation theory [19], which hold extensive practical applications, heavily relies on the concepts of modular spaces. The comprehensive structure of modular spaces, functioning both as Banach spaces and possessing the modular equivalent of norms or metric concepts, holds crucial significance in practical applications. This paper aims to elucidate the orthogonal stability of the cubic functional equation within modular space and random normed space.

Initially, our focus will be on understanding terminology, defining key concepts, establishing notation, and understanding common attributes within the subject matter. Then we are conveying these definitions in this paper as follows:

Definition 1.1. [31] *Assume $\mathcal{V}$ ($dim\mathcal{V} \geq 2$) be a real vector space associated with a binary relation $\perp$ having the following characteristics:*
$(O1)$ *totality of $\perp$ for zero: $\upsilon_1 \perp 0$, $0 \perp \upsilon_1$ for all $\upsilon_1 \in \mathcal{V}$;*
$(O2)$ *independence: if $\upsilon_1, \upsilon_2 \in \mathcal{V} - \{0\}$, $\upsilon_1 \perp \upsilon_2$, then, υ_1, υ_2 are linearly independent;*
$(O3)$ *homogeneity: if $\upsilon_1, \upsilon_2 \in \mathcal{V}$, $\upsilon_1 \perp \upsilon_2$, then, $\alpha\upsilon_1 \perp \beta\upsilon_2$ for all $\alpha, \beta \in R$;*
$(O4)$ *Thalesian Property: if $\upsilon_1 \in \mathcal{W}$ where $\mathcal{W}$ is a 2-dimensional subspace of $\mathcal{V}$ and $\lambda \in R^+$, then there exists $\upsilon_0 \in \mathcal{W}$ such that $\upsilon_1 \perp \upsilon_0$ and $\upsilon_1 + \upsilon_0 \perp \lambda\upsilon_1 - \upsilon_0$.*

The pair $(\mathcal{V}, \perp)$ is called an orthogonality space.

Definition 1.2. [10] *Let us consider a vector space $\mathcal{V}$, for arbitrary $\upsilon_1, \upsilon_2 \in \mathcal{V}$, a functional $\rho : \mathcal{V} \to [0, \infty]$ is called modular if it satisfies the following characteristics:*

(1) $\rho(\upsilon_1) = 0 \iff \upsilon_1 = 0$,
(2) $\rho(\alpha\upsilon_1) = \rho(\upsilon_1)$, *where* $|\alpha| = 1$,
(3) $\rho(\alpha\upsilon_1 + \beta\upsilon_2) \leq \rho(\upsilon_1) + \rho(\upsilon_2) \iff \alpha + \beta = 1$ *and* $\alpha, \beta \geq 0$.

Definition 1.3. [30] *Let us consider a vector space $\mathcal{V}$, for arbitrary $\upsilon_1, \upsilon_2 \in \mathcal{V}$, a functional $\rho : \mathcal{V} \to [0, \infty]$ is called convex modular if it satisfies the following characteristics:*

(1) $\rho(\upsilon_1) = 0 \iff \upsilon_1 = 0$,
(2) $\rho(\alpha\upsilon_1) = \rho(\upsilon_1)$, *where* $|\alpha| = 1$,
(3) $\rho(\alpha\upsilon_1 + \beta\upsilon_2) \leq \alpha\rho(\upsilon_1) + \beta\rho(\upsilon_2) \iff \alpha + \beta = 1$ *and* $\alpha, \beta \geq 0$.

The vector space $\mathcal{V}_\rho = \{\upsilon_1 \in \mathcal{V} : \rho(\lambda\upsilon_1) \to 0$ as $\lambda \to 0\}$ is called modular space.

Definition 1.4. [30] *Let $\{\upsilon_n\}$ be a sequence in $\mathcal{V}_\rho$ and υ be a point in $\mathcal{V}_\rho$ then the sequence $\{\upsilon_n\}$ is called $\rho-$ convergent to υ if and only if $\rho(\upsilon_n - \upsilon) \to 0$ as $n \to \infty$.*

Definition 1.5. [11] *Let* $\{\upsilon_n\}$ *be a sequence in* $\mathcal{V}_\rho$ *and is called* $\rho-$ *Cauchy if* $\rho(\upsilon_n - \upsilon_m) \to 0$ *as* $n, m \to \infty$.

Definition 1.6. [11] *If any* $\rho-$ *Cauchy sequence is* $\rho-$ *convergent to an element of subset* $\mathcal{W}$ *of* $\mathcal{V}_\rho$ *then, the subset* $\mathcal{W}$ *is called* $\rho-$ *complete.*

Remark. *Fatou property is satisfied by the modular* ρ *if and only if* $\rho(\upsilon) \le \lim_{n\to\infty} \inf \rho(\upsilon_n)$ *whenever the sequence* $\{\upsilon_n\}$ *is* $\rho-$ *convergent to* υ. *Moreover, if* ρ *is convex modular on* $\mathcal{V}$ *and* $|\alpha| \le 1$ *then,* $\rho(\alpha\upsilon) \le |\alpha|\rho(\upsilon)$.

Definition 1.7. [29] *A mapping* $T : [0,1] \to [0,1]$ *is said to be continuous t-norm, if mapping* T *has the following properties:*

(1) T *is commutative as well as associative.*
(2) T *is continuous.*
(3) $T(\alpha_1, 1) = \alpha_1 \quad \forall \quad \alpha_1 \in [0,1]$.
(4) $T(\alpha_1, \alpha_2) \le T(\alpha_3, \alpha_4)$ *whenever* $(\alpha_1 \le \alpha_3)$ *and* $(\alpha_2 \le \alpha_4)$ $\forall \quad \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1]$.

Remark. *Let* T *be a t-norm and* $\{\upsilon_m\}$ *be a sequence of numbers in* $[0,1]$ *then* $T_{i=1}^m \upsilon_i$ *is defined recurrently by*

$$T_{i=1}^m \upsilon_i = \begin{cases} \upsilon_1, & \text{if } m = 2. \\ T(T_{i=1}^{m-1}\upsilon_i, \upsilon_m), & \text{if } m \ge 2 \end{cases}$$

Definition 1.8. [29] *A triplet* $(\mathcal{W}, \mu^\rho, T)$ *is called random normed space if it satisfies the following conditions:*

(1) $\mu^\rho{}_\upsilon(t) = \epsilon_0(t) \quad \forall \quad t > 0 \Longleftrightarrow \upsilon = 0$;
(2) $\mu^\rho{}_{\alpha\upsilon}(t) = \mu^\rho{}_\upsilon(\frac{t}{|\alpha|}) \quad \forall \quad \upsilon \in \mathcal{W}, \alpha \ne 0$;
(3) $\mu^\rho{}_{\upsilon_1+\upsilon_2}(t+s) \ge T(\mu^\rho{}_{\upsilon_1}(t), \mu^\rho{}_{\upsilon_2}(s)) \quad \forall \quad \upsilon_1, \upsilon_2 \in \mathcal{W}$ *and all* $t, s \ge 0$.

Here, $\mathcal{W}, T$ *are vector space and continuous* t*-norm respectively and* μ^ρ *is a mapping from* $\mathcal{W}$ *into* $D^+ \subset \Delta^+$ *(space of distribution functions).*

Theorem 1.9. [9] *Consider a complete generalized metric space* $(\mathcal{W}, d)$ *and a strictly contractive mapping* $J : \mathcal{W} \to \mathcal{W}$ *with Lipschitz constant* $L < 1$. *Then for each given vector* $\upsilon \in \mathcal{W}$, *either*

$$d(J^n\upsilon, J^{n+1}\upsilon) = \infty$$

for all non-negative integers n *or there exists a positive integer* n_0 *such that*

(1) $d(J^n\upsilon, J^{n+1}\upsilon) < \infty, \quad \forall \quad n \ge n_0$;
(2) *the sequence* $\{J^n\upsilon\}$ *converges to a fixed point* $\upsilon^\star$;
(3) $\upsilon^\star$ *is the unique fixed point* J *in the set* $Z = \{z, \in \mathcal{W} | d(J^{n_0}\upsilon, z) < \infty\}$;
(4) $d(z, z^\star) \le \frac{1}{1-L} d(z, Jz)$ *for all* $z \in Z$.

This article is structured into five sections. The first section functions as an introduction, setting the groundwork for the subsequent discussions. The second section delves into investigating the orthogonal stability of the cubic functional equation within modular space, employing both direct and fixed point methods for analysis. Similarly, in the third section, we explore the orthogonal stability of the cubic functional equation in random norm space, employing direct and fixed point methods for analysis as well. In fourth and fifth section, we provide the experimental results and comparative evaluation of the results respectively.

