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CONVERGENCE THEOREMS OF FIXED POINT FOR GENERALIZED ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we study the convergence of common fixed point for generalized asymptotically quasi-nonexpansive mappings in real Banach spaces and give the necessary and sufficient condition for convergence of three-step iterative sequence with errors for such maps. The results obtained in this paper extend and improve some recent known results.

1. INTRODUCTION

It is well known that the concept of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [3] who proved that every asymptotically nonexpansive self-mapping of nonempty closed bounded and convex subset of a uniformly convex Banach space has fixed point. In 1973, Petryshyn and Williamson [8] gave necessary and sufficient conditions for Mann iterative sequence (cf. [6]) to converge to fixed points of quasi-nonexpansive mappings. In 1997, Ghosh and Debnath [2] extended the results of Petryshyn and Williamson [8] and gave necessary and sufficient conditions for Ishikawa iterative sequence to converge to fixed points for quasi-nonexpansive mappings.

Qihou [10] extended results of [2, 8] and gave necessary and sufficient conditions for Ishikawa iterative sequence with errors to converge to fixed point of asymptotically quasi-nonexpansive mappings.

In 2003, Zhou et al. [16] introduced a new class of generalized asymptotically nonexpansive mapping and gave a necessary and sufficient condition for the modified Ishikawa and Mann iterative sequences to converge to fixed points for the class of mappings. Atsushiba [1] studied the necessary and sufficient condition for the convergence of iterative sequences to a common fixed point of the finite family of asymptotically nonexpansive mappings in Banach spaces. Suzuki [12], Zeng and Yao [15] discussed a necessary and sufficient condition for common fixed points of two nonexpansive mappings and a finite family of nonexpansive mappings, and

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proved some convergence theorems for approximating a common fixed point, respectively.

Recently, Lan [5] introduced a new class of generalized asymptotically quasi-nonexpansive mappings and gave necessary and sufficient condition for the 2-step modified Ishikawa iterative sequences to converge to fixed points for the class of mappings.

More recently, Nantadilok [7] extend and improve the result of Lan [5] and gave a necessary and sufficient condition for convergence of common fixed point for three-step iterative sequence with errors $\{x_n\}$ for generalized asymptotically quasi-nonexpansive mappings, which was defined as follows:

$$x_1 \in C;$$

$$\begin{aligned} z_n &= (1-c_n)x_n + c_n T_3^n x_n + v_n, \\ y_n &= (1-b_n)x_n + b_n T_2^n z_n + u_n, \\ x_{n+1} &= (1-a_n)x_n + a_n T_1^n y_n + w_n, \qquad n \geq 1, \end{aligned} \tag{1}$$

where $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are sequences in C and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are sequences in $[0,1]$ satisfying some conditions.

The aim of this paper is to obtain a convergence result of three-step iteration scheme with bounded errors for generalized asymptotically quasi-nonexpansive mappings in real Banach spaces which extend and improve some recent known results.

Let X be a normed space, C be a nonempty closed convex subset of X, and $T_i\colon C \to C$, $(i = 1,2,3)$ be three generalized asymptotically quasi-nonexpansive mappings with respect to $\{r_{in}\}$ and $\{s_{in}\}$ with $\sum_{n=1}^{\infty r_n} < \infty$ and $\sum_{n=1}^{\infty s_n} < \infty$ where $r_n = \max\{r_{1n}, r_{2n}, r_{3n}\}$, $s_n = \max\{s_{1n}, s_{2n}, s_{3n}\}$. Define a sequence $\{x_n\}$ in C as follows:

$$x_1 \in C;$$

$$\begin{aligned} z_n &= (1-\gamma_n-\nu_n)x_n + \gamma_n T_3^n x_n + \nu_n u_n, \\ y_n &= (1-\beta_n-\mu_n)x_n + \beta_n T_2^n z_n + \mu_n v_n, \\ x_{n+1} &= (1-\alpha_n-\lambda_n)x_n + \alpha_n T_1^n y_n + \lambda_n w_n, \qquad n \geq 1, \end{aligned} \tag{2}$$

where $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$, $\sum_{n=1}^{\infty} \lambda_n < +\infty$, $\sum_{n=1}^{\infty} \mu_n < +\infty$ and $\sum_{n=1}^{\infty} \nu_n < +\infty$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C.

