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CERTAIN SUBCLASSES OF STRONGLY STARLIKE AND STRONGLY CONVEX FUNCTIONS DEFINED BY CONVOLUTION

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Abstract. In the present paper certain subclasses of strongly starlike and strongly convex functions defined by convolution with the generalized Hurwitz -Lerch Zeta function are investigated. Some inclusion relations are also mentioned as special cases of our main results.

1. Introduction And Preliminaries

Let $\mathcal{A}$ denote the class of analytic functions $f(z)$ defined in the open unit disk $D = \{ z \in C; \ |z| < 1\}$ by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

If $g \in \mathcal{A}$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

then the Hadamard product (or convolution) $f * g$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

We say that a function $f \in \mathcal{A}$ is starlike of order α and belongs to the class $S^*(\alpha)$,if it satisfies the inequality:

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in D; 0 \le \alpha < 1). \tag{1.2}$$

A function $f \in \mathcal{A}$ is called strongly starlike of order α and $f \in S_s^*(\beta)$, if it satisfies the inequality:

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\beta \quad (z \in D; 0 < \beta \le 1).$$

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The class $\mathcal{K}$ of convex functions of order α, is a subclass of $\mathcal{A}$ where the functions $f \in \mathcal{A}$ satisfy the inequality:

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in D; 0 \leq \alpha < 1). \tag{1.3}$$

We denote by $\mathcal{K}_c(\beta)$ a class of strongly convex functions of order β , if the following inequality holds:

$$\left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi}{2}\beta \quad (z \in D; 0 \leq \alpha < 1).$$

A function $f(z) \in \mathcal{A}$ is called strongly starlike of order β and type α (say $f \in S_s^*(\alpha, \beta)$), if it satisfies the inequality:

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \alpha\right)\right| < \frac{\pi}{2}\beta \quad (z \in D; 0 \leq \alpha < 1, 0 < \beta \leq 1). \tag{1.4}$$

Also, that a function $f(z) \in \mathcal{A}$ is in the class of strongly convex functions of order β and type α (denoted by $f \in \mathcal{K}_c(\alpha, \beta)$), if it satisfies the following inequality:

$$\left|\arg\left(1+\frac{zf''(z)}{f'(z)} - \alpha\right)\right| < \frac{\pi}{2}\beta \quad (z \in D; 0 \leq \alpha < 1, 0 < \beta \leq 1). \tag{1.5}$$

It readily follows that

$$f(z) \in \mathcal{K}_c(\alpha, \beta) \quad \Longleftrightarrow \quad zf'(z) \in S_s^*(\alpha, \beta)$$

and we note that

$$S_s^*(0, \beta) = S_s^*(\beta); \mathcal{K}_c(0, \beta) = \mathcal{K}_c(\beta)$$

and

$$S_s^*(\alpha, 1) = S_s^*(\alpha); \mathcal{K}_c(\alpha, 1) = \mathcal{K}_c(\alpha).$$

Srivastava and Attiya [13] introduced and investigated following family of linear operator which was further studied by Li [6] and Prajapat and Goyal [11]. This operator is defined in terms of the Hadamard product of two analytic functions by

$$J_{\lambda,\mu}f = H_{\lambda,\mu} * f(z) \quad (z \in D; \lambda \in C, \mu \in C\backslash Z_0^-; f \in A), \tag{1.6}$$

where

$$H_{\lambda,\mu} = (1+\mu)^{\lambda}[\varphi(z, \lambda, \mu) - \mu^{-\lambda}] \ (z \in D) \tag{1.7}$$

and ϕ is the generalized Hurwitz-Lerch Zeta function [14] defined by

$$\phi(z, \lambda, \mu) = \sum_{k=0}^{\infty} \frac{z^k}{(\mu+k)^{\lambda}} \tag{1.8}$$

$$(\lambda \in C, \mu \in C/Z_0^- \quad when \quad |z| < 1, \operatorname{Re}(\lambda) > 1 \; when \; |z| = 1).$$

The function $J_{\lambda,\mu}f(z)$ is also in the class $\mathcal{A}$, since by using (1.1), we can write

$$J_{\lambda,\mu}f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+\mu}{k+\mu}\right)^{\lambda} a_k z^k. \tag{1.9}$$