2. Orthogonal Stability of Cubic Functional Equation in Modular space

Aboutaib et al.[1] established the Ulam stability results for the general linear functional equation in modular space. Motivated by their insights and research contributions, this section aims to investigate the stability of the given orthogonal cubic functional equation within modular space by employing the direct and fixed point methods respectively. Two key concepts are important when studying modulars. A modular ρ is said to possess the Fatou property if $\rho(\upsilon) \leq \liminf_{n\to\infty} \rho(\upsilon_n)$ for any sequence $\{\upsilon_n\}$ that ρ-converges to to υ. Additionally,ρ satisfies the Δ_2-condition if there exists a constant $\kappa \geq 0$ such that $\rho(2\upsilon) \leq \kappa\rho(\upsilon)$ for all $\upsilon \to \mathcal{V}_\rho$. We assume that the modular function ρ possesses the Fatou property and satisfies the Δ_2 condition with a certain condition $0 < \kappa \leq 2$. For the sake of convenience, we define

$$\begin{aligned}
&Df(\upsilon_1,\upsilon_2,\upsilon_3) = f(2\upsilon_1+3\upsilon_2+4\upsilon_3) - 3f(\upsilon_1+3\upsilon_2+4\upsilon_3)\\
&-f(-\upsilon_1+3\upsilon_2+4\upsilon_3) - 2f(\upsilon_1+3\upsilon_2) - 2f(\upsilon_1+4\upsilon_3)\\
&+6f(\upsilon_1-3\upsilon_2) + 6f(\upsilon_1-4\upsilon_3) + 3f(3\upsilon_2+4\upsilon_3)\\
&-16[f(\upsilon_1-\frac{3}{2}\upsilon_2) + f(\upsilon_1-2\upsilon_3)] + 18f(\upsilon_1) + 6f(3\upsilon_2) + 6f(4\upsilon_3),
\end{aligned}$$

where, $\upsilon_1,\upsilon_2,\upsilon_3$ are mutually orthogonal.

Definition 2.1. *Let $\mathcal{V}$ be an orthogonality space and $\mathcal{U}$ be a real Banach space. A mapping $f : \mathcal{V} \to \mathcal{U}$ is called orthogonally cubic if it satisfies the so called orthogonally cubic functional equation*

$$\begin{aligned}
&f(2\upsilon_1+3\upsilon_2+4\upsilon_3) - 3f(\upsilon_1+3\upsilon_2+4\upsilon_3) - f(-\upsilon_1+3\upsilon_2+4\upsilon_3) - 2f(\upsilon_1+3\upsilon_2)\\
&-2f(\upsilon_1+4\upsilon_3) + 6f(\upsilon_1-3\upsilon_2) + 6f(\upsilon_1-4\upsilon_3) + 3f(3\upsilon_2+4\upsilon_3)\\
&-16[f(\upsilon_1-\frac{3}{2}\upsilon_2) + f(\upsilon_1-2\upsilon_3)] + 18f(\upsilon_1) + 6f(3\upsilon_2) + 6f(4\upsilon_3) = 0
\end{aligned}$$

for all $\upsilon_1,\upsilon_2,\upsilon_3 \in \mathcal{V}$ with $\upsilon_1,\upsilon_2,\upsilon_3$ are mutually orthogonal.

Theorem 2.2. *Let $\mathcal{V}_\rho$ be a $\rho-$ complete modular space and $(\mathcal{W}, ||.||)$ be a real normed space with dimension greater than two. Suppose $f : \mathcal{W} \to \mathcal{V}_\rho$ is an odd mapping satisfying following inequality*

$$\rho(Df(\upsilon_1,\upsilon_2,\upsilon_3)) \leq \epsilon(||\upsilon_1||^p + ||\upsilon_2||^p + ||\upsilon_3||^p) \tag{2.1}$$

where, $\upsilon_1,\upsilon_2,\upsilon_3$ are mutually orthogonal and $\upsilon_1,\upsilon_2,\upsilon_3 \in \mathcal{W}$ and $0 < p < 3$. Then, there exists unique orthogonally cubic mapping $Q_c : \mathcal{W} \to \mathcal{V}_\rho$ such that

$$\rho(f(\upsilon_1) - Q_c(\upsilon_1)) \leq \frac{\epsilon}{8-\kappa 2^{p-1}}||\upsilon_1||. \tag{2.2}$$

Proof. Taking $(\upsilon_1,\upsilon_2,\upsilon_3) = (0,0,0)$ in (2.1), we obtain, $f(0) = 0$ and putting $\upsilon_2 = \upsilon_3 = 0$ in (2.1), then we get

$$\rho(f(2\upsilon_1) - 8f(\upsilon_1)) \leq ||\upsilon_1||^p$$

$$\implies \rho(\frac{f(2\upsilon_1)}{8} - f(\upsilon_1)) \leq \frac{\epsilon}{8}||\upsilon_1||^p. \tag{2.3}$$

Replacing υ_1 by $2\upsilon_1$ in (2.3)

$$\rho(\frac{f(2^2\upsilon_1)}{8} - f(2\upsilon_1)) \leq \frac{2^p\epsilon}{8}||\upsilon_1||^p, \tag{2.4}$$

therefore,

$$\rho(\frac{f(2^2\upsilon_1)}{8^2} - \frac{f(2\upsilon_1)}{8}) \leq \frac{2^{p-3}\epsilon}{8}||\upsilon_1||^p, \tag{2.5}$$

using equations (2.3) and (2.5), we obtain

$$\rho(\frac{f(2^2\upsilon_1)}{8^2} - f(\upsilon_1)) \leq \frac{\kappa}{2}\rho(\frac{f(2^2\upsilon_1)}{8^2} - \frac{f(2\upsilon_1)}{8})$$
$$+\frac{\kappa}{2}\rho(\frac{f(2\upsilon_1)}{8} - f(\upsilon_1)) \leq \frac{\epsilon}{8}(1 + \frac{\kappa}{2}.2^{p-3})||\upsilon_1||^p.$$

By using mathematical induction, we can show that

$$\rho\bigg(\frac{f(2^n\upsilon_1)}{8^n} - f(\upsilon_1)\bigg) \leq \frac{\epsilon}{8}\sum_{i=0}^{n-1}\bigg(\frac{\kappa}{2}\bigg)^i 2^{i(p-3)}||\upsilon_1||^p. \tag{2.6}$$

Equation(2.6) is true for $n = 1$. Assume that result in equation (2.6) is true for n and we will show that result is also true for $n+1$. Therefore, we have

$$\begin{aligned}
&\rho\bigg(\frac{f(2^{n+1}\upsilon_1)}{8^{n+1}} - f(\upsilon_1)\bigg)\\
&= \rho\bigg(\frac{f(2^{n+1}\upsilon_1)}{8^{n+1}} - \frac{f(2\upsilon_1)}{8} + \frac{f(2\upsilon_1)}{8} - f(\upsilon_1)\bigg)\\
&\leq \frac{\kappa}{2}\rho\bigg(\frac{f(2\upsilon_1)}{8} - f(\upsilon_1)\bigg) + \frac{\kappa}{2.8}\rho\bigg(\frac{f(2^{n+1}\upsilon_1)}{8^n} - f(2\upsilon_1)\bigg)\\
&\leq \frac{\kappa}{2}\frac{\epsilon}{8}||\upsilon_1||^p + \frac{\epsilon}{8}\sum_{i=0}^{n-1}\bigg(\frac{\kappa}{2}\bigg)^{i+1} 2^{\{(i+1)(p-3)\}}||\upsilon_1||^p\\
&\leq \frac{\epsilon}{8}\sum_{i=0}^{n}\bigg(\frac{\kappa}{2}\bigg)^i 2^{i(p-3)}||\upsilon_1||^p\\
&= \frac{\epsilon}{8}\bigg(\frac{1-(\kappa 2^{p-4})^n}{1-(\kappa 2^{p-4})}\bigg)||\upsilon_1||^p.
\end{aligned} \tag{2.7}$$

Replacing υ_1 by $2^m\upsilon_1$ in (2.6), we get

$$\rho\bigg(\frac{f(2^{n+m}\upsilon_1)}{8^n} - f(2^m\upsilon_1)\bigg) \leq \frac{\epsilon}{8}\bigg(\frac{1-(\kappa 2^{p-4})^n}{1-(\kappa 2^{p-4})}\bigg)2^{mp}||\upsilon_1||^p. \tag{2.8}$$