2. PRELIMINARIES

In the sequel, we need the following definitions and lemmas for our main results in this paper.

Definition 2.1 (See [5]). Let X be a real Banach space, C be a nonempty subset of X and $F(T)$ denotes the set of fixed points of T. A mapping $T\colon C \to C$ is said to be

(1) asymptotically nonexpansive if there exists a sequence $\{r_n\} \subset [0, \infty)$ with $r_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq (1 + r_n)\|x - y\|, \tag{3}$$

(2) asymptotically quasi-nonexpansive if (4) holds for all $x \in C$ and $y \in F(T)$;

(3) generalized quasi-nonexpansive with respect to $\{s_n\}$, if there exists a sequence $\{s_n\} \subset [0, 1)$ with $s_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - p\| \leq \|x - p\| + s_n\|x - T^n x\|, \tag{4}$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$,

(4) generalized asymptotically quasi-nonexpansive with respect to $\{r_n\}$ and $\{s_n\} \subset [0, 1)$ with $r_n \to 0$ and $s_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - p\| \leq (1 + r_n)\|x - p\| + s_n\|x - T^n x\|, \tag{5}$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

Remark 2.2. It is easy to see that,

(i) if $s_n = 0$ for all $n \geq 1$, then the generalized asymptotically quasi-nonexpansive mapping reduces to the usual asymptotically quasi-nonexpansive mapping.

(ii) if $r_n = s_n = 0$ for all $n \geq 1$, then the generalized asymptotically quasi-nonexpansive mapping reduces to the usual quasi-nonexpansive mapping.

(iii) if $r_n = 0$ for all $n \geq 1$, then the generalized asymptotically quasi-nonexpansive mapping reduces to the generalized quasi-nonexpansive mapping.

Lan [5] has shown that the generalized asymptotically quasi-nonexpansive mapping is not a generalized quasi-nonexpansive mapping.

We have the following example shows that a generalized asymptotically quasi-nonexpansive mapping is not a generalized quasi-nonexpansive mapping.

Example [11]. Let $X = \ell_\infty$ with the norm $\|.\|$ defined by

$$\|x\| = \sup_{i \in N} |x_i|, \qquad \forall x = (x_1, x_2, \dots, x_n, \dots) \in X,$$

and $C = \{x = (x_1, x_2, \dots, x_n, \dots) \in X: \ x_i \geq 0, \ x_1 \geq x_i, \ \forall \ i \in N \ and \ x_2 = x_1\}$. Then C is a nonempty subset of X.

Now, for any $x = (x_1, x_2, \dots, x_n, \dots) \in C$, define a mapping $T\colon C \to C$ as follows

$$T(x) = (0, 2x_1, 0, \dots, 0, \dots).$$

It is easy to see that T is a generalized asymptotically quasi-nonexpansive mapping. In fact, for any $x = (x_1, x_2, \dots, x_n, \dots) \in C$, taking $T(x) = x$, that is,

$$(0, 2x_1, 0, \dots, 0, \dots) = (x_1, x_2, \dots, x_n, \dots).$$

Then we have $F(T) = \{0\}$ and $T^n(x) = (0, 0, \dots, 0, \dots)$, $\forall n = 2, 3, \dots.$ for all $r_1, s_1 \in [0, 1)$ with $r_1 + s_1 \geq 1$, we have

$$\begin{aligned}
&\|T(x) - p\| - (1 + r_1)\|x - p\| - s_1\|x - T(x)\| \\
&\quad = \|(0, 2x_1, 0, \dots, 0, \dots)\| - (1+r_1)\|(x_1, x_2, \dots, x_n, \dots)\| - s_1\|(x_1, x_2, \dots, x_n, \dots)\| \\
&\quad = 2x_1 - (1 + r_1)x_1 - s_1 x_1 \leq 0.
\end{aligned}$$

And

$$\begin{aligned}
\|T^n(x) - p\| - (1 + r_n)\|x - p\| - s_n\|x - T^n(x)\| &= 0 - (1 + r_n)x_1 - s_n x_1 \\
&\leq 0.
\end{aligned}$$

