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CERTAIN CLASSES OF p-VALENT ANALYTIC FUNCTIONS WITH RESPECT TO SYMMETRIC AND CONJUGATE POINTS INVOLVING A CALCULUS OPERATOR

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Abstract. In this paper, we consider classes $TS_s(\alpha,\beta,\nu,\delta,p)$,$TS_c(\alpha,\beta,\nu,\delta,p)$ and $TS_{sc}(\alpha,\beta,\nu,\delta,p)$ of p-valent analytic functions with respect to symmetric, conjugate and symmetric conjugate points respectively. Necessary and sufficient coefficient conditions for functions belonging to these classes are obtained. In addition, coefficient estimates, growth theorems, closure theorems, extreme points, integral transform, radii of starlikeness, convexity are also found for functions belonging to these classes.

1. Introduction

Let $A(p)$ denotes the class of functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \ (p \in N = \{1, 2, 3....\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disk $\mathrm{U} = \{z : z \in C \text{ and } |z| < 1\}$and $T(p)$ denotes the subclass of $A(p)$ whose members are of the form:

$$f(z) = z^p - \sum_{k=1}^{\infty} |a_{p+k}| z^{p+k} \ (p \in N = \{1, 2, 3....\}). \tag{1.2}$$

We denote $A(1) \equiv A$.

Let S_s^* and C_s respectively denote the class of starlike and convex functions $f(z) \in A$ with respect to symmetric points satisfying

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0 \text{ and } \mathrm{Re}\left\{\frac{(zf'(z))'}{(f(z) - f(-z))'}\right\} > 0 \ \ , z \in \mathrm{U} \tag{1.3}$$

respectively which were introduced and studied by Sakaguchi [13] and Das and singh [3] respectively.

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Let S_c^* and S_{sc}^* respectively denote the class of starlike functions $f(z) \in A$ with respect to conjugate and symmetric conjugate points satisfying

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)+\overline{f(\overline{z})}}\right\} > 0 \text{ and } \operatorname{Re}\left\{\frac{zf'(z)}{f(z)-\overline{f(-\overline{z})}}\right\} > 0$$

respectively which were introduced by El-Ashwah and Thomas [5]. The class S_{sc}^* is also studied by Chen et al. [2] (see also [17], [18] and [12]).

We denote by $S_s^*(p)$ and $C_s(p)$ respectively the class of starlike and convex functions $f(z) \in A(p)$ with respect to symmetric points satisfying

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)+(-1)^{-p}f(-z)}\right\} > 0 \text{ and } \operatorname{Re}\left\{\frac{(zf'(z))'}{(f(z)+(-1)^{-p}f(-z))'}\right\} > 0 \qquad (1.4)$$

respectively. Note that $S_s^*(1) \equiv S_s^*$, $C_s(1) \equiv C_s$.

We also denote by $S_c^*(p)$ and $S_{sc}^*(p)$ respectively the class of starlike functions $f(z) \in A(p)$ with respect to conjugate and symmetric conjugate points satisfying

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)+\overline{f(\overline{z})}}\right\} > 0 \text{ and } \operatorname{Re}\left\{\frac{zf'(z)}{f(z)+(-1)^{-p}\overline{f(-\overline{z})}}\right\} > 0$$

respectively. Further we denote by $C_c(p)$ and $C_{sc}(p)$ respectively the class of convex functions $f(z) \in A(p)$ with respect to conjugate and symmetric conjugate points satisfying

$$\operatorname{Re}\left\{\frac{(zf'(z))'}{(f(z)+\overline{f(\overline{z})})'}\right\} > 0 \text{ and } \operatorname{Re}\left\{\frac{(zf'(z))'}{(f(z)+(-1)^{-p}\overline{f(-\overline{z})})'}\right\} > 0$$

respectively.

In this paper, we consider an operator $I^{\delta,\nu}$ for $\nu > -1, \delta \in R$, which is defined for an analytic function $f(z)$ by

$$\begin{aligned} I^{\delta,\nu} f(z) &= \frac{1}{\Gamma(\delta)} \int_0^z (z-t)^{\delta-1} t^{\nu} f(t)\, dt, \ \delta > 0 \\ &= \frac{1}{\Gamma(1+\delta)} \frac{d}{dz} \int_0^z (z-t)^{\delta} t^{\nu} f(t)\, dt, \ -1 < \delta \\ &= \frac{d^{\lambda}}{dz^{\lambda}} I^{\gamma,\nu} f(z), \ \gamma - \lambda = \delta \le -1, \ -1 < \gamma \le 0, \ \lambda \in N. \end{aligned}$$

The multiplicities of $(z-t)^{\delta-1}, (z-t)^{\gamma}$ are removed by considering $\log(z-t)$ to be real if $(z-t) > 0$.

The operator $I^{\delta,\nu} f(z)$ is studied recently by Kim and Srivastava [11], Dziok [4] and the image of z^k under this operator is given by

$$I^{\delta,\nu} z^k = \frac{\Gamma(\nu+1+k)}{\Gamma(\nu+1+\delta+k)} z^{\delta+\nu+k} \qquad (1.5)$$

for positive $\nu + 1 + k > -\delta$.

Definition 1.1. *With the use of (1.5), a normalized operator* $\tilde{I}_p^{\delta,\nu} : A(p) \to A(p)$ *for* $\nu > -1-p$, $\delta + \nu > -1-p$ *is defined by*

$$\tilde{I}_p^{\delta,\nu} f(z) = \frac{\Gamma(\nu+1+\delta+p)}{\Gamma(\nu+1+p)} z^{-\delta-\nu} I_p^{\delta,\nu} f(z). \qquad (1.6)$$

The series expansion of $\tilde{I}_p^{\delta,\nu} f(z)$ is given by

$$\tilde{I}_p^{\delta,\nu} f(z) = z^p + \sum_{k=1}^{\infty} \theta_{p+k}(\delta,\nu) a_{p+k} z^{p+k}, \tag{1.7}$$

where for convenience

$$\theta_{p+k}(\delta,\nu) = \frac{(\nu+1+p)_k}{(\nu+1+\delta+p)_k}. \tag{1.8}$$

The symbol $(\lambda)_k$ is called Pochhammer symbol defined as

$$(\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \lambda(\lambda+1)...(\lambda+k-1) \text{ and } (\lambda)_0 = 1.$$

Note that $\tilde{I}_p^{0,0} f(z) \equiv f(z)$, $\tilde{I}_p^{-1,0} f(z) \equiv \frac{zf'(z)}{p}$.