Hence,

$$\rho\bigg(\frac{f(2^{n+m}\upsilon_1)}{8^{n+m}} - \frac{f(2^m\upsilon_1)}{8^m}\bigg) \leq \frac{\epsilon}{8}\bigg(\frac{1-(\kappa 2^{p-4})^n}{1-(\kappa 2^{p-4})}\bigg)2^{m(p-3)}||\upsilon_1||^p. \tag{2.9}$$

If $m, n \to \infty$, we get the sequence $\{\frac{f(2^m\upsilon_1)}{8^m}\}$ is a $\rho-$ convergent in $\rho-$ complete modular space $\mathcal{V}_\rho$ and we will define the mapping $Q_c = lim_{n\to\infty}\frac{f(2^m\upsilon_1)}{8^m}$ from $\mathcal{W}$ into $\mathcal{V}_{\rho'}$ satisfying

$$\rho(Q_c(\upsilon_1) - f(\upsilon_1)) \leq \frac{\epsilon}{8-\kappa 2^{p-1}}||\upsilon_1||^p \tag{2.10}$$

for all $\upsilon_1 \in \mathcal{W}$, since ρ has Fatou's property.
For all $\upsilon_1, \upsilon_2, \upsilon_3 \in \mathcal{W}$ with $\upsilon_i \perp \upsilon_j (i \neq j \& i, j = 1, 2, 3)$, by applying (2.1) and $(O3)$,

we get

$$\rho\bigg(\frac{f(2^n(2\upsilon_1+3\upsilon_2+4\upsilon_3))}{8^n}-3\frac{f(2^n(\upsilon_1+3\upsilon_2+4\upsilon_3))}{8^n}-\frac{f(2^n(-\upsilon_1+3\upsilon_2+4\upsilon_3))}{8^n}$$
$$-2\frac{f(2^n(\upsilon_1+3\upsilon_2))}{8^n}-2\frac{f(2^n(\upsilon_1+4\upsilon_3))}{8^n}+6\frac{f(2^n(\upsilon_1-3\upsilon_2))}{8^n}+6\frac{f(2^n(\upsilon_1-4\upsilon_3))}{8^n}$$
$$+3\frac{f(2^n(\upsilon_2+4\upsilon_3))}{8^n}-16[f(2^n(\upsilon_1-\frac{3}{2}\upsilon_2))+f(2^n(\upsilon_1-2\upsilon_3))]+18f(2^n(\upsilon_1))$$
$$+6f(2^n(3\upsilon_2))+6f(2^n(4\upsilon_3))\bigg)\leq\frac{2^{np}}{8^n}\epsilon(||\upsilon_1||^p+||\upsilon_2||^p+||\upsilon_3||^p).$$

To prove Q_c satisfies $Df(\upsilon_1,\upsilon_2,\upsilon_3)=0$, taking $n\to\infty$ in above inequality, we get $DQ_c(\upsilon_1,\upsilon_2,\upsilon_3)=0$ for all $\upsilon_1,\upsilon_2,\upsilon_3\in\mathcal{W}$ with $\upsilon_1,\upsilon_2,\upsilon_3$ are orthogonal. Hence, $Q_c:\mathcal{W}\to\mathcal{V}_\rho$ is an orthogonally cubic mapping. Now, we will show the uniqueness of the mapping, for this we assume another cubic mapping $Q'_c:\mathcal{W}\to\mathcal{V}_\rho$ satisfying inequality (2.10).

$$\begin{aligned}\rho(Q_c(\upsilon_1)-Q'_c(\upsilon_1)) &= \rho\bigg(\frac{Q_c(2^m\upsilon_1)}{8^m}-\frac{Q'_c(2^m\upsilon_1)}{8^m}\bigg)\\ &\leq \frac{\kappa}{2.8^m}[\rho(Q_c(2^m\upsilon_1)-f(2^m\upsilon_1))\\ &+ \rho(Q_c(2^m\upsilon_1)-f(2^m\upsilon_1))]\\ &\leq \frac{\kappa\epsilon 2^{m(p-3)}}{8-\kappa 2^{p-1}}||\upsilon_1||^p.\end{aligned}$$

Now, taking $m\to\infty$, we get $Q_c=Q'_c$. □

Corollary 2.3. *Let $\mathcal{V}_\rho$ be a $\rho-$ complete modular space and $(\mathcal{W},||.||)$ be a real normed space with dimension greater than two. Suppose $f:\mathcal{W}\to\mathcal{V}_\rho$ is an odd mapping satisfying following inequality*

$$\rho(Df(\upsilon_1,\upsilon_2,\upsilon_3))\leq\epsilon \tag{2.11}$$

where, $\upsilon_1,\upsilon_2,\upsilon_3$ are mutually orthogonal and $\upsilon_1,\upsilon_2,\upsilon_3\in\mathcal{W}$. Then, there exists unique orthogonally cubic mapping $Q_c:\mathcal{W}\to\mathcal{V}_\rho$ such that

$$\rho(f(\upsilon_1)-Q_c(\upsilon_1))\leq\frac{\epsilon}{8-\kappa 2^{p-1}}. \tag{2.12}$$

Theorem 2.4. *Let us consider an odd mapping $f:\mathcal{W}\to\mathcal{V}_\rho$ which satisfies an inequality*

$$\rho(D(f(\upsilon_1,\upsilon_2,\upsilon_3)))\leq\phi(\upsilon_1,\upsilon_2,\upsilon_3) \tag{2.13}$$

and $f(0)=0$ where, $\upsilon_1,\upsilon_2,\upsilon_3$ are mutually orthogonal. Define the function $\phi:\mathcal{W}^3\to[0,\infty)$ satisfies the inequality

$$\phi(2\upsilon_1,2\upsilon_2,2\upsilon_3)\leq 8L\phi(\upsilon_1,\upsilon_2,\upsilon_3)\quad\forall\quad\upsilon_1,\upsilon_2,\upsilon_3\in\mathcal{W}, \tag{2.14}$$

where the constant L lie between 0 and 1. Then, there is exactly one and only one cubic mapping $Q_c:\mathcal{W}\to\mathcal{V}_\rho$ which satisfies the inequality

$$\rho(f(\upsilon_1)-Q_c(\upsilon_1))\leq\frac{1}{8(1-L)}\phi(\upsilon_1,0,0). \tag{2.15}$$

Proof. Putting $(\upsilon_1, \upsilon_2, \upsilon_3) = (\upsilon_1, 0, 0)$ in (2.11) and by using the property $(O1)$, we can write

$$\rho(f(2\upsilon_1) - 8f(\upsilon_1)) \leq \phi(\upsilon_1, 0, 0). \tag{2.16}$$

Also,

$$\rho(\frac{f(2\upsilon_1)}{8}) - f(\upsilon_1)) \leq \frac{1}{8}\phi(\upsilon_1, 0, 0). \tag{2.17}$$

Consider a set

$$F_{\tilde{\rho}} = \{f_1 : \mathcal{W} \to \mathcal{V}_\rho; f_1(0) = 0\}$$

the setting of the convex modular $\tilde{\rho}$ which is defined on $F_{\tilde{\rho}}$ in such a manner

$$\tilde{\rho}(f_1) = \inf\{\lambda > 0 : \rho(f_1(\upsilon_1)) \leq \lambda\phi(\upsilon_1, 0, 0)\}.$$

There are only sufficient to prove the condition for convex modular $\tilde{\rho}$ with $\alpha+\beta = 1$ is

$$\tilde{\rho}(\alpha f_1 + \beta f_2) \leq \alpha\tilde{\rho}(f_1) + \beta\tilde{\rho}(f_2).$$

Now, let for each $\epsilon > 0$, there are λ_1 and λ_2 such that

$$\lambda_1 \leq \tilde{\rho}(f_1) + \epsilon; \quad \rho(f_1(\upsilon_1)) \leq \lambda_1\phi(\upsilon_1, 0, 0)$$

and

$$\lambda_2 \leq \tilde{\rho}(f_2) + \epsilon; \quad \rho(f_2(\upsilon_1)) \leq \lambda_2\phi(\upsilon_1, 0, 0).$$

If $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$, we have

$$\rho(\alpha f_1(\upsilon_1) + \beta f_2(\upsilon_2)) \leq \alpha\rho(f_1(\upsilon_1)) + \beta\rho(f_2(\upsilon_1)),$$

hence

$$\tilde{\rho}(\alpha f_1 + \beta f_2) \leq \alpha\tilde{\rho}(f_1) + \beta\tilde{\rho}(f_2) + (\alpha + \beta)\epsilon.$$