Recently, a further extension of the Srivastava Attiya operator $J_{\lambda,\mu}$ was introduced and investigated by Darus and Al-Shaqsi [3] (see also Xiang et al.[15]) which is defined by

$$J_{\lambda,\mu}^{c,d}f(z) = z + \sum_{K=2}^{\infty} \left(\frac{1+\mu}{k+\mu}\right)^{\lambda} \frac{c!\,(k+d-2)\,!}{(d-2)\,!\,(k+c-1)\,!} a_k z^k. \tag{1.10}$$

$$\left(z \in D, \lambda \in C, \mu \in C\backslash Z_0^-;\ f \in A;\ c > -1 \ and\ \ d > 0\right).$$

It may be noted here that the operator $J^{c,d}_{\lambda,\mu}$ contains the known Choi-Saigo-Srivastava operator [2], the Srivastava-Attiya operator [13], the Owa and Srivastava integral operator [9], the generalized Benardi-Libera-Livingston integral operator, the operator, closely related to the multiplier transformation studied by Flett [4] and Li [6], fractional differintegral operator studied by Patel and Mishra [10] and several other operators (see also [1], [5] and [7]). We now observe some special cases of the operator (1.10) which are given below.

$$J^{1,2}_{0,\mu} f(z) = f(z), \tag{1.11}$$

$$J^{1,2}_{1,0} f(z) = z + \sum_{k=2}^{\infty} \frac{1}{k} a_k z^k = \int_0^{\infty} \frac{f\left(t\right)}{t} dt, \tag{1.12}$$

$$\begin{aligned} J^{1,2}_{1,b} f(z) =&\ z + \sum_{k=2}^{\infty} \left(\tfrac{1+b}{k+b}\right) a_k z^k \\ &= \tfrac{1+b}{z^b} \int_0^z t^{b-1} f(t) dt = F(f)(z),\ b > -1, \end{aligned} \tag{1.13}$$

$$J^{1,2}_{\lambda,1} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{2}{k+1}\right)^{\lambda} a_{\mathrm{k}} z^k = I^{\lambda} f(z), \tag{1.14}$$

where (1.12) and (1.13) are the well known Libera, generalized Benardi- Libera-Livingston integral operators and (1.14) represents the operator closely related to the multiplier transformation studied by Flett [4].
Using (1.10), it is easy to show that

$$z(J^{c+1,d}_{\lambda,\mu} f)'(z) = (c+1) J^{c,d}_{\lambda,\mu} f\left(z\right) - c J^{c+1,d}_{\lambda,\mu} f\left(z\right), \tag{1.15}$$

$$z(J^{c,d}_{\lambda+1,\mu} f)'(z) = (\mu+1) J^{c,d}_{\lambda,\mu} f\left(z\right)\ - \mu J^{c,d}_{\lambda+1,\mu} f\left(z\right), \tag{1.16}$$

$$z(J^{c,d}_{\lambda,\mu} f)'(z) \ \ = d J^{c,d+1}_{\lambda,\mu} f\left(z\right) - \left(d{-}1\right) J^{c,d}_{\lambda,\mu} f\left(z\right). \tag{1.17}$$

Definition 1. We define a subclass of strongly starlike functions $S^*_s\left(\alpha,\beta\right)$ by

$$S^*_s\left(c,d;\lambda,\mu;\alpha,\beta\right) = \left\{ f : f \in A, J^{c,d}_{\lambda,\mu} f(z) \in S^*_s(\alpha,\beta) and \frac{z(J^{c,d}_{\lambda,\mu} f)'(z)}{(J^{c,d}_{\lambda,\mu} f)(z)} \neq \alpha;\ z \in D \right\}. \tag{1.18}$$

Definition 2. We define a subclass of strongly convex functions $\mathcal{K}_c\left(\alpha,\beta\right)$ by

$$\mathcal{K}_c\left(c,d;\lambda,\mu;\alpha,\beta\right) = \left\{ f : f \in A, J^{c,d}_{\lambda,\mu} f(z) \in \mathcal{K}_c(\alpha,\beta)\ and\ \frac{(z(J^{c,d}_{\lambda,\mu} f)')'(z)}{(J^{c,d}_{\lambda,\mu} f)'(z)} \neq \alpha;\ \ z \in D \right\}. \tag{1.19}$$

From the above two definitions, the following relation holds:

$$f\left(z\right) \in \mathcal{K}_c\left(c,d;\lambda,\mu;\alpha,\beta\right) \Longleftrightarrow z f'\left(z\right) \in S^*_s\left(c,d;\lambda,\mu;\alpha,\beta\right). \tag{1.20}$$

Recently, Prajapat, Raina and Srivastava [12] and Prajapat and Goyal [11] have studied some inclusion relations for certain subclasses of strongly starlike and strongly convex functions involving a family of fractional integral and Srivastava Attiya operators respectively. In the present work we shall pursue similar considerations and investigate some inclusion relations for the newly defined subclasses stated in Definitions 1 and 2 above.