Motivated with the recent work of Huang et al.[8], Wang et al.[19] and to unify the studies in [14], Sudharsan et al.[16], Khairnar and More [10] and Halim et al.[6], [7] as well to obtain some new results, we define following sub classes of $A(p)$ involving a calculus operator $\tilde{I}_p^{\delta,\nu} f(z)$.

Definition 1.2. *A function $f(z) \in A(p)$ is said to be in $S_s(\alpha,\beta,\nu,\delta,p)$ class if it satisfies for $0 \le \alpha \le 1, 0 < \beta \le 1, \nu > -1-p, \delta+\nu > -1-p$,*

$$\left| \frac{2z(\tilde{I}_p^{\delta,\nu} f(z))'}{\tilde{I}_p^{\delta,\nu} f(z) + (-1)^{-p} \tilde{I}_p^{\delta,\nu} f(-z)} - p \right| < \beta \left| \frac{2\alpha z(\tilde{I}_p^{\delta,\nu} f(z))'}{\tilde{I}_p^{\delta,\nu} f(z) + (-1)^{-p} \tilde{I}_p^{\delta,\nu} f(-z)} + p \right|. \tag{1.9}$$

Definition 1.3. *A function $f(z) \in A(p)$ is said to be in $S_c(\alpha,\beta,\nu,\delta,p)$ class if it satisfies for $0 \le \alpha \le 1, 0 < \beta \le 1, \nu > -1-p, \delta+\nu > -1-p$,*

$$\left| \frac{2z(\tilde{I}_p^{\delta,\nu} f(z))'}{\tilde{I}_p^{\delta,\nu} f(z) + \overline{\tilde{I}_p^{\delta,\nu} f(\overline{z})}} - p \right| < \beta \left| \frac{2\alpha z(\tilde{I}_p^{\delta,\nu} f(z))'}{\tilde{I}_p^{\delta,\nu} f(z) + \overline{\tilde{I}_p^{\delta,\nu} f(\overline{z})}} + p \right|. \tag{1.10}$$

Definition 1.4. *A function $f(z) \in A(p)$ is said to be in $S_{sc}(\alpha,\beta,\nu,\delta,p)$ class if it satisfies for $0 \le \alpha \le 1, 0 < \beta \le 1, \nu > -1-p, \delta+\nu > -1-p$,*

$$\left| \frac{2z(\tilde{I}_p^{\delta,\nu} f(z))'}{\tilde{I}_p^{\delta,\nu} f(z) + (-1)^{-p} \overline{\tilde{I}_p^{\delta,\nu} f(-\overline{z})}} - p \right| < \beta \left| \frac{2\alpha z(\tilde{I}_p^{\delta,\nu} f(z))'}{\tilde{I}_p^{\delta,\nu} f(z) + (-1)^{-p} \overline{\tilde{I}_p^{\delta,\nu} f(-\overline{z})}} + p \right|. \tag{1.11}$$

We denote by $TS_s(\alpha,\beta,\nu,\delta,p) \equiv S_s(\alpha,\beta,\nu,\delta,p) \cap T(p)$, $TS_c(\alpha,\beta,\nu,\delta,p) \equiv S_c(\alpha,\beta,\nu,\delta,p) \cap T(p)$, $TS_{sc}(\alpha,\beta,\nu,\delta,p) \equiv S_{sc}(\alpha,\beta,\nu,\delta,p) \cap T(p)$. We further denote $TS_s(\alpha,\beta,0,0,p) \equiv TS_s^*(\alpha,\beta,p)$ and $TS_s(\alpha,\beta,0,-1,p) \equiv TS_s^c(\alpha,\beta,p)$ which were studied by Sharma and Awasthi [14] for $p = 1$. Also we denote $TS_c(\alpha,\beta,0,0,p) \equiv TS_c^*(\alpha,\beta,p)$ and $TS_c(\alpha,\beta,0,-1,p) \equiv TS_c^c(\alpha,\beta,p)$.

In this paper, we give necessary and sufficient coefficient conditions for functions belonging to the classes $TS_s(\alpha,\beta,\nu,\delta,p)$, $TS_c(\alpha,\beta,\nu,\delta,p)$ and $TS_{sc}(\alpha,\beta,\nu,\delta,p)$. Coefficient estimate, growth theorem, closure theorem, extreme points, integral transform, radii of starlikeness, convexity are also found for functions belonging to these classes.

2. Coefficient Conditions

In this section, we give necessary and sufficient coefficient conditions for functions belonging to $TS_s(\alpha,\beta,\nu,\delta,p), TS_c(\alpha,\beta,\nu,\delta,p)$ and $TS_{sc}(\alpha,\beta,\nu,\delta,p)$ classes.

Theorem 2.1. *Let $f(z)\in T(p)$ of the form (1.2) satisfies*

$$\sum_{k=1}^{\infty}\left[\frac{(p+k)(1+\alpha\beta)}{p\beta(1+\alpha)}+\frac{(\beta-1)\{1+(-1)^k\}}{2\beta(1+\alpha)}\right]\frac{(\nu+1+p)_k}{(\nu+1+\delta+p)_k}|a_{p+k}|\le 1, \quad (2.1)$$

or, equivalently

$$\sum_{k=p+1}^{\infty}\left[\frac{k(1+\alpha\beta)}{p\beta(1+\alpha)}+\frac{(\beta-1)\{1+(-1)^{k-p}\}}{2\beta(1+\alpha)}\right]\frac{(\nu+1+p)_{k-p}}{(\nu+1+\delta+p)_{k-p}}|a_k|\le 1, \quad (2.2)$$

for $0\le\alpha\le 1$, $0<\beta\le 1$, $\nu>-1-p$, $\delta+\nu>-1-p$, if and only if $f(z)\in TS_s(\alpha,\beta,\nu,\delta,p)$.