Therefore, we obtain

$$\tilde{\rho}(\alpha f_1 + \beta f_2) \leq \alpha\tilde{\rho}(f_1) + \beta\tilde{\rho}(f_2).$$

Furthermore, Δ_2- condition is satisfied by the convex modular $\tilde{\rho}$ with $0 < \kappa \leq 2$. Consider a $\tilde{\rho}-$ Cauchy sequence $\{f_{1n}\}$ in $F_{\tilde{\rho}}$. For a given $\epsilon > 0$ there is $n' \in N$ in such a way $\tilde{\rho}(f_n - f_m) \leq \epsilon \quad \forall \quad n, m \geq n'$. Therefore, we get

$$\rho(f_n(\upsilon_1) - f_m(\upsilon_1)) \leq \epsilon\phi(\upsilon_1, 0, 0) \quad \forall \quad n, m \geq n'. \tag{2.18}$$

The last inequality shows that $\{f_n(\upsilon_1)\}$ is a $\rho-$ Cauchy sequence if υ_1 is an arbitrary point in $\mathcal{W}$. Also the sequence $\{f_n(\upsilon_1)\}$ is a $\rho-$convergent in $\mathcal{V}_\rho$, then the function $f_1 : \mathcal{W} \to \mathcal{V}_\rho$ is defined as

$$f_1(\upsilon) = lim_{n\to\infty} f_{1n}(\upsilon_1).$$

Taking $m \to \infty$ in (2.16), we get

$$\tilde{\rho}(f_{1n} - f_1) \leq \epsilon \quad \forall \quad n > n'.$$

Therefore, in $F_{\tilde{\rho}}$ the sequence $\{f_{1n}\}$ is $\tilde{\rho}-$ convergent which implies that $F_{\tilde{\rho}}$ is $\tilde{\rho}-$ complete. Now, considering a linear transformation $T : F_{\tilde{\rho}} \to F_{\tilde{\rho}}$ defined as

$$Tf_1(\upsilon_1) = \frac{1}{8}f_1(2\upsilon_1) \quad \forall \quad f_1 \in F_{\tilde{\rho}}.$$

Let $\lambda \in [0, \infty)$ be any constant and $f_1, f_2 \in F_{\tilde{\rho}}$ with $\tilde{\rho}(f_1 - f_2) \leq \lambda$ and by the definition of modular $\tilde{\rho}$, we have

$$\rho(f_1(\upsilon_1) - f_2(\upsilon_1)) \leq \lambda\phi(\upsilon_1, 0, 0).$$

Therefore, by the above result and assumption, we obtain

$$\begin{aligned}\rho(\frac{f_1(2\upsilon_1)}{8}-\frac{f_2(2\upsilon_1)}{8}) &\leq \frac{1}{8}\rho(f_1(2\upsilon_1)-f_2(2\upsilon_1))\\ &\leq \frac{1}{8}\lambda\phi(2\upsilon_1,0,0)\leq\lambda L\phi(\upsilon_1,0,0).\end{aligned}$$

Hence,

$$\tilde{\rho}(Tf_1-Tf_2)\leq L\tilde{\rho}(f_1-f_2),$$

this shows that mapping T is $\tilde{\rho}-$ strict contraction. Now, we shall prove that the mapping T satisfies the condition of [[16]theorem 3.4].
Exchange υ_1 by $2\upsilon_1$ in (2.15)

$$\rho\bigg(\frac{f(2^2\upsilon_1)}{8}-f(2\upsilon_1)\bigg)\leq\frac{1}{8}\phi(2\upsilon_1,0,0). \tag{2.19}$$

Now,

$$\begin{aligned}&\rho\bigg(\frac{f(2^2\upsilon_1)}{8^2}-f(\upsilon_1)\bigg)=\rho\bigg(\frac{1}{8}\bigg(\frac{f(2^2\upsilon_1)}{8}-8f(\upsilon_1)\bigg)\bigg)\\ &\leq\frac{1}{8}\rho\bigg(\frac{f(2^2\upsilon_1)}{8}-f(2\upsilon_1)+f(2\upsilon_1)-8f(\upsilon_1)\bigg)\\ &\leq\frac{\kappa}{2.8}\bigg[\rho\bigg(\frac{f(2^2\upsilon_1)}{8}-f(2\upsilon_1)\bigg)+\rho\bigg(f(2\upsilon_1)-8f(\upsilon_1)\bigg)\bigg]\\ &\leq\frac{\kappa}{2.8^2}\phi(2\upsilon_1,0,0)+\frac{\kappa}{2.8}\phi(\upsilon_1,0,0)\\ &\leq\frac{\kappa}{2.8^2}\phi(2\upsilon_1,0,0)+\frac{1}{8}\phi(\upsilon_1,0,0).\end{aligned} \tag{2.20}$$

Using mathematical induction and equations (2.15) and (2.18), the authors can conclude that

$$\rho\bigg(\frac{f(2^n\upsilon_1)}{8^n}-f(\upsilon_1)\bigg)\leq\sum_{i=1}^{n}\frac{\kappa^{i-1}}{2^{i-1}8^i}\phi(2^{i-1}\upsilon_1,0,0). \tag{2.21}$$

Now suppose that result (2.19) is true for n and we will show that result is also true for $n+1$. Therefore, we have

$$\begin{aligned}&\rho\bigg(\frac{f(2^{n+1}\upsilon_1)}{8^{n+1}}-f(\upsilon_1)\bigg)=\rho\bigg(\frac{1}{8}\bigg(\frac{f(2^{n+1}\upsilon_1)}{8^n}-8f(\upsilon_1)\bigg)\bigg)\\ &\leq\frac{1}{8}\rho\bigg(\frac{f(2^{n+1}\upsilon_1)}{8^n}-8f(\upsilon_1)\bigg)\\ &\leq\frac{\kappa}{2.8}\bigg[\rho\bigg(\frac{f(2^{n+1}\upsilon_1)}{8^n}-f(2\upsilon_1)\bigg)+\rho\bigg(f(2\upsilon_1)-8f(\upsilon_1)\bigg)\bigg]\\ &\leq\frac{\kappa}{2.8}\bigg[\sum_{i=1}^{n}\frac{\kappa^{i-1}}{2^{i-1}8^i}\phi(2^i\upsilon_1,0,0)+\phi(\upsilon_1,0,0)\bigg]\\ &\leq\sum_{i=0}^{n}\frac{\kappa^i}{2^i8^{i+1}}\phi(2^i\upsilon_1,0,0)=\sum_{i=1}^{n+1}\frac{\kappa^{i-1}}{2^{i-1}8^i}\phi(2^{i-1}\upsilon_1,0,0).\end{aligned}$$

Hence, the result (2.19) is held for natural numbers n. Therefore,

$$\begin{aligned}\rho\left(\frac{f(2^n\upsilon_1)}{8^n}-f(\upsilon_1)\right) &\leq \sum_{i=1}^{n}\frac{\kappa^{i-1}}{2^{i-1}8^i}\phi(2^{i-1}\upsilon_1,0,0)\\ &\leq \frac{1}{8}\sum_{i=1}^{n}L^{i-1}\phi(\upsilon_1,0,0)=\frac{(1-L^n)\phi(\upsilon_1,0,0)}{8(1-L)}\\ &\leq \frac{\phi(\upsilon_1,0,0)}{8(1-L)}. \end{aligned}\tag{2.22}$$

Now, the authors will explain the following condition

$$\delta_{\tilde{\rho}}(f)=\sup\{\tilde{\rho}(T^nf-(T^mf):n,m\in N\}<\infty.$$

By using inequality (2.19), we have

$$\begin{aligned}&\rho\left(\frac{f(2^n\upsilon_1)}{8^n}-\frac{f(2^m\upsilon_1)}{8^m}\right)\\ &=\rho\left(\frac{f(2^n\upsilon_1)}{8^n}-f(\upsilon_1)+f(\upsilon_1)-\frac{f(2^m\upsilon_1)}{8^m}\right)\\ &\leq\frac{\kappa}{2}\left[\rho\left(\frac{f(2^n\upsilon_1)}{8^n}-f(\upsilon_1)\right)+\rho\left(f(\upsilon_1)-\frac{f(2^m\upsilon_1)}{8^m}\right)\right]\\ &\leq\frac{\kappa}{8(1-L)}\phi(\upsilon_1,0,0).\end{aligned}$$