For all $n = 2, 3, \dots$, $\{r_n\}$ and $\{s_n\} \subset [0, 1)$ with $r_n \to 0$ and $s_n \to 0$ as $n \to \infty$, and so T is a generalized asymptotically quasi-nonexpansive mapping. However, T is not a generalized quasi-nonexpansive mapping. Since

$$\begin{aligned}
\|T(x) - p\| - \|x - p\| - s_1\|x - T(x)\| &= 2x_1 - x_1 - s_1 x_1 \\
&> 0, \quad \forall s_1 \in [0, 1).
\end{aligned}$$

Lemma 2.3 (See [13]). Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \qquad n \geq 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 (See [5]). Let C be nonempty closed subset of a Banach space X and $T\colon C \to C$ be a generalized asymptotically quasi-nonexpansive mapping with the fixed point set $F(T) \neq \emptyset$. Then $F(T)$ is closed subset in C.

3. MAIN RESULTS

In this section, we establish some strong convergence theorems of three-step iteration scheme with bounded errors for generalized asymptotically quasi-nonexpansive mappings in a real Banach space.

Lemma 3.1. Let X be a real arbitrary Banach space, C be a nonempty closed convex subset of X. Let $T_i\colon C \to C$, $(i = 1, 2, 3)$ be generalized asymptotically quasi-nonexpansive mappings with respect to $\{r_{in}\}$ and $\{s_{in}\}$ such that $\sum_{n=1}^{\infty} \frac{r_n+2s_n}{1-s_n} < \infty$ where $r_n = \max\{r_{1n}, r_{2n}, r_{3n}\}$, $s_n = \max\{s_{1n}, s_{2n}, s_{3n}\}$. Let $\{x_n\}$ be the sequence defined by (2) with the restrictions $\sum_{n=1}^{\infty} \lambda_n < +\infty$, $\sum_{n=1}^{\infty} \mu_n < +\infty$ and $\sum_{n=1}^{\infty} \nu_n < +\infty$. If $\mathcal{F} = \cap_{i=1}^{3} F(T_i) \neq \emptyset$. Then

(**i**) $\|x_{n+1} - p\| \leq (1 + h_n)\|x_n - p\| + \theta_n$,

for all $p \in \mathcal{F}$ and $n \geq 1$, where $h_n = k_n^3 - 1$, $k_n = \frac{1+r_n+s_n}{1-s_n}$ and $\theta_n = M_2(\lambda_n + \mu_n + \nu_n)$ with $\sum_{n=1}^{\infty} h_n < \infty$ and $\sum_{n=1}^{\infty} \theta_n < \infty$.

(**ii**) there exists a constant $M' > 0$ such that

$$\|x_{n+m} - p\| \leq M'\|x_n - p\| + M' \sum_{k=n}^{n+m-1} \theta_k,$$

for all $p \in \mathcal{F}$ and $n, m \geq 1$.

(**iii**) $\lim_{n\to\infty} \|x_n - p\|$ exists.

Proof. Let $p \in \mathcal{F}$, then it follows from (5), we have

$$\begin{aligned}
\|x_n - T_3^n x_n\| &\leq \|x_n - p\| + \|T_3^n x_n - p\| \\
&\leq \|x_n - p\| + (1 + r_{3n})\|x_n - p\| + s_{3n}\|x_n - T_3^n x_n\| \\
&\leq (2 + r_{3n})\|x_n - p\| + s_{3n}\|x_n - T_3^n x_n\| \\
&\leq (2 + r_n)\|x_n - p\| + s_n\|x_n - T_3^n x_n\|
\end{aligned}$$

which implies that

$$\|x_n - T_3^n x_n\| \leq \frac{2 + r_n}{1 - s_n}\|x_n - p\|. \tag{6}$$

Similarly, we have

$$\begin{aligned}
\|y_n - T_1^n y_n\| &\leq \|y_n - p\| + \|T_1^n y_n - p\| \\
&\leq \|y_n - p\| + (1 + r_{1n})\|y_n - p\| + s_{1n}\|y_n - T_1^n y_n\| \\
&\leq (2 + r_{1n})\|y_n - p\| + s_{1n}\|y_n - T_1^n y_n\| \\
&\leq (2 + r_n)\|y_n - p + s_n\|y_n - T_1^n y_n\|
\end{aligned}$$

which implies that

$$\|y_n - T_1^n y_n\| \leq \frac{2 + r_n}{1 - s_n}\|y_n - p\|, \tag{7}$$

and also

$$\|z_n - T_2^n z_n\| \leq \frac{2 + r_n}{1 - s_n}\|z_n - p\|. \tag{8}$$

Since $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C, for any given $p \in \mathcal{F}$, we can set

$$M = \max\{\sup_{n\geq 1}\|u_n - p\|, \sup_{n\geq 1}\|v_n - p\|, \sup_{n\geq 1}\|w_n - p\|\}.$$