Proof. Let $f(z)\in TS_s(\alpha,\beta,\nu,\delta,p)$, then from Definition 1.2, we get

$$\left|\frac{2z(\tilde{I}_p^{\delta,\nu}f(z))'}{\tilde{I}_p^{\delta,\nu}f(z)+(-1)^{-p}\tilde{I}_p^{\delta,\nu}f(-z)}-p\right|-\beta\left|\frac{2\alpha z(\tilde{I}_p^{\delta,\nu}f(z))'}{\tilde{I}_p^{\delta,\nu}f(z)+(-1)^{-p}\tilde{I}_p^{\delta,\nu}f(-z)}+p\right|<0$$

or,

$$\begin{aligned}&\Bigg|2pz^p-2\sum_{k=1}^{\infty}(p+k)\theta_{p+k}(\delta,\nu)|a_{p+k}|z^{p+k}-2pz^p\\ &+p\sum_{k=1}^{\infty}\theta_{p+k}(\delta,\nu)\{1+(-1)^k\}|a_{p+k}|z^{p+k}\Bigg|\\ &-\beta\Bigg|2\alpha pz^p-2\alpha\sum_{k=1}^{\infty}(p+k)\theta_{p+k}(\delta,\nu)|a_{p+k}|z^{p+k}\\ &+2pz^p-p\sum_{k=1}^{\infty}\theta_{p+k}(\delta,\nu)\{1+(-1)^k\}|a_{p+k}|z^{p+k}\Bigg|<0.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\sum_{k=1}^{\infty}\left(2(p+k)-p\{1+(-1)^k\}\right)\theta_{p+k}(\delta,\nu)|a_{p+k}||z^{p+k}|-\\ &\beta\left[2p(1+\alpha)|z^p|-\sum_{k=1}^{\infty}2(p+k)\alpha\theta_{p+k}(\delta,\nu)|a_{p+k}||z^{p+k}|-p\sum_{k=1}^{\infty}\theta_{p+k}(\delta,\nu)\{1+(-1)^k\}|a_{p+k}||z^{p+k}|\right]\\ \le\ &0\end{aligned}$$

or,

$$\sum_{k=1}^{\infty}\left(2(p+k)(1+\alpha\beta)+p(\beta-1)\{1+(-1)^k\}\right)\theta_{p+k}(\delta,\nu)|a_{p+k}|\le 2p\beta(1+\alpha)$$

or,

$$\sum_{k=1}^{\infty}\left[\frac{(p+k)(1+\alpha\beta)}{p\beta(1+\alpha)}+\frac{(\beta-1)\{1+(-1)^k\}}{2\beta(1+\alpha)}\right]\theta_{p+k}(\delta,\nu)|a_{p+k}|\le 1.$$

For converse part, the method of Clunie and Keogh [1] is used for $|z|<1$.

Consider,

$$\begin{aligned}
& \left|2z(\tilde{I}_p^{\delta,\nu} f(z))' - p\left(\tilde{I}_p^{\delta,\nu} f(z) + (-1)^{-p}\tilde{I}_p^{\delta,\nu} f(-z)\right)\right| \\
& -\beta\left|2\alpha z(\tilde{I}_p^{\delta,\nu} f(z))' + p\left(\tilde{I}_p^{\delta,\nu} f(z) + (-1)^{-p}\tilde{I}_p^{\delta,\nu} f(-z)\right)\right| \\
= & \left|\sum_{k=1}^{\infty} -\left(2(p+k) - p\{1+(-1)^k\}\right)\theta_{p+k}(\delta,\nu)|a_{p+k}|z^{p+k}\right| - \beta \\
& \left|2p(1+\alpha)z^p - \sum_{k=1}^{\infty} 2(p+k)\alpha\theta_{p+k}(\delta,\nu)|a_{p+k}|z^{p+k} - p\beta\sum_{k=1}^{\infty}\theta_{p+k}(\delta,\nu)\{1+(-1)^k\}|a_{p+k}|z^{p+k}\right| \\
< & \sum_{k=1}^{\infty}\left[2(p+k)(1+\alpha\beta) + p(\beta-1)\{1+(-1)^k\}\right]\theta_{p+k}(\delta,\nu)|a_{p+k}| - 2p\beta(1+\alpha) \leq 0,
\end{aligned}$$

if (2.1) holds. Hence

$$\left|\frac{2z(\tilde{I}_p^{\delta,\nu} f(z))'}{\tilde{I}_p^{\delta,\nu} f(z) + (-1)^{-p}\tilde{I}_p^{\delta,\nu} f(-z)} - p\right| < \beta\left|\frac{2\alpha z(\tilde{I}_p^{\delta,\nu} f(z))'}{\tilde{I}_p^{\delta,\nu} f(z) + (-1)^{-p}\tilde{I}_p^{\delta,\nu} f(-z)} + p\right|.$$

Thus, $f(z) \in TS_s(\alpha,\beta,\nu,\delta,p)$. The result is sharp for the function:

$$f_k(z) = z^p - \frac{2p\beta(1+\alpha)\,(\nu+1+\delta+p)_k}{[2(p+k)(1+\alpha\beta) + p(\beta-1)\{1+(-1)^k\}]\,(\nu+1+p)_k} z^{p+k} \qquad (k \geq 1).$$

□

For $\delta = \nu = 0$ and $\delta = -1, \nu = 0$ respectively in Theorem 2.1, we get following results.