Therefore, we get

$$\tilde{\rho}(T^nf-T^mf)\leq\frac{\kappa}{8(1-L)}.$$

With the help of the definition of $\delta_{\tilde{\rho}}(f)$, we obtain $\delta_{\tilde{\rho}}(f)<\infty$ and we get the $\tilde{\rho}-$ convergence limit of the sequence $\{T^nf\}$ is Q_c by using the [[16]lemma 3.3]. Since Fatou's property is satisfied by ρ, then the inequality (2.20) give $\tilde{\rho}(TQ_c-f)<\infty$. Now, exchange υ_1 through $2^n\upsilon_1$ in equation (2.15)

$$\rho(\frac{f(2^{n+1}\upsilon_1)}{8}-f(2^n\upsilon_1))\leq\frac{1}{8}\phi(2^n\upsilon_1,0,0).$$

Hence,

$$\begin{aligned}\rho\left(\frac{f(2^{n+1}\upsilon_1)}{8^{n+1}}-\frac{f(2^n\upsilon_1)}{8^n}\right) &= \rho\left(\frac{1}{8^n}\left(\frac{f(2^{n+1}\upsilon_1)}{8}-f(2^n\upsilon_1)\right)\right)\\ &\leq \frac{1}{8^n}\rho\left(\frac{f(2^{n+1}\upsilon_1)}{8}-f(2^n\upsilon_1)\right)\\ &\leq \frac{1}{8^{n+1}}\phi(2^n\upsilon_1,0,0)\leq\frac{L^n}{8}\phi(\upsilon_1,0,0)\\ &\leq \phi(\upsilon_1,0,0).\end{aligned}$$

Hence, the authors obtain that $\tilde{\rho}(TQ_c-Q_c)<\infty$. Also using the result [[16]theorem3.4], we proved that $Q_c\in F_{\tilde{\rho}}$ is the fixed point of T. Now, replacing the υ_1,υ_2 and υ_3 by $2^n\upsilon_1,2^n\upsilon_2$ and $2^n\upsilon_3$ respectively and using the property(O3), we get

$$\begin{aligned}&\rho\left(\frac{Df(2^n\upsilon_1,2^n\upsilon_2,2^n\upsilon_3)}{8^n}\right)\leq\frac{1}{8^n}\rho(Df(2^n\upsilon_1,2^n\upsilon_2,2^n\upsilon_3))\\ &\leq\frac{1}{8^n}\phi(2^n\upsilon_1,2^n\upsilon_2,2^n\upsilon_3)\leq L^n\phi(\upsilon_1,\upsilon_2,\upsilon_3).\end{aligned}$$

Consider limit as $n \to \infty$, we obtain $DQ_c(\upsilon_1, \upsilon_2, \upsilon_3) = 0$, here $0 < L < 1$. Now by using (2.20) inequality, we obtain

$$\tilde{\rho}(Q_c - f) \leq \frac{1}{8(1-L)}.$$

Now, we shall show that the mapping Q_c is unique. For this, we consider another mapping Q'_c,

$$\begin{aligned} \tilde{\rho}(Q_c - Q'_c) &= \tilde{\rho}(TQ_c - TQ'_c) \leq \frac{\kappa}{2}[\tilde{\rho}(TQ_c - f) + \tilde{\rho}(TQ'_c - f)] \\ &\leq \frac{\kappa}{8(1-L)} < \infty. \end{aligned}$$

We know T is strict contraction mapping, then we obtain

$$\tilde{\rho}(Q_c - Q'_c) = \tilde{\rho}(TQ_c - TQ'_c) \leq L\tilde{\rho}(Q_c - Q'_c),$$

this shows that $\tilde{\rho}(Q_c - Q'_c) = 0$ which implies $Q_c = Q'_c$. □

Corollary 2.5. *Let us consider a Banach space $(\mathcal{V}, ||.||)$ and an odd mapping $f : \mathcal{W} \to \mathcal{V}$ which satisfies an inequality*

$$||D(f(\upsilon_1, \upsilon_2, \upsilon_3)|| \leq \phi(\upsilon_1, \upsilon_2, \upsilon_3) \tag{2.23}$$

and $f(0) = 0$, where, $\upsilon_1, \upsilon_2, \upsilon_3$ are mutually orthogonal. Define the function $\phi : \mathcal{W}^3 \to [0, \infty)$ satisfies the inequality

$$\phi(2\upsilon_1, 2\upsilon_2, 2\upsilon_3) \leq 8L\phi(\upsilon_1, \upsilon_2, \upsilon_3) \quad \forall \quad \upsilon_1, \upsilon_2, \upsilon_3 \in \mathcal{W}, \tag{2.24}$$

where constant L lies between 0 and 1. Then, there is exactly one and only one cubic mapping $Q_c : \mathcal{W} \to \mathcal{V}$ which satisfied the inequality

$$||f(\upsilon_1) - Q_c(\upsilon_1)|| \leq \frac{1}{8(1-L)}\phi(\upsilon_1, 0, 0). \tag{2.25}$$

Example 2.6. *Consider an orlicz function ρ which satisfies the Δ_2 condition with $0 < \kappa < 2$. Also suppose a mapping $f : \mathcal{W} \to L^\rho$ satisfying the inequality*

$$\int_R \Phi(|Df(\upsilon_1, \upsilon_2, \upsilon_3)|)dm(t) \leq \Phi(\upsilon_1, \upsilon_2, \upsilon_3)$$

where $f(0) = 0$ and $\Phi : R \to [0, \infty)$ is a given function such that

$$\Phi(k\upsilon_1, k\upsilon_2, k\upsilon_3) \leq k^3 L\Phi(\upsilon_1, \upsilon_2, \upsilon_3)$$

for all $\upsilon_1, \upsilon_2, \upsilon_3 \in \mathcal{W}$ and a constant $0 < L < 1$. Then there exists a unique cubic mapping $Q_c : \mathcal{W} \to L^\rho$ such that

$$\int_R \Phi|Q_c(\upsilon_1) - f(\upsilon_1)|dm(t) \leq \frac{1}{2k^3(1-L)}\Phi(\upsilon_1, 0, 0).$$

Here, m denotes the Lebesgue measure in R and L^ρ is an Orlic space.

3. Orthogonal Stability of Cubic Functional Equation in Random Normed Space

Benzaruala et al.[5] established the Ulam stability results for the general linear functional equation in random normed space. Motivated by their insights and research contributions, this section aims to investigate the stability of the given orthogonal cubic functional equation within random normed space by employing the direct and fixed point methods respectively.

Theorem 3.1. *let $\mathcal{W}_1$ be a linear orthogonality space, $(\mathcal{W}_2, \mu^\rho, min)$ be completely random normed space and $(\mathcal{W}_3, \mu'^\rho, min)$ be a random normed space, and define a function $\psi : \mathcal{W}_1 \times \mathcal{W}_1 \times \mathcal{W}_1 \to \mathcal{W}_3$ such that*

$$\mu'^\rho{}_{\psi(2\upsilon_1,0,0)}(t) \geq \mu'^\rho{}_{\alpha\psi(\upsilon_1,0,0)}(t), \quad 0<\alpha<8, t>0, \upsilon_1 \in \mathcal{W}_1 \tag{3.1}$$

and $\lim_{n\to\infty} \mu'^\rho{}_{\psi(2^n\upsilon_1,2^n\upsilon_1,2^n\upsilon_1)} = 1$. If an odd mapping $f : \mathcal{W}_1 \to \mathcal{W}_2$ satisfying $f(0) = 0$ and

$$\mu^\rho{}_\Theta(t) \geq \mu'^\rho{}_{\psi(\upsilon_1,\upsilon_2,\upsilon_3)}(t), \tag{3.2}$$

for all $\upsilon_1, \upsilon_2, \upsilon_3 \in \mathcal{W}_1$ and $\upsilon_1, \upsilon_2, \upsilon_3$ are mutually orthogonal, where

$$\begin{aligned} \Theta &= f(2\upsilon_1+3\upsilon_2+4\upsilon_3) - 3f(\upsilon_1+3\upsilon_2+4\upsilon_3) \\ &- f(-\upsilon_1+3\upsilon_2+4\upsilon_3) - 2f(\upsilon_1+3\upsilon_2) - 2f(\upsilon_1+4\upsilon_3) \\ &+ 6f(\upsilon_1-3\upsilon_2) + 6f(\upsilon_1-4\upsilon_3) + 3f(3\upsilon_2+4\upsilon_3) \\ &- 16[f(\upsilon_1-\frac{3}{2}\upsilon_2) + f(\upsilon_1-2\upsilon_3)] + 18f(\upsilon_1) + 6f(3\upsilon_2) + 6f(4\upsilon_3), \end{aligned}$$

then, there is exactly one and only one cubic mapping $Q_c : \mathcal{W}_1 \to \mathcal{W}_2$ in such a way

$$\mu^\rho{}_{f(\upsilon_1)-Q_c(\upsilon_1)}(t) \geq \mu'^\rho{}_{\psi(\upsilon_1,0,0)}((8-\alpha)t). \tag{3.3}$$