It follows from (2), (5) and (6) that

$$\begin{aligned}
\|z_n - p\| &= \|(1-\gamma_n-\nu_n)x_n + \gamma_n T_3^n x_n + \nu_n u_n - p\| \\
&= \|(1-\gamma_n-\nu_n)(x_n - p) + \gamma_n(T_3^n x_n - p) + \nu_n(u_n - p)\| \\
&\leq (1-\gamma_n-\nu_n)\|x_n - p\| + \gamma_n\|T_3^n x_n - p\| + \nu_n\|u_n - p\| \\
&\leq (1-\gamma_n-\nu_n)\|x_n - p\| + \gamma_n[(1+r_{3n})\|x_n - p\| + s_{3n}\|x_n - T_3^n x_n\|] \\
&\quad +\nu_n M \\
&\leq (1-\gamma_n-\nu_n)\|x_n - p\| + \gamma_n[(1+r_n)\|x_n - p\| + s_n\|x_n - T_3^n x_n\|] \\
&\quad +\nu_n M \\
&\leq (1-\gamma_n-\nu_n)\|x_n - p\| + \gamma_n[(1+r_n)\|x_n - p\| + s_n\frac{2+r_n}{1-s_n}\|x_n - p\|] \\
&\quad +\nu_n M \\
&\leq (1-\gamma_n-\nu_n)\|x_n - p\| + \gamma_n(\frac{1+r_n+s_n}{1-s_n})\|x_n - p\| + \nu_n M \\
&\leq \{1 + \gamma_n(\frac{1+r_n+s_n}{1-s_n} - 1) - \nu_n\}\|x_n - p\| + \nu_n M \\
&\leq \{1 + \gamma_n(\frac{1+r_n+s_n}{1-s_n} - 1)\}\|x_n - p\| + \nu_n M \\
&\leq \frac{1+r_n+s_n}{1-s_n}\|x_n - p\| + \nu_n M. \qquad (9)
\end{aligned}$$

Again from (2), (7) and (9), we have

$$\begin{aligned}
\|y_n - p\| &\leq (1-\beta_n-\mu_n)\|x_n - p\| + \beta_n\|T_2^n z_n - p\| + \mu_n\|v_n - p\| \\
&\leq (1-\beta_n-\mu_n)\|x_n - p\| + \beta_n[(1+r_{2n})\|z_n - p\| + s_{2n}\|z_n - T_2^n z_n\|] + \mu_n M \\
&\leq (1-\beta_n-\mu_n)\|x_n - p\| + \beta_n[(1+r_n)\|z_n - p\| + s_n\|z_n - T_2^n z_n\|] + \mu_n M \\
&\leq (1-\beta_n-\mu_n)\|x_n - p\| + \beta_n[(1+r_n)\|z_n - p\| + s_n\frac{2+r_n}{1-s_n}\|z_n - p\|] + \mu_n M \\
&\leq (1-\beta_n-\mu_n)\|x_n - p\| + \beta_n(\frac{1+r_n+s_n}{1-s_n})\|z_n - p\| + \mu_n M \\
&\leq (1-\beta_n-\mu_n)\|x_n - p\| + \beta_n(\frac{1+r_n+s_n}{1-s_n})\{\frac{1+r_n+s_n}{1-s_n}\|x_n - p\| + \nu_n M\} \\
&\quad +\mu_n M
\end{aligned}$$