Corollary 2.2. *Let $f(z) \in T(p)$ of the form (1.2) satisfies*

$$\sum_{k=1}^{\infty}\left[\frac{(p+k)(1+\alpha\beta)}{\beta p(1+\alpha)} + \frac{(\beta-1)\{1+(-1)^k\}}{2\beta(1+\alpha)}\right]|a_{p+k}| \leq 1.$$

for $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, if and only if $f(z) \in TS_s^*(\alpha,\beta,p)$.

Corollary 2.3. *Let $f(z) \in T(p)$ of the form (1.2) satisfies*

$$\sum_{k=1}^{\infty}(p+k)\left[\frac{(p+k)(1+\alpha\beta)}{\beta p^2(1+\alpha)} + \frac{(\beta-1)\{1+(-1)^k\}}{2\beta p(1+\alpha)}\right]|a_{p+k}| \leq 1.$$

for $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, if and only if $f(z) \in TS_s^c(\alpha,\beta,p)$.

Corollary 2.4. *Let $f(z) \in TS_s(\alpha,\beta,\nu,\delta,p)$, then for $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$*

$$|a_{p+k}| \leq \frac{2p\beta(1+\alpha)\,(\nu+1+\delta+p)_k}{[2(p+k)(1+\alpha\beta) + p(\beta-1)\{1+(-1)^k\}]\,(\nu+1+p)_k} \quad (k \geq 1).$$

On similar lines of Theorem 2.1, we can easily prove following Theorems for $TS_c(\alpha,\beta,\nu,\delta,p)$ and $TS_{sc}(\alpha,\beta,\nu,\delta,p)$ classes.

Theorem 2.5. *Let $f(z) \in T(p)$ of the form (1.2) satisfies*

$$\sum_{k=1}^{\infty}[k(1+\alpha\beta)+p\beta(1+\alpha)]\frac{(\nu+1+p)_k}{(\nu+1+\delta+p)_k}|a_{p+k}| \leq p\beta(1+\alpha). \tag{2.3}$$

for $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, $\nu > -1-p$, $\delta+\nu > -1-p$, if and only if $f(z) \in TS_c(\alpha,\beta,\nu,\delta,p)$.

Theorem 2.6. *Let $f(z) \in T(p)$ of the form (1.2) satisfies (2.1) for $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, $\nu > -1-p$, $\delta+\nu > -1-p$, if and only if $f(z) \in TS_{sc}(\alpha,\beta,\nu,\delta,p)$.*

3. Coefficient Estimates

In this section, we obtain coefficient estimates for functions to be in $TS_s(\alpha,\beta,\nu,\delta,p), TS_c(\alpha,\beta,\nu,\delta,p)$ and $TS_{sc}(\alpha,\beta,\nu,\delta,p)$ classes.

Theorem 3.1. *Let $f(z) \in TS_s(\alpha,\beta,\nu,\delta,p)$, then for any $\zeta \in \mathbb{C}$,*

$$\begin{aligned}\left|a_{p+2}-\zeta a_{p+1}^2\right| &\leq \frac{(1+\alpha p)\beta(\nu+p+2)}{(p+1)\theta_{p+1}(\delta,\nu)(\nu+\delta+p+2)} \times \\ &\quad max\left[1, \beta\left\{\alpha+\frac{\zeta(1+\alpha p)(\nu+p+2)}{(p+1)\theta_{p+1}(\delta,\nu)(\nu+\delta+p+2)}\right\}\right].\end{aligned} \tag{3.1}$$

The estimate(3.1) is sharp.

Proof. By hypothesis, we write

$$\frac{2z(\tilde{I}_p^{\delta,\nu}f(z))'}{\tilde{I}_p^{\delta,\nu}f(z)+(-1)^{-p}\tilde{I}_p^{\delta,\nu}f(-z)} = \frac{1-\beta w(z)}{1+\alpha\beta w(z)}, \tag{3.2}$$

where $w(z) = \sum_{k=1}^{\infty} w_k z^k$ is a bounded analytic function satisfying the condition $w(0) = 0$ and $|w(z)| < 1$ for $z \in$U. Writing corresponding series expansions in (3.2), we get

$$\begin{aligned}&\left(2p-2\sum_{k=1}^{\infty}(p+k)\theta_{p+k}(\delta,\nu)|a_{p+k}|z^k\right)\left(1+\alpha\beta\sum_{k=1}^{\infty}w_k z^k\right)\\ =\ &\left(2-\sum_{k=1}^{\infty}\theta_{p+k}(\delta,\nu)\{1+(-1)^k\}|a_{p+k}|z^k\right)\left(1-\beta\sum_{k=1}^{\infty}w_k z^k\right).\end{aligned}$$

Equating the coefficients of z and z^2 on both sides, we obtain

$$p\alpha\beta w_1 - (p+1)|a_{p+1}|\theta_{p+1}(\delta,\nu) = -\beta w_1$$

or,

$$|a_{p+1}| = \frac{\beta w_1(1+\alpha p)}{(p+1)\theta_{p+1}(\delta,\nu)} \tag{3.3}$$

and

$$\begin{aligned}&p\alpha\beta w_2 - (p+2)\theta_{p+2}(\delta,\nu)|a_{p+2}| - \alpha\beta^2 w_1^2(1+\alpha p)\\ =\ &-\beta w_2 - \theta_{p+2}(\delta,\nu)|a_{p+2}|\end{aligned}$$

or,

$$|a_{p+2}| = \frac{(1+\alpha p)\beta\left\{w_2-\alpha\beta w_1^2\right\}}{(p+1)\theta_{p+2}(\delta,\nu)}. \tag{3.4}$$