Proof. Taking $(\upsilon_1, \upsilon_2, \upsilon_3) = (\upsilon_1, 0, 0)$ in equation (3.2)

$$\mu^\rho{}_{\left(\frac{f(2\upsilon_1)}{8} - f(\upsilon_1)\right)}(t) \geq \mu'^\rho{}_{\psi(\upsilon_1,0,0)}(8t), \quad \forall \quad \upsilon_1 \in \mathcal{W}_1, t>0, \tag{3.4}$$

substituting $\upsilon_1 = 2^n\upsilon_1$ in equation (3.4), we get

$$\begin{aligned} \mu^\rho{}_{\left(\frac{f(2^{n+1}\upsilon_1)}{8^{n+1}} - \frac{f(2^n\upsilon_1)}{8^n}\right)}(t) &\geq \mu'^\rho{}_{\psi(2^n\upsilon_1,0,0)}(8^{n+1}t) \\ &\geq \mu'^\rho{}_{\psi(\upsilon_1,0,0)}(\frac{8^{n+1}}{\alpha^n}t). \end{aligned} \tag{3.5}$$

Also,

$$\frac{f(2^n\upsilon_1)}{8^n} - f(\upsilon_1) = \sum_{k=0}^{n-1}\left(\frac{f(2^{k+1}\upsilon_1)}{8^{k+1}} - \frac{f(2^k\upsilon_1)}{8^k}\right).$$

Now, using the equation (3.5), we get

$$\begin{aligned} \mu^\rho{}_{\left(\frac{f(2\upsilon_1)}{8} - f(\upsilon_1)\right)}\left(t\sum_{k=0}^{n-1}\frac{\alpha^k}{8^{k+1}}\right) &\geq T_{k=0}^{n-1}\left(\mu'^\rho{}_{\psi(\upsilon_1,0,0)(t)}\right) \\ &= \mu'^\rho{}_{\psi(\upsilon_1,0,0)}(t), \end{aligned} \tag{3.6}$$

therefore,

$$\mu^{\rho}{}_{\left(\frac{f(2^n v_1)}{8^n}-f(v_1)\right)}(t) \geq \mu'^{\rho}{}_{\psi(v_1,0,0)}\left(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{8^{k+1}}}\right). \tag{3.7}$$

In equation(3.7), substituting $v_1 = 2^m v_1$, we get

$$\mu^{\rho}{}_{\left(\frac{f(2^{n+m} v_1)}{8^{n+m}}-\frac{f(2^m v_1)}{8^m}\right)}(t) \geq \mu'^{\rho}{}_{\psi(v_1,0,0)}\left(\frac{t}{\sum_{k=m}^{n+m}\frac{\alpha^k}{8^{k+1}}}\right). \tag{3.8}$$

Since,

$$\lim_{m,n\to\infty} \mu'^{\rho}{}_{\psi(v_1,0,0)}\left(\frac{t}{\sum_{k=m}^{n+m}\frac{\alpha^k}{8^{k+1}}}\right) = 1.$$

Hence, the sequence $\{\frac{f(2^n v_1)}{8^n}\}$ is cauchy in complete random normed space $(\mathcal{W}_2, \mu^{\rho}, min)$ and $Q_c(v_1)$ is the convergence point of the sequence.
Now, for every $\delta > 0$, we find

$$\begin{aligned}
&\mu^{\rho}_{(Q_c(v_1)-f(v_1))}(t+\delta)\\
&\geq T\left(\mu^{\rho}{}_{(Q_c(v_1)-\frac{f(2^n v_1)}{8^n})}(\delta), \mu^{\rho}{}_{(\frac{f(2^n v_1)}{8^n}-f(v_1))}(t)\right) &(3.9)\\
&\geq T\left(\mu^{\rho}{}_{(Q_c(v_1)-\frac{f(2^n v_1)}{8^n})}(\delta), \mu'^{\rho}{}_{\psi(v_1,0,0)}\left(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{8^{k+1}}}\right)\right). &(3.10)
\end{aligned}$$

Putting limit $n \to \infty$, we get

$$\mu^{\rho}_{(Q_c(v_1)-f(v_1))}(t+\delta) \geq \mu'^{\rho}{}_{\psi(v_1,0,0)}(t(8-\alpha)). \tag{3.11}$$

Taking $\delta \to 0$, because δ is arbitrary, we obtain

$$\mu^{\rho}_{(Q_c(v_1)-f(v_1))}(t) \geq \mu'^{\rho}{}_{\psi(v_1,0,0)}(t(8-\alpha)). \tag{3.12}$$

By replacing v_1, v_2, v_3 with $2^n v_1, 2^n v_2, 2^n v_3$ in equation (3.2), we obtain

$$\mu^{\rho}{}_{\Theta_1}(t) \geq \mu'^{\rho}{}_{\psi(2^n v_1,{}^n 2 v_2,2^n v_3)}(8^n t), \tag{3.13}$$

where

$$\begin{aligned}
\Theta_1 &= \frac{f(2^n(2v_1+3v_2+4v_3))}{8^n} - 3\frac{f(2^n(v_1+3v_2+4v_3))}{8^n}\\
&- \frac{f(2^n(-v_1+3v_2+4v_3))}{8^n} - 2\frac{f(2^n(v_1+3v_2))}{8^n} - 2\frac{f(2^n(v_1+4v_3))}{8^n}\\
&+ 6\frac{f(2^n(v_1-3v_2))}{8^n} + 6\frac{f(2^n(v_1-4v_3))}{8^n} + 3\frac{f(2^n(3v_2+4v_3))}{8^n}\\
&- 16[\frac{f(2^n(v_1-\frac{3}{2}v_2))}{8^n} + \frac{f(2^n(v_1-2v_3))}{8^n}]\\
&+ 18\frac{f(2^n v_1)}{8^n} + 6\frac{f(3.2^n v_2)}{8^n} + 6\frac{f(4.2^n v_3)}{8^n}.
\end{aligned}$$

As, $\lim_{n\to\infty} \mu'^{\rho}{}_{\psi(2^n v_1,{}^n 2 v_2,2^n v_3)}(8^n t) = 1$, then we can say that mapping Q_c is cubic. Now, we prove the uniqueness of cubic mapping Q_c, for this, we assume that there is another cubic mapping $Q'_c : \mathcal{W}'_1 \to \mathcal{W}_2$ which fulfills (3.3). Now,

$$\begin{aligned}
&\mu^{\rho}_{(Q_c(\upsilon_1)-Q'_c(\upsilon_1))}(t) = \lim_{n\to\infty}\left\{\mu^{\rho}_{(\frac{Q_c(2^n\upsilon_1)}{8^n}-\frac{Q'_c(2^n\upsilon_1)}{8^n})}(t), \mu^{\rho}_{(\frac{Q_c(2^n\upsilon_1)}{8^n}-\frac{Q'_c(2^n\upsilon_1)}{8^n})}(t)\right\}\\
&\geq \min\left\{\mu^{\rho}_{(\frac{Q_c(2^n\upsilon_1)}{8^n}-\frac{f(2^n\upsilon_1)}{8^n})}(\frac{t}{2}), \mu^{\rho}_{(\frac{Q_c(2^n\upsilon_1)}{8^n}-\frac{f(2^n\upsilon_1)}{8^n})}(\frac{t}{2})\right\}\\
&\geq {\mu'}^{\rho}_{\psi(2^n\upsilon_1,0,0)}(\frac{8^n t}{2}(8-\alpha)) \geq {\mu'}^{\rho}_{\psi(\upsilon_1,0,0)}((\frac{8^n t(8-\alpha)}{2\alpha^n}). \qquad (3.14)
\end{aligned}$$

As, $\lim_{n\to\infty}((\frac{8^n t(8-\alpha)}{2\alpha^n}) = \infty$. Therefore, we obtain ${\mu'}^{\rho}_{\psi(\upsilon_1,0,0)}((\frac{8^n t(8-\alpha)}{2\alpha^n}) = 1$. Hence, $\mu^{\rho}_{(Q_c(\upsilon_1)-Q'_c(\upsilon_1))}(t) = 1$. So, $Q_c(\upsilon_1) = Q'_c(\upsilon_1)$. □

Next, we will prove the results using fixed point method.