$$\begin{aligned}
&\leq (1-\beta_n-\mu_n)\|x_n-p\|+\beta_n(\frac{1+r_n+s_n}{1-s_n})^2\|x_n-p\| \\
&\quad +\beta_n(\frac{1+r_n+s_n}{1-s_n})\nu_n M+\mu_n M \\
&\leq [1+\beta_n\{(\frac{1+r_n+s_n}{1-s_n})^2-1\}-\mu_n]\|x_n-p\|+\beta_n(\frac{1+r_n+s_n}{1-s_n})\nu_n M \\
&\quad +\mu_n M \\
&\leq [1+\beta_n\{(\frac{1+r_n+s_n}{1-s_n})^2-1\}]\|x_n-p\|+(\frac{1+r_n+s_n}{1-s_n})\nu_n M \\
&\quad +\mu_n M \\
&\leq (\frac{1+r_n+s_n}{1-s_n})^2\|x_n-p\|+(\mu_n+\nu_n)M(\frac{1+r_n+s_n}{1-s_n}) \\
&\leq (\frac{1+r_n+s_n}{1-s_n})^2\|x_n-p\|+(\mu_n+\nu_n)M_1, \qquad (10)
\end{aligned}$$

where $M_1=\sup_{n\geq 1}\{\frac{1+r_n+s_n}{1-s_n}\}M$ and from (2), (8) and (10), we have

$$\begin{aligned}
\|x_{n+1}-p\| &\leq (1-\alpha_n-\lambda_n)\|x_n-p\|+\alpha_n\|T_1^n y_n-p\|+\lambda_n\|w_n-p\| \\
&\leq (1-\alpha_n-\lambda_n)\|x_n-p\|+\alpha_n[(1+r_{1n})\|y_n-p\|+s_{1n}\|y_n-T_1^n y_n\|+\lambda_n M \\
&\leq (1-\alpha_n-\lambda_n)\|x_n-p\|+\alpha_n[(1+r_n)\|y_n-p\|+s_n\|y_n-T_1^n y_n\|+\lambda_n M \\
&\leq (1-\alpha_n-\lambda_n)\|x_n-p\|+\alpha_n[(1+r_n)\|y_n-p\|+s_n\frac{2+r_n}{1-s_n}\|y_n-p\|]+\lambda_n M \\
&\leq (1-\alpha_n-\lambda_n)\|x_n-p\|+\alpha_n(\frac{1+r_n+s_n}{1-s_n})\|y_n-p\|+\lambda_n M \\
&\leq (1-\alpha_n-\lambda_n)\|x_n-p\|+\alpha_n(\frac{1+r_n+s_n}{1-s_n})\{(\frac{1+r_n+s_n}{1-s_n})^2\|x_n-p\| \\
&\quad +(\mu_n+\nu_n)M_1\}+\lambda_n M \\
&\leq [1+\alpha_n\{(\frac{1+r_n+s_n}{1-s_n})^3-1\}-\lambda_n]\|x_n-p\|+\alpha_n(\frac{1+r_n+s_n}{1-s_n})(\mu_n+\nu_n)M_1 \\
&\quad +\lambda_n M \\
&\leq [1+\alpha_n\{(\frac{1+r_n+s_n}{1-s_n})^3-1\}]\|x_n-p\|+(\frac{1+r_n+s_n}{1-s_n})(\mu_n+\nu_n)M_1 \\
&\quad +(\frac{1+r_n+s_n}{1-s_n})\lambda_n M_1 \\
&\leq (\frac{1+r_n+s_n}{1-s_n})^3\|x_n-p\|+(\lambda_n+\mu_n+\nu_n)(\frac{1+r_n+s_n}{1-s_n})M_1 \\
&\leq (\frac{1+r_n+s_n}{1-s_n})^3\|x_n-p\|+(\lambda_n+\mu_n+\nu_n)M_2 \\
&= (1+h_n)\|x_n-p\|+\theta_n, \qquad (11)
\end{aligned}$$

where $M_2=\sup_{n\geq 1}\{\frac{1+r_n+s_n}{1-s_n}\}M_1$, $h_n=k_n^3-1$, $k_n=\frac{1+r_n+s_n}{1-s_n}$ and $\theta_n=(\lambda_n+\mu_n+\nu_n)M_2$. This completes the prove of part (i).