Now, for any complex number ζ, we write

$$\begin{aligned} \left|a_{p+2}-\zeta a_{p+1}^2\right| &= \left|\frac{(1+\alpha p)\beta\left\{w_2-\alpha\beta w_1^2\right\}}{(p+1)\theta_{p+2}(\delta,\nu)}-\zeta\frac{\beta^2 w_1^2(1+\alpha p)^2}{(p+1)^2\theta_{p+1}^2(\delta,\nu)}\right| \qquad (3.5)\\ &= \frac{(1+\alpha p)\beta(\nu+p+2)}{(p+1)\theta_{p+1}(\delta,\nu)(\nu+\delta+p+2)}\times\\ &\quad \left|w_2-\beta\left\{\alpha+\zeta\frac{(1+\alpha p)\beta(\nu+p+2)}{(p+1)\theta_{p+1}(\delta,\nu)(\nu+\delta+p+2)}\right\}w_1^2\right|\\ &= \frac{(1+\alpha p)\beta(\nu+p+2)}{(p+1)\theta_{p+1}(\delta,\nu)(\nu+\delta+p+2)}\left|w_2-\xi w_1^2\right|, \end{aligned}$$

where

$$\xi=\beta\left\{\alpha+\zeta\frac{(1+\alpha p)\beta(\nu+p+2)}{(p+1)\theta_{p+1}(\delta,\nu)(\nu+\delta+p+2)}\right\} \qquad (3.6)$$

It is known from the result of Keogh and Marker [8] that, for every complex number ξ,

$$\left|w_2-\xi w_1^2\right|\leq \max\{1,|\xi|\},$$

and the estimate is sharp for the functions $f_0(z)=z^p$ and $f_1(z)=z^{p+1}$ for $|\xi|\geq 1$ and for $|\xi|<1$ respectively. From (3.5), it follows that

$$\left|a_{p+2}-\zeta a_{p+1}^2\right|\leq\frac{(1+\alpha p)\beta(\nu+p+2)}{(p+1)\theta_{p+1}(\delta,\nu)(\nu+\delta+p+2)}\max\{1,|\xi|\},$$

where ξ is given by (3.6). □

For $\delta=\nu=0$ in Theorem 3.1, we get following result.

Corollary 3.2. *Let $f(z)\in TS_s^*(\alpha,\beta,p)$, then for any $\zeta\in\mathbb{C}$,*

$$\left|a_{p+2}-\zeta a_{p+1}^2\right|\leq\frac{(1+\alpha p)\beta}{(p+1)}max\left[1,\beta\left\{\alpha+\frac{(1+\alpha p)\beta}{(p+1)}\right\}\right] \qquad (3.7)$$

and estimate (3.7) is sharp.

Taking $p=1$ in above corollary, we get the result studied by Sharma and Awasthi [15].

On similar lines of Theorem 3.1, we can easily prove following Theorems for $TS_c(\alpha,\beta,\nu,\delta,p)$ and $TS_{sc}(\alpha,\beta,\nu,\delta,p)$ classes.

Theorem 3.3. *Let $f(z)\in TS_c(\alpha,\beta,\nu,\delta,p)$,then for any $\zeta\in\mathbb{C}$,*

$$\begin{aligned} \left|a_{p+2}-\zeta a_{p+1}^2\right| &\leq \frac{(1+\alpha p)\beta(\nu+p+2)}{(p+1)\theta_{p+1}(\delta,\nu)(\nu+\delta+p+2)}\times \qquad (3.8)\\ &\quad max\left[1,\left|\frac{\beta}{p}\left\{\frac{(1+p\alpha+\alpha)}{(p\alpha+1)}+\zeta\frac{(p+1)(1+\alpha p)\beta(\nu+p+2)}{p\theta_{p+1}(\nu+\delta+p+2)}\right\}\right|\right]. \end{aligned}$$

The estimate (3.8) is sharp.

Theorem 3.4. *Let $f(z)\in TS_{sc}(\alpha,\beta,\nu,\delta,p)$, then for any $\zeta\in\mathbb{C}$, we obtain (3.1) and the estimate(3.1) is sharp.*

4. Growth Theorems

In this section, we obtain growth results of $f(z) \in TS_s(\alpha,\beta,\nu,\delta,p), TS_c(\alpha,\beta,\nu,\delta,p)$ and $TS_{sc}(\alpha,\beta,\nu,\delta,p)$ classes.

Theorem 4.1. *Let $f(z) \in T(p)$ of the form (1.2) be in the class $TS_s(\alpha,\beta,\nu,\delta,p)$, then for $|z| = r < 1$.*

$$r^p - \frac{p\beta(1+\alpha)}{(p+1)(1+\alpha\beta)} r^{p+1} \leq |\tilde{I}_p^{\delta,\nu} f(z)| \leq r^p + \frac{p\beta(1+\alpha)}{(p+1)(1+\alpha\beta)} r^{p+1}. \tag{4.1}$$

Proof. In (2.1), let $[2(p+k)(1+\alpha\beta) - p(1-\beta)\{1+(-1)^k\}] =: A_{p+k}$ for $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, it is noted that $A_{p+k+1} - A_{p+k} > 0$ thus, using (2.1), we get

$$2(p+1)(1+\alpha\beta)\sum_{k=1}^{\infty}\theta_{p+k}(\delta,\nu)|a_{p+k}|$$

$$\leq \sum_{k=1}^{\infty}[2(p+k)(1+\alpha\beta) - p(1-\beta)\{1+(-1)^k\}]\theta_{p+k}(\delta,\nu)|a_{p+k}|$$

$$\leq 2p\beta(1+\alpha)$$

which gives

$$\sum_{k=1}^{\infty}\theta_{p+k}(\delta,\nu)|a_{p+k}| \leq \frac{p\beta(1+\alpha)}{(p+1)(1+\alpha\beta)}.$$

Therefore,

$$|\tilde{I}_p^{\delta,\nu} f(z)| \leq r^p + \frac{p\beta(1+\alpha)}{(p+1)(1+\alpha\beta)} r^{p+1}$$

and

$$|\tilde{I}_p^{\delta,\nu} f(z)| \geq r^p - \frac{p\beta(1+\alpha)}{(p+1)(1+\alpha\beta)} r^{p+1}.$$

This proves (4.1). □

Using Theorem 2.5 and Theorem 2.6, we obtain following results for $TS_c(\alpha,\beta,\nu,\delta,p)$ and $TS_{sc}(\alpha,\beta,\nu,\delta,p)$ classes.