Theorem 3.2. *let $\mathcal{W}_1$ be a linear orthogonality space, $(\mathcal{W}_2, \mu^{\rho}, min)$ be completely random normed space and $(\mathcal{W}_3, {\mu'}^{\rho}, min)$ be a random normed space, and define a function $\psi : \mathcal{W}_1 \times \mathcal{W}_1 \times \mathcal{W}_1 \to \mathcal{W}_3$ such that*

$$ {\mu'}^{\rho}_{\psi(2\upsilon_1,0,0)}(t) \geq {\mu'}^{\rho}_{\alpha\psi(\upsilon_1,0,0)}(t), \quad 0<\alpha<8, t>0, \upsilon_1 \in \mathcal{W}_1. \qquad (3.15)$$

If an odd mapping $f : \mathcal{W}_1 \to \mathcal{W}_2$ satisfying $f(0) = 0$ and

$$ {\mu}^{\rho}_{\Theta}(t) \geq {\mu'}^{\rho}_{\psi(\upsilon_1,\upsilon_2,\upsilon_3)}(t), \qquad (3.16)$$

for all $\upsilon_1, \upsilon_2, \upsilon_3 \in \mathcal{W}_1$ and $\upsilon_1, \upsilon_2, \upsilon_3$ are mutually orthogonal, where

$$\begin{aligned}
\Theta &= f(2\upsilon_1 + 3\upsilon_2 + 4\upsilon_3) - 3f(\upsilon_1 + 3\upsilon_2 + 4\upsilon_3)\\
&- f(-\upsilon_1 + 3\upsilon_2 + 4\upsilon_3) - 2f(\upsilon_1 + 3\upsilon_2) - 2f(\upsilon_1 + 4\upsilon_3)\\
&+ 6f(\upsilon_1 - 3\upsilon_2) + 6f(\upsilon_1 - 4\upsilon_3) + 3f(3\upsilon_2 + 4\upsilon_3)\\
&- 16[f(\upsilon_1 - \frac{3}{2}\upsilon_2) + f(\upsilon_1 - 2\upsilon_3)] + 18f(\upsilon_1) + 6f(3\upsilon_2) + 6f(4\upsilon_3),
\end{aligned}$$

then, there is exactly one and only one cubic mapping $Q_c : \mathcal{W}_1 \to \mathcal{W}_2$ in such a way

$$ {\mu}^{\rho}_{f(\upsilon_1)-Q_c(\upsilon_1)}(t) \geq {\mu'}^{\rho}_{\psi(\upsilon_1,0,0)}((8-\alpha)t). \qquad (3.17)$$

Proof. Taking $(\upsilon_1, \upsilon_2, \upsilon_3) = (\upsilon_1, 0, 0)$ in equation (3.16)

$$ \mu^{\rho}_{\left(\frac{f(2\upsilon_1)}{8}-f(\upsilon_1)\right)}(t) \geq {\mu'}^{\rho}_{\psi(\upsilon_1,0,0)}(8t), \quad \forall \quad \upsilon_1 \in \mathcal{W}_1, t>0. \qquad (3.18)$$

Now, assume $\mathcal{H}$ is a set of all odd mappings $h : \mathcal{W}_1 \to \mathcal{W}_2$ having condition $h(0) = 0$. Also, define a generalized metric on $\mathcal{H}$ such that

$$ d(h,k) = \inf\{\beta \in R^+ : {\mu}^{\rho}_{h(\upsilon_1)-k(\upsilon_1)}(\beta t) \geq {\mu'}^{\rho}_{\psi(\upsilon_1,0,0)}(t)\}.$$

By using the result[[20]lemma2.1], we can say that $(\mathcal{H}, d)$ is generalized metric space. Now, a mapping $J : \mathcal{H} \to \mathcal{H}$ is defined as

$$ Jh(\upsilon_1) = \frac{h(2\upsilon_1)}{8}; \quad h \in \mathcal{H}.$$

Let, $f, g \in \mathcal{H}$ is taken such that $d(f,g) < \epsilon$. Hence,

$$\begin{aligned} {\mu^\rho}_{Jg(\upsilon_1)-Jf(\upsilon_1)}\left(\frac{\alpha\beta}{8}t\right) &= {\mu^\rho}_{\frac{g(2\upsilon_1)}{8}-\frac{f(2\upsilon_1)}{8}}\left(\frac{\alpha\beta}{8}t\right) \\ &= {\mu^\rho}_{g(2\upsilon_1)-f(2\upsilon_1)}(\alpha\beta t) \\ &\geq {\mu'^\rho}_{\psi(2\upsilon_1,0,0)}(2t) \geq {\mu'^\rho}_{\psi(\upsilon_1,0,0)}(t). \end{aligned}$$

If $d(f,g) < \epsilon$, then we get $d(Jf, Jg) < \frac{\alpha}{8}\epsilon$. Therefore,

$$d(Jf, Jg) < \frac{\alpha}{8} d(f,g).$$

This implies that mapping J is strictly self-contractive with Lipschtiz constant $\frac{\alpha}{8}$. Now using equation (3.18), we obtain

$${\mu^\rho}_{Jf(\upsilon_1)-f(\upsilon_1)}\left(\frac{1}{8}t\right) \geq {\mu'^\rho}_{\psi(\upsilon_1,0,0)}(t).$$

This implies that $d(Jf', f) \leq \frac{1}{8}$. Now from theorem (1), there exist a unique mapping $Q_c : \mathcal{W}_1 \to \mathcal{W}_2$ such that J has a fixed point as Q_c.
As $m \to \infty$, $d(J^m g, Q_c) \to 0$ which means

$$\lim_{m\to\infty} \frac{f(2^m \upsilon_1)}{8^m} = Q_c \quad \forall \quad \upsilon_1 \in \mathcal{W}_1.$$

Now, using equations (3.15) and (3.16), we get

$$\begin{aligned} {\mu^\rho}_{\frac{Df(2^m\upsilon_1,2^m\upsilon_2,2^m\upsilon_3)}{8}}(t) &\geq {\mu'^\rho}_{\psi(2^m\upsilon_1,2^m\upsilon_2,2^m\upsilon_3)}(8^m t) \\ &= {\mu'^\rho}_{\psi(2^m\upsilon_1,2^m\upsilon_2,2^m\upsilon_3)}(\alpha^m(\frac{8}{\alpha})^m t) \\ &\geq {\mu'^\rho}_{\psi(\upsilon_1,\upsilon_2,\upsilon_3)}((\frac{8}{\alpha})^m t). \end{aligned} \tag{3.19}$$

Taking $m \to \infty$ in equation (3.19), we obtain that ${\mu^\rho}_{DQ_c(\upsilon_1,\upsilon_2,\upsilon_3)}(t) = 1$, this implies $DQ_c(\upsilon_1,\upsilon_2,\upsilon_3) = 0$. Therefore, cubic nature is followed by mapping $Q_c : \mathcal{W}_1 \to \mathcal{W}_2$. As, in set $\mathcal{H}_1 = \{g \in \mathcal{H}, d(f,g) < \infty\}$, J has unique fixed point Q_c where $Q_c : \mathcal{W}_1 \to \mathcal{W}_2$ is unique mapping satisfying

$${\mu^\rho}_{f(\upsilon_1)-Q_c(\upsilon_1)}(\beta t) \geq {\mu'^\rho}_{\psi(\upsilon_1,0,0)}(t).$$

Now, by using the fixed point alternative, we get

$$d(f, Q_c) \leq \frac{1}{1-L} d(f, Jf) \leq \frac{1}{8(1-L)} = \frac{1}{8(1-\frac{\alpha}{8})}.$$

This implies that

$${\mu^\rho}_{f(\upsilon_1)-Q_c(\upsilon_1)}(\frac{1}{8-\alpha}t) \geq {\mu'^\rho}_{\psi(\upsilon_1,0,0)}(t).$$

Therefore,

$${\mu^\rho}_{f(\upsilon_1)-Q_c(\upsilon_1)}(t) \geq {\mu'^\rho}_{\psi(\upsilon_1,0,0)}((8-\alpha)t). \tag{3.20}$$