(ii) If $x\geq 0$ then $1+x\leq e^x$. Thus, from part (i) for any positive integer $m,n\geq 1$, we have

$$
\begin{aligned}
\|x_{n+m}-p\| &\leq (1+h_{n+m-1})\|x_{n+m-1}-p\|+\theta_{n+m-1} \\
&\leq e^{h_{n+m-1}}\|x_{n+m-1}-p\|+\theta_{n+m-1} \\
&\leq e^{h_{n+m-1}}\{e^{h_{n+m-2}}\|x_{n+m-2}-p\|+\theta_{n+m-2}\}+\theta_{n+m-1} \\
&\leq e^{(h_{n+m-1}+h_{n+m-2})}\|x_{n+m-2}-p\|+e^{h_{n+m-1}}\theta_{n+m-2}+\theta_{n+m-1} \\
&\leq e^{(h_{n+m-1}+h_{n+m-2})}\|x_{n+m-2}-p\|+e^{h_{n+m-1}}(\theta_{n+m-1}+\theta_{n+m-2}) \\
&\leq \dots \\
&\leq \dots \\
&\leq e^{(h_{n+m-1}+h_{n+m-2}+\dots+h_n)}\|x_n-p\| \\
&\quad +e^{(h_{n+m-1}+h_{n+m-2}+\dots+h_n)}(\theta_{n+m-1}+\theta_{n+m-2}+\dots+\theta_n) \\
&\leq e^{(\sum_{k=n}^{n+m-1} h_k)}\|x_n-p\|+e^{(\sum_{k=n}^{n+m-1} h_k)}\sum_{k=n}^{n+m-1}\theta_k.
\end{aligned}
$$

Setting $M' = e^{(\sum_{k=n}^{n+m-1} h_k)}$. Hence the above inequality reduces to

$$\|x_{n+m}-p\| \leq M'\|x_n-p\|+M'\sum_{k=n}^{n+m-1}\theta_k.$$

This completes the proof of part (ii).

(iii) From (i), we have

$$\|x_{n+1}-p\| \leq (1+h_n)\|x_n-p\|+\theta_n.$$

where $M_2 = \sup_{n\geq 1}\{\frac{1+r_n+s_n}{1-s_n}\}M_1$, $h_n = k_n^3-1$, $k_n = \frac{1+r_n+s_n}{1-s_n}$ and $\theta_n = (\lambda_n+\mu_n+\nu_n)M_2$. Since $k_n-1 = \frac{r_n+2s_n}{1-s_n}$, the assumption $\sum_{n=1}^{\infty}\frac{r_n+2s_n}{1-s_n} < \infty$ implies that $\lim_{n\to\infty} k_n = 1$. By assumptions, it follows that $\sum_{n=1}^{\infty} h_n < \infty$ and $\sum_{n=1}^{\infty}\theta_n < \infty$. It follows from Lemma 2.3 that the limit $\lim_{n\to\infty}\|x_n-p\|$ exists. This completes the proof of part (iii).

Theorem 3.2. Let X be a real arbitrary Banach space, C be a nonempty closed convex subset of X. Let $T_i\colon C\to C$, $(i=1,2,3)$ be generalized asymptotically quasi-nonexpansive mappings with respect to $\{r_{in}\}$ and $\{s_{in}\}$ such that $\sum_{n=1}^{\infty}\frac{r_n+2s_n}{1-s_n} < \infty$ where $r_n = \max\{r_{1n}, r_{2n}, r_{3n}\}$, $s_n = \max\{s_{1n}, s_{2n}, s_{3n}\}$. Let $\{x_n\}$ be the sequence defined by (2) and some $a, b \in (0,1)$ with the following restrictions:

(i) $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$;

(ii) $\sum_{n=1}^{\infty}\lambda_n < +\infty$, $\sum_{n=1}^{\infty}\mu_n < +\infty$, $\sum_{n=1}^{\infty}\nu_n < +\infty$.

If $\mathcal{F} = \cap_{i=1}^{3} F(T_i) \neq \emptyset$. Then the iterative sequence $\{x_n\}$ converges strongly to a common fixed point p of T_1, T_2 and T_3 if and only if

$$\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0,$$

where $d(x, \mathcal{F})$ denotes the distance between x and the set $\mathcal{F}$.

Proof. The necessity is obvious and it is omitted. Now we prove the sufficiency. From Lemma 3.1 (i) we have

$$\|x_{n+1} - p\| \leq (1 + h_n)\|x_n - p\| + \theta_n, \qquad n \geq 1.$$

Therefore

$$d(x_{n+1}, \mathcal{F}) \leq (1 + h_n)d(x_n, \mathcal{F}) + \theta_n,$$

where $h_n = k_n^3 - 1$, $k_n = \frac{1+r_n+s_n}{1-s_n}$ and $\theta_n = (\lambda_n + \mu_n + \nu_n)M_2$. Since $k_n - 1 = \frac{r_n+2s_n}{1-s_n}$, the assumption $\sum_{n=1}^{\infty} \frac{r_n+2s_n}{1-s_n} < \infty$ implies that $\lim_{n\to\infty} k_n = 1$. By assumptions, it follows that $\sum_{n=1}^{\infty} h_n < \infty$ and $\sum_{n=1}^{\infty} \theta_n < \infty$. By Lemma 2.3 and $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$, we get that $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$.