Theorem 4.2. *Let $f(z) \in T(p)$ of the form (1.2) be in the class $TS_c(\alpha,\beta,\nu,\delta,p)$, then for $|z| = r < 1$*

$$r^p - \frac{p\beta(1+\alpha)}{[(1+\alpha\beta)+p\beta(1+\alpha)]} r^{p+1} \leq |\tilde{I}_p^{\delta,\nu} f(z)| \leq r^p + \frac{p\beta(1+\alpha)}{[(1+\alpha\beta)+p\beta(1+\alpha)]} r^{p+1}. \tag{4.2}$$

Theorem 4.3. *Let $f(z) \in T(p)$ of the form (1.2) be in the class $TS_{sc}(\alpha,\beta,\nu,\delta,p)$, then for $|z| = r < 1$.*

$$r^p - \frac{p\beta(1+\alpha)}{(p+1)(1+\alpha\beta)} r^{p+1} \leq |\tilde{I}_p^{\delta,\nu} f(z)| \leq r^p + \frac{p\beta(1+\alpha)}{(p+1)(1+\alpha\beta)} r^{p+1}.$$

5. Extreme Points

Theorem 5.1. *Let* $f_0(z) = z^p$ *and*

$$f_k(z) = z^p - \frac{2p\beta(1+\alpha)}{[2(p+k)(1+\alpha\beta)+(\beta-1)\{1+(-1)^k\}]}\frac{(\nu+1+\delta+p)_k}{(\nu+1+p)_k}z^{p+k}\ (k \geq 1), \tag{5.1}$$

then $f(z) \in TS_s(\alpha, \beta, \nu, \delta, p)$, *if and only if it can be expressed in the form*
$f(z) = \sum_{k=0}^{\infty} \lambda_k\ f_k(z)$, *where* $\lambda_k \geq 0$ *and* $\sum_{k=0}^{\infty} \lambda_k = 1$.

Proof. Suppose

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty}\lambda_k\ f_k(z) \\ &= z^p - \sum_{k=1}^{\infty}\frac{2p\beta(1+\alpha)\lambda_k}{[2(p+k)(1+\alpha\beta)+p(\beta-1)\{1+(-1)^k\}]}\frac{(\nu+1+\delta+p)_k}{(\nu+1+p)_k}z^{p+k}. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k=1}^{\infty}[2(p+k)(1+\alpha\beta)+p(\beta-1)\{1+(-1)^k\}]\frac{(\nu+1+p)_k}{(\nu+1+\delta+p)_k} \\ &\quad\times \frac{2p\beta(1+\alpha)\lambda_k}{[2(p+k)(1+\alpha\beta)+p(\beta-1)\{1+(-1)^k\}]}\frac{(\nu+1+\delta+p)_k}{(\nu+1+p)_k} \\ &= \sum_{k=1}^{\infty}2p\beta(1+\alpha)\lambda_k \leq 2p\beta(1+\alpha)\sum_{k=1}^{\infty}\lambda_k = 2p\beta(1+\alpha)(1-\lambda_0) \\ &\leq 2p\beta(1+\alpha). \end{aligned}$$

Hence, from (2.1), $f(z) \in TS_s(\alpha, \beta, \nu, \delta, p)$. Conversely, suppose that $f(z) \in TS_s(\alpha, \beta, \nu, \delta, p)$. Then from Corollary 2.4, we obtain

$$|a_{p+k}| \leq \frac{2p\beta(1+\alpha)}{[2(p+k)(1+\alpha\beta)+p(\beta-1)\{1+(-1)^k\}]}\frac{(\nu+1+\delta+p)_k}{(\nu+1+p)_k} \quad (k \geq 1).$$

Setting

$$\lambda_k = \frac{[2(p+k)(1+\alpha\beta)+p(\beta-1)\{1+(-1)^k\}]|a_{p+k}|}{2p\beta(1+\alpha)}\frac{(\nu+1+p)_k}{(\nu+1+\delta+p)_k} \quad (k \geq 1)$$

and

$$1 - \sum_{k=1}^{\infty}\lambda_k = \lambda_0.$$

On using (5.1), we get

$$f(z) = \sum_{k=0}^{\infty}\lambda_k\ f_k(z).$$

□

On similar lines of Theorem 5.1, we can easily prove following Theorems for $TS_c(\alpha, \beta, \nu, \delta, p)$ and $TS_{sc}(\alpha, \beta, \nu, \delta, p)$ classes.

Theorem 5.2. *Let* $f_0(z) = z^p$ *and*

$$f_k(z) = z^p - \frac{p\beta(1+\alpha)}{[k(1+\alpha\beta)+p\beta(1+\alpha)]}\frac{(\nu+1+\delta+p)_k}{(\nu+1+p)_k}z^{p+k}\ (k \geq 1), \tag{5.2}$$

then $f(z) \in TS_c(\alpha, \beta, \nu, \delta, p)$ *if and only if it can be expressed in the form* $f(z) = \sum_{k=0}^{\infty}\lambda_k\ f_k(z)$, *where* $\lambda_k \geq 0$ *and* $\sum_{k=0}^{\infty}\lambda_k = 1$.

Theorem 5.3. *Let* $f_0(z) = z^p$ *and*

$$f_k(z) = z^p - \frac{2p\beta(1+\alpha)}{[2(p+k)(1+\alpha\beta)+(\beta-1)\{1+(-1)^k\}]}\frac{(\nu+1+\delta+p)_k}{(\nu+1+p)_k}z^{p+k}\ (k \geq 1),$$

then $f(z) \in TS_{sc}(\alpha, \beta, \nu, \delta, p)$, *if and only if it can be expressed in the form* $f(z) = \sum_{k=0}^{\infty}\lambda_k\ f_k(z)$, *where* $\lambda_k \geq 0$ *and* $\sum_{k=0}^{\infty}\lambda_k = 1$.