This completes the proof. □

Corollary 3.3. *let $\mathcal{W}_1$ be a linear space, $(\mathcal{W}_2, \mu^\rho, min)$ be completely random normed space and $(\mathcal{W}_3, \mu'^\rho, min)$ be a random normed space. If an odd mapping $f : \mathcal{W}_1 \to \mathcal{W}_2$ satisfying $f(0) = 0$ and*

$${\mu^\rho}_{\Theta}(t) \geq \frac{t}{t + \xi||\upsilon_0||},$$

where

$$\begin{aligned}\Theta &= f(2\upsilon_1+3\upsilon_2+4\upsilon_3)-3f(\upsilon_1+3\upsilon_2+4\upsilon_3)\\ &- f(-\upsilon_1+3\upsilon_2+4\upsilon_3)-2f(\upsilon_1+3\upsilon_2)-2f(\upsilon_1+4\upsilon_3)\\ &+ 6f(\upsilon_1-3\upsilon_2)+6f(\upsilon_1-4\upsilon_3)+3f(3\upsilon_2+4\upsilon_3)\\ &- 16[f(\upsilon_1-\frac{3}{2}\upsilon_2)+f(\upsilon_1-2\upsilon_3)]+18f(\upsilon_1)+6f(3\upsilon_2)+6f(4\upsilon_3),\end{aligned}$$

then, there is exactly one and only one cubic mapping $Q_c : \mathcal{W}_1 \to \mathcal{W}_2$ *in such a way*

$$\mu^{\rho}{}_{f(\upsilon_1)-Q_c(\upsilon_1)}(t) \geq \frac{(8-\alpha)t}{(8-\alpha)t+\xi||\upsilon_0||}.$$

Theorem 3.4. *let* $\mathcal{W}_1$ *be a linear space,* $(\mathcal{W}_2, \mu^{\rho}, min)$ *be completely random normed space and* $(\mathcal{W}_3, \mu'^{\rho}, min)$ *be a random normed space, and define a function* $\psi : \mathcal{W}_1 \times \mathcal{W}_1 \times \mathcal{W}_1 \to \mathcal{W}_3$ *such that*

$$\mu'^{\rho}{}_{\psi(\frac{\upsilon_1}{2},0,0)}(t) \geq \mu'^{\rho}{}_{\alpha\psi(\upsilon_1,0,0)}(t), \quad \forall \quad \upsilon_1 \in \mathcal{W}_1, t>0. \tag{3.21}$$

If an odd mapping $f : \mathcal{W}_1 \to \mathcal{W}_2$ *satisfying* $f(0) = 0$ *and*

$$\mu^{\rho}{}_{\Theta}(t) \geq \mu'^{\rho}{}_{\psi(\upsilon_1,\upsilon_2,\upsilon_3)}(t), \tag{3.22}$$

where

$$\begin{aligned}\Theta &= f(2\upsilon_1+3\upsilon_2+4\upsilon_3)-3f(\upsilon_1+3\upsilon_2+4\upsilon_3)\\ &- f(-\upsilon_1+3\upsilon_2+4\upsilon_3)-2f(\upsilon_1+3\upsilon_2)-2f(\upsilon_1+4\upsilon_3)\\ &+ 6f(\upsilon_1-3\upsilon_2)+6f(\upsilon_1-4\upsilon_3)+3f(3\upsilon_2+4\upsilon_3)\\ &- 16[f(\upsilon_1-\frac{3}{2}\upsilon_2)+f(\upsilon_1-2\upsilon_3)]+18f(\upsilon_1)+6f(3\upsilon_2)+6f(4\upsilon_3),\end{aligned}$$

then, there is exactly one and only one cubic mapping $Q_c : \mathcal{W}_1 \to \mathcal{W}_2$ *in such a way*

$$\mu^{\rho}{}_{f(\upsilon_1)-Q_c(\upsilon_1)}(t) \geq \mu'^{\rho}{}_{\psi(\upsilon_1,0,0)}((\alpha-8)t). \tag{3.23}$$

Proof. The proof is the same as above. □

4. Results of Experiment

In this section, we discuss the graphs of both the exact solution and the approximate solution for the given equation. It is simple to demonstrate that function $f(\upsilon) = \upsilon^3$ is an exact solution to cubic equation. For experimental purposes, we explored an alternative function $Q_c(\upsilon) = \upsilon^3 + \upsilon^3(log|\upsilon|)$, which differs from a cubic function.These two functions were graphed using matlab, and it was observed that the graphs of both functions, $f(\upsilon)$ and $Q_c(\upsilon)$, coincide at multiple points. This suggests that $Q_c(\upsilon)$ serves as an approximate solution to the given cubic equation. Now, next table shows that the behaviour of the exact solution, approximate solution and difference between their values between -1 and 1. Also, the graphs of the functions $f(\upsilon)$ (in cyan colour) and $Q_c(\upsilon)$ (in red colour) are shown in Figure 1.

Error of Approximation

Values of v	**Exact Solution** $f(v)$	**Approximate Solution** $Q_c(v)$	**Difference** $\lvert f(v)-Q_c(v)\rvert$
-1	-1	-1	0
-0.9	-0.729	-0.706	0.023
-0.8	-0.512	-0.477	0.035
-0.7	-0.343	-0.306	0.037
-0.6	-0.216	-0.182	0.034
-0.5	-0.125	-0.099	0.026
-0.4	-0.064	-0.046	0.018
-0.3	-0.027	-0.017	0.010
-0.2	-0.008	-0.004	-0.004
-0.1	-0.001	-0.0003	0.0002
0.1	-0.001	-0.003	0.0002
0.2	0.008	0.004	0.004
0.3	0.027	0.017	0.010
0.4	0.064	0.046	0.018
0.5	0.125	0.099	0.026
0.6	0.216	0.182	0.034
0.7	0.343	0.306	0.037
0.8	0.512	0.477	0.035
0.9	0.729	0.706	0.023
1	1	1	0

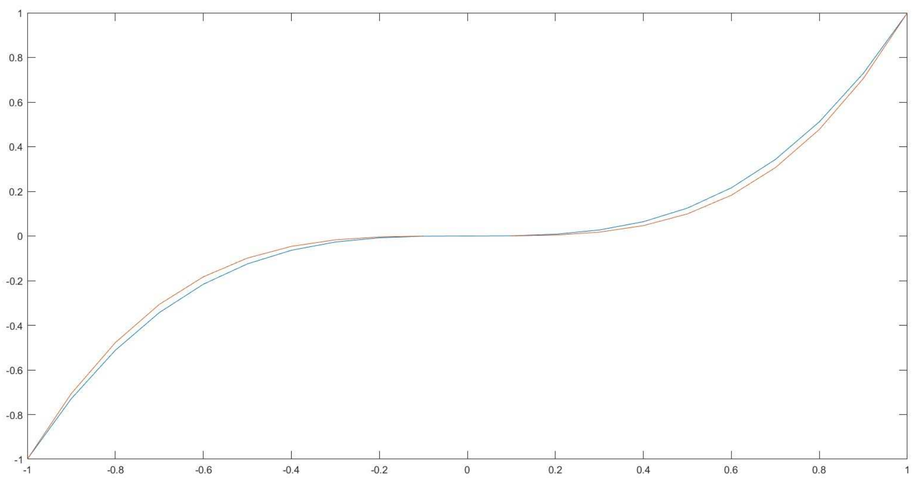

FIGURE 1. Graph of $f(v)$ and $Q_c(v)$

5. Comparative evaluation of the results

In this study, we addressed a new cubic functional equation and established its stability across the spaces, including modular space and random normed space. The main results obtained are summarized as follows:

Corollary No.	**Space setting**	**Stability result**
Corollary 2.3	Modular Space	$\rho(f(v_1) - Q_c(v_1)) \leq \frac{\epsilon}{8-\kappa 2^{p-1}}$
Corollary 3.3	Random Normed Space	$\mu^{\rho}{}_{f(v_1)-Q_c(v_1)}(t) \geq \frac{(8-\alpha)t}{(8-\alpha)t+\xi\|\|v_0\|\|}$

Upon comparing the results presented in the table, it is evident that the approximate solution closely aligns with the exact solution within the framework of Modular space since the upper bound $\frac{\epsilon}{8-\kappa 2^{p-1}}$ is less when compared with upper bound in random normed space. The stability results concerning Hyers-Ulam stability regarding upper bound are obtained in Corollaries 1 and 3.

Conclusion

In conclusion, our research presents a new three-dimensional cubic functional equation, and our focus lies in assessing its orthogonal stability. Using the direct method and fixed point method, we explore stability within modular space and random norm space. Our findings deepen the understanding of the behavior of the equation and provide insights for potential applications in various mathematical contexts, contributing to the field of functional analysis.

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

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