Next, we prove that $\{x_n\}$ is a Cauchy sequence. From Lemma 3.1 (ii), we have

$$\|x_{n+m} - p\| \leq M'\|x_n - p\| + M' \sum_{k=n}^{n+m-1} \theta_k, \tag{12}$$

for all $p \in \mathcal{F}$ and $n, m \geq 1$. Since $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$ for each $\varepsilon > 0$, there exists a natural number n_1 such that for all $n \geq n_1$,

$$d(x_n, \mathcal{F}) < \frac{\varepsilon}{8M'}, \qquad \sum_{n=n_1}^{\infty} \theta_n < \frac{\varepsilon}{2M'}. \tag{13}$$

By the first inequality of (13), we know that there exists $p_1 \in \mathcal{F}$ such that

$$\|x_{n_1} - p_1\| < \frac{\varepsilon}{4M'}. \tag{14}$$

From (12), (13) and (14), for all $n \geq n_1$, we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p_1\| + \|x_n - p_1\| \\
&\leq [M'\|x_{n_1} - p_1\| + M' \sum_{k=n_1}^{n_1+m-1} \theta_k] + M'\|x_{n_1} - p_1\| \\
&= 2M'\|x_{n_1} - p_1\| + M' \sum_{k=n_1}^{n_1+m-1} \theta_k \\
&< 2M'.\frac{\varepsilon}{4M'} + M'.\frac{\varepsilon}{2M'} = \varepsilon,
\end{aligned}$$

which shows that $\{x_n\}$ is a Cauchy sequence in X. Thus the completeness of X implies that $\{x_n\}$ must be convergent. Let $\lim_{n\to\infty} x_n = p$, that is, $\{x_n\}$ converges to p. Then $p \in C$, because C is a closed subset of X. By Lemma 2.4 we know that the set $\mathcal{F}$ is closed. From the continuity of $d(x_n, \mathcal{F})$ with

$$d(x_n, \mathcal{F}) \to 0 \quad \textit{and} \quad x_n \to p \quad \textit{as} \quad n \to \infty,$$

we get

$$d(p, \mathcal{F}) = 0$$

and so $p \in \mathcal{F} = \cap_{i=1}^{3} F(T_i)$, that is, p is a common fixed point of T_1, T_2 and T_3. This completes the proof.

In Theorem 3.2, if $T_1 = T_2 = T_3 = T$, we obtain the following result:

Theorem 3.3. Let X be a real arbitrary Banach space, C be a nonempty closed convex subset of X. Let $T\colon C \to C$ be generalized asymptotically quasi-nonexpansive mapping with respect to $\{r_n\}$ and $\{s_n\}$ such that $\sum_{n=1}^{\infty} \frac{r_n+2s_n}{1-s_n} < \infty$. Let $\{x_n\}$ be the sequence defined as:

$$x_1 \in C;$$
$$\begin{aligned} z_n &= (1 - \gamma_n - \nu_n)x_n + \gamma_n T^n x_n + \nu_n u_n, \\ y_n &= (1 - \beta_n - \mu_n)x_n + \beta_n T^n z_n + \mu_n v_n, \\ x_{n+1} &= (1 - \alpha_n - \lambda_n)x_n + \alpha_n T^n y_n + \lambda_n w_n, \qquad n \geq 1, \end{aligned}$$

where $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C and some $a, b \in (0, 1)$ with the following restrictions:

(i) $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$;

(ii) $\sum_{n=1}^{\infty} \lambda_n < +\infty$, $\sum_{n=1}^{\infty} \mu_n < +\infty$, $\sum_{n=1}^{\infty} \nu_n < +\infty$.

If $F(T) \neq \emptyset$. Then the iterative sequence $\{x_n\}$ converges strongly to a fixed point p of T if and only if

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

Remark 3.4. Our results extend and improve some recent corresponding results announced by Lan [5] and Nantadilok [7].

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