6. Closure Theorem for $TS_s(\alpha, \beta, \nu, \delta, p), TS_c(\alpha, \beta, \nu, \delta, p)$ and $TS_{sc}(\alpha, \beta, \nu, \delta, p)$

Theorem 6.1. *Let* $f(z)$ *be of the form (1.2) and* $g(z)$ *given by* $g(z) = z^p - \sum_{k=1}^{\infty}|b_{p+k}|z^{p+k}$ *are in* $TS_s(\alpha, \beta, \nu, \delta, p)$ *class, then* $h(z) = z^p - \frac{1}{2}\sum_{k=1}^{\infty}|a_{p+k}+b_{p+k}|z^{p+k}$ *is also in* $TS_s(\alpha, \beta, \nu, \delta, p)$ *class.*

Proof. Since $f(z)$ and $g(z)$ are in $TS_s(\alpha, \beta, \nu, \delta, p)$, by Theorem 2.1, we get

$$\sum_{k=1}^{\infty}[2(p+k)(1+\alpha\beta)+p(\beta-1)\{1+(-1)^k\}]\theta_{p+k}(\delta, \nu)\ |a_{p+k}| \leq 2p\beta(1+\alpha) \tag{6.1}$$

and

$$\sum_{k=1}^{\infty}[2(p+k)(1+\alpha\beta)+p(\beta-1)\{1+(-1)^k\}]\theta_{p+k}(\delta, \nu)\ |b_{p+k}| \leq 2p\beta(1+\alpha) \tag{6.2}$$

Then, using (6.1) and (6.2), we have

$$\frac{1}{2}\sum_{k=1}^{\infty}[2(p+k)(1+\alpha\beta)+p(\beta-1)\{1+(-1)^k\}]\theta_{p+k}(\delta, \nu)\ |a_{p+k}+b_{p+k}| \leq 2p\beta(1+\alpha),$$

which implies that $h(z) \in TS_s(\alpha, \beta, \nu, \delta, p)$. □

On similar lines of Theorem 6.1, we can easily prove following Theorems for $TS_c(\alpha, \beta, \nu, \delta, p)$ and $TS_{sc}(\alpha, \beta, \nu, \delta, p)$ classes.

Theorem 6.2. *If* $f(z)$ *and* $g(z) = z^p - \sum_{k=1}^{\infty}|b_{p+k}|z^{p+k} \in TS_c(\alpha, \beta, \nu, \delta, p)$,*then* $h(z) = z^p - \frac{1}{2}\sum_{k=1}^{\infty}|a_{p+k}+b_{p+k}|z^{p+k}$ *is also in* $TS_c(\alpha, \beta, \nu, \delta, p)$.

Theorem 6.3. *If* $f(z)$ *and* $g(z) = z^p - \sum_{k=1}^{\infty}|b_{p+k}|z^{p+k} \in TS_{sc}(\alpha, \beta, \nu, \delta, p)$, *then* $h(z) = z^p - \frac{1}{2}\sum_{k=1}^{\infty}|a_{p+k}+b_{p+k}|z^{p+k}$ *is also in* $TS_{sc}(\alpha, \beta, \nu, \delta, p)$.

7. Integral Operator

Definition 7.1. *Let $f \in T(p)$, an integral operator $R_c(f)$ with $c > -p$, is defined by*

$$R_c(f) = \frac{(c+p)}{z^c}\int_0^z t^{c-1} f(t)dt, z \in U. \tag{7.1}$$

Theorem 7.1. *Let the function $f(z)$ of the form (1.2) be in the class $TS_s(\alpha, \beta, \nu, \delta, p)$, then $R_c(f)$ defined by (7.1) be also in the class $TS_s(\alpha, \beta, \nu, \delta, p)$.*

Proof. From (7.1), we get

$$R_c f(z) = z^p - \sum_{k=1}^{\infty} \frac{c+p}{c+p+k} |a_{p+k}| z^{p+k}.$$

Therefore by hypothesis

$$\begin{aligned} & \sum_{k=1}^{\infty} [2(p+k)(1+\alpha\beta) + p(\beta-1)\{1+(-1)^k\}]\theta_{p+k}(\delta,\nu) \frac{c+p}{c+p+k} |a_{p+k}| \\ \leq & \sum_{k=1}^{\infty} [2(p+k)(1+\alpha\beta) + p(\beta-1)\{1+(-1)^k\}]\theta_{p+k}(\delta,\nu) |a_{p+k}| \\ \leq & \ 2p\beta(1+\alpha). \end{aligned}$$

Which by Theorem 2.1 implies that $R_c(f) \in TS_s(\alpha, \beta, \nu, \delta, p)$. □

On similar lines of Theorem 7.1, we can easily prove following Theorems for $TS_c(\alpha, \beta, \nu, \delta, p)$ and $TS_{sc}(\alpha, \beta, \nu, \delta, p)$ classes.