To establish our main results, we shall apply the following lemma:
Lemma 1[8] *Let a function* $p(z)$ *be analytic in* D *with* $p(0) = 1$, $p'(0) = 0$ *and* $p(z) \neq 0 \quad (z \in D)$*. If there exists a point* $z_0 \in D$ *such that*

$$|\arg(p(z))| < \frac{\pi}{2}\beta \quad (|z| < |z_0|) \quad \textit{and} \quad |\arg(p(z_0))| = \frac{\pi}{2}\beta \quad (0 < \beta \leq 1),$$

then

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta,$$

where

$$k \geq \frac{1}{2}(a + \frac{1}{a}) \geq 1 \quad \textit{when} \quad \arg(p(z_0)) = \frac{\pi}{2}\beta,$$
$$k \leq -\frac{1}{2}(a + \frac{1}{a}) \leq -1 \quad \textit{when} \quad \arg(p(z_0)) = -\frac{\pi}{2}\beta,$$

and

$$(p(z_0))^{1/\beta} = \pm ia \quad (a > 0).$$

2. Main Results

Our first main result is given as follows:
Theorem 1. *Let* $0 \leq \alpha < 1,\ 0 < \beta\ \leq 1, \lambda \in C$ *and* $\min\{\mu + \alpha,\ c + 1,\ d\} > 0$, *then*

$$S_s^*(c, d; \lambda, \mu\,; \alpha, \beta) \subset S_s^*(c, d; \lambda + 1.\mu\,; \alpha, \beta)\,. \tag{2.1}$$

Proof. Following [12], let us assume that the function f belongs the class $S_s^*(c, d; \lambda, \mu; \alpha, \beta)$ and define a function $p(z)$ by

$$p(z) = \frac{1}{1\text{-}\alpha}\left(\frac{z(J^{c,d}_{\lambda+1,\mu}f)'(z)}{(J^{c,d}_{\lambda+1,\mu}f)(z)} - \alpha\right) \quad (z \in D). \tag{2.2}$$

The function (2.2) is analytic in the unit disk D and $p(0) = 1$. Differentiation and using (1.16), we find that

$$\frac{1}{1\text{-}\alpha}\left(\frac{z(J^{c,d}_{\lambda,\mu}f)'(z)}{(J^{c,d}_{\lambda,\mu}f)(z)} - \alpha\right) = p(z) \ + \frac{zp'(z)}{\mu + \alpha + (1{-}\alpha)p(z)}. \tag{2.3}$$

Hence, the relations (2.2) and (2.3) imply that $p(z) \neq 0$ and $|\arg(p(z))| < \frac{\pi}{2}\beta$ for $0 < \beta \leq 1$ in D. Otherwise, there exists a z_0 in D where the function $p(z)$ satisfies the conditions of Lemma 1 and in the case when $\arg(p(z_0)) = \frac{\pi}{2}\beta$ and $(p(z_0))^{1/\beta} = ia$, we get

$$\begin{aligned}
&\arg\left(\tfrac{1}{(1\text{-}\alpha)}\left(\tfrac{z(J^{c,d}_{\lambda,\mu}f)'(z_0)}{(J^{c,d}_{\lambda,\mu}f)(z_0)} - \alpha\right)\right)\\
&= \arg(p(z_0)) + \arg\left(1 + \tfrac{zp'(z_0)/p(z_0)}{\mu+\alpha+(1\text{-}\alpha)p(z_0)}\right)\\
&= \tfrac{\pi}{2}\beta + \tan^{-1}\left(\tfrac{k\beta(\mu+\alpha+(1\text{-}\alpha)a^\beta \cos\frac{\pi\beta}{2})}{(\mu+\alpha)^2+(1\text{-}\alpha)^2a^{2\beta}+2(\mu+\alpha)(1\text{-}\alpha)a^\beta\cos\frac{\pi\beta}{2}+k\beta(1\text{-}\alpha)a^\beta\sin\frac{\pi\beta}{2}}\right)\\
&\geq \tfrac{\pi}{2}\beta \quad \left(k \geq \tfrac{1}{2}(a + \tfrac{1}{a}) \geq 1 \ \textit{and} \ 0 < \beta \leq 1\right).
\end{aligned} \tag{2.4}$$