Theorem 7.2. *Let the function $f(z)$ of the form (1.2) be in the class $TS_c(\alpha, \beta, \nu, \delta, p)$, then $R_c(f)$ defined by (7.1) be also in the class $TS_c(\alpha, \beta, \nu, \delta, p)$.*

Theorem 7.3. *Let the function $f(z)$ of the form (1.2) be in the class $TS_{sc}(\alpha, \beta, \nu, \delta, p)$, then $R_c(f)$ defined by (7.1) be also in the class $TS_{sc}(\alpha, \beta, \nu, \delta, p)$.*

8. Radii of starlikeness and Convexity

Theorem 8.1. *Let the function $f(z)$ of the form (1.2) be in the class $TS_s(\alpha, \beta, \nu, \delta, p)$, then $f(z)$ is starlike in the disk $|z| = r_1 < 1$, where*

$$r_1 = \inf_{k\geq 1} \left[\left(\frac{(p+k)(1+\alpha\beta)}{p\beta(1+\alpha)} + \frac{(\beta-1)\{1+(-1)^k\}}{2\beta(1+\alpha)} \right) \frac{p}{(p+k)} \theta_{p+k}(\delta,\nu) \right]^{\frac{1}{k}}. \tag{8.1}$$

Proof. To estabilsh the required result, it is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - p \right| \leq p \quad \text{for } |z| < 1$$

or equivalently,

$$\left| \frac{\sum\limits_{k=1}^{\infty} k |a_{p+k}| z^k}{1 - \sum\limits_{k=1}^{\infty} |a_{p+k}| z^k} \right| \leq p$$

which is equivalent to show that

$$\frac{\sum_{k=1}^{\infty}(p+k)|a_{p+k}||z^k|}{p} \le 1. \tag{8.2}$$

As $f(z) \in TS_s(\alpha,\beta,\nu,\delta,p)$, we have from Theorem 2.1

$$\sum_{k=1}^{\infty}\left[\frac{(p+k)(1+\alpha\beta)}{p\beta(1+\alpha)} + \frac{(\beta-1)\{1+(-1)^k\}}{2\beta(1+\alpha)}\right]\theta_{p+k}(\delta,\nu)|a_{p+k}| \le 1.$$

Thus (8.2) is true, if

$$\frac{(p+k)|z^k|}{p} \le \left[\frac{(p+k)(1+\alpha\beta)}{p\beta(1+\alpha)} + \frac{(\beta-1)\{1+(-1)^k\}}{2\beta(1+\alpha)}\right]\theta_{p+k}(\delta,\nu)$$

or,

$$r_1 = \inf_{k\ge 1}.\left[\left(\frac{(p+k)(1+\alpha\beta)}{p\beta(1+\alpha)} + \frac{(\beta-1)\{1+(-1)^k\}}{2\beta(1+\alpha)}\right)\frac{p}{(p+k)}\theta_{p+k}(\delta,\nu)\right]^{\frac{1}{k}}$$

which is the required result. □

On similar lines of Theorem 8.1, we can easily prove following Theorems for $TS_c(\alpha,\beta,\nu,\delta,p)$ and $TS_{sc}(\alpha,\beta,\nu,\delta,p)$ classes.

Theorem 8.2. *Let the function $f(z)$ of the form (1.2) be in the class $TS_c(\alpha,\beta,\nu,\delta,p)$, then $f(z)$ is starlike in the disk $|z| = r_1 < 1$, where*

$$r_1 = \inf_{k\ge 1}.\left[\left(\frac{k(1+\alpha\beta)+p\beta(1+\alpha)}{p\beta(1+\alpha)}\right)\frac{p}{(p+k)}\theta_{p+k}(\delta,\nu)\right]^{\frac{1}{k}}.$$

Theorem 8.3. *Let the function $f(z)$ of the form (1.2) be in the class $TS_{sc}(\alpha,\beta,\nu,\delta,p)$, then $f(z)$ is starlike in the disk $|z| = r_1 < 1$, where*

$$r_1 = \inf_{k\ge 1}.\left[\left(\frac{(p+k)(1+\alpha\beta)}{p\beta(1+\alpha)} + \frac{(\beta-1)\{1+(-1)^k\}}{2\beta(1+\alpha)}\right)\frac{p}{(p+k)}\theta_{p+k}(\delta,\nu)\right]^{\frac{1}{k}}.$$

Similarly we can proved the following Results.

Theorem 8.4. *Let the function $f(z)$ of the form (1.2) be in the class $TS_s(\alpha,\beta,\nu,\delta,p)$, then $f(z)$ is convex in the disk $|z| = r_2 < 1$, where*

$$r_2 = \inf_{k\ge 1}.\left[\left(\frac{(p+k)(1+\alpha\beta)}{p\beta(1+\alpha)} + \frac{(\beta-1)\{1+(-1)^k\}}{2\beta(1+\alpha)}\right)\frac{p^2}{(p+k)^2}\theta_{p+k}(\delta,\nu)\right]^{\frac{1}{k}}. \tag{8.3}$$

Theorem 8.5. *Let the function $f(z)$ of the form (1.2) be in the class $TS_c(\alpha,\beta,\nu,\delta,p)$, then $f(z)$ is convex in the disk $|z| = r_2 < 1$, where*

$$r_2 = \inf_{k\ge 1}.\left[\left(\frac{k(1+\alpha\beta)+p\beta(1+\alpha)}{p\beta(1+\alpha)}\right)\frac{p^2}{(p+k)^2}\theta_{p+k}(\delta,\nu)\right]^{\frac{1}{k}}. \tag{8.4}$$

Theorem 8.6. *Let the function $f(z)$of the form (1.2) be in the class $TS_{sc}(\alpha,\beta,\nu,\delta,p)$, then $f(z)$ is convex in the disk $|z| = r_2 < 1$, where*

$$r_2 = \inf_{k\ge 1}.\left[\left(\frac{(p+k)(1+\alpha\beta)}{p\beta(1+\alpha)} + \frac{(\beta-1)\{1+(-1)^k\}}{2\beta(1+\alpha)}\right)\frac{p^2}{(p+k)^2}\theta_{p+k}(\delta,\nu)\right]^{\frac{1}{k}}.$$

In order to establish the required results in Theorem 8.4, 8.5 and 8.6, it is sufficient to show that,

$$\left|1 + \frac{zf''(z)}{f'(z)} - p\right| \leq p \quad \text{for } |z| < 1.$$

Remark. *The results of Sharma and Awasthi* [14] *follows by taking* $\delta = \nu = 0, p = 1$ *and* $\delta = -1, \nu = 0, p = 1$ *respectively in Theorems 2.1, 4.1, 5.1, 6.1 and 8.4.*

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