But this leads to a contradiction, as $f \in S_s^*(c, d; \lambda, \mu; \alpha, \beta)$.
On the same lines, we can show that when $\arg(p(z_0)) = -\frac{\pi}{2}\beta$ and $(p(z_0))^{1/\beta} = -ia$,

$$\arg\left(\frac{1}{(1\text{-}\alpha)}\left(\frac{z(J^{c,d}_{\lambda,\mu}f)'(z_0)}{(J^{c,d}_{\lambda,\mu}f)(z_0)} - \alpha\right)\right) \ \leq\ -\frac{\pi}{2}\beta \quad \left(k \leq -\frac{1}{2}(a + \frac{1}{a}) \leq -1 \ \textit{and} \quad 0 < \beta \leq 1\right).$$

This is again a contradiction to the fact that $f \in S_s^*(c,d;\lambda,\mu;\alpha,\beta)$. Therefore, $|\arg(p(z))| < \frac{\pi}{2}\beta$ in D and $f \in S_s^*(c,d;\lambda+1,\mu;\alpha,\beta)$, which completes the proof of Theorem 1.

Theorem 2. *Let* $0 \le \alpha < 1,\ 0 < \beta \le 1\ \lambda \in C$ *and* $\min\{\mu+\alpha,\ c+1,\ d\} > 0$*, then*

$$\mathcal{K}_c(c,d;\lambda,\mu;\alpha,\beta) \subset \mathcal{K}_c(c,d;\lambda+1,\mu\,;\alpha,\beta).$$

Proof. In view of Theorem 1 and relation (1.20), we find that

$$\begin{aligned} f(z) \in \mathcal{K}_c(c,d;\lambda,\mu;\alpha,\beta) &\Longrightarrow zf'(z) \in S_s^*(c,d;\lambda,\mu;\alpha,\beta) \\ &\Longrightarrow zf'(z) \in S_s^*(c,d;\lambda+1,\mu;\alpha,\beta) \\ &\Longrightarrow f(z) \in \mathcal{K}_c(c,d;\lambda+1,\mu\,;\alpha,\beta). \end{aligned}$$

This completes the proof.

Remark 1. Above two theorems imply the following inclusions:

$$S_s^*(c,d;\lambda,\mu;\alpha,\beta) \subset S_s^*(c,d;\lambda+1,\mu;\alpha,\beta)\ldots.. \subset S_s^*(c,d;\lambda+n,\mu;\alpha,\beta),$$

$$\mathcal{K}_c(c,d;\lambda,\mu;\alpha,\beta) \subset \mathcal{K}_c(c,d;\lambda+1,\mu;\alpha,\beta)\ldots.. \subset \mathcal{K}_c(c,d;\lambda+n,\mu;\alpha,\beta),$$

for $n \in N$.

Our next result follows by taking into account the relation (1.15) and is given by:

Theorem 3. *Let* $0 \le \alpha < 1, 0 < \beta \ \le 1, \lambda \in C, \mu \in C\backslash Z_0^-$ *and* $\min\{c+\alpha,\ c+1,\ d\} > 0$*, then*

$$S_s^*(c,d;\lambda,\mu;\alpha,\beta) \subset S_s^*(c+1,d;\lambda,\mu;\alpha,\beta).$$

Proof. The above inclusion can easily be proved by applying the relation (1.15) and the method followed in Theorem 1.

Theorem 4. *Let* $0 \le \alpha < 1, 0 < \beta \ \le 1, \lambda \in C, \mu \in C\backslash Z_0^-$ *and* $\min\{c+\alpha,\ c+1,\ d\} > 0$*, then*

$$\mathcal{K}_c(c,d;\lambda,\mu;\alpha,\beta) \subset \mathcal{K}_c(c+1,d;\lambda,\mu;\alpha,\beta).$$

Proof. With the help of (1.15) and Theorem 3, the result given by Theorem 4 can easily be proved by following the proof of Theorem 2.

Next, we prove the following result.

Theorem 5. *Let* $0 \le \alpha < 1, 0 < \beta \ \le 1, \lambda \in C, \mu \in C\backslash Z_0^-$ *and* $\min\{c+1,\ d,\ d+\alpha-1\} > 0$*, then*

$$S_s^*(c,d+1;\lambda,\mu;\alpha,\beta) \subset S_s^*(c,d;\lambda,\mu;\alpha,\beta).$$

Proof. First we set

$$p(z) = \frac{1}{1\text{-}\alpha}\left(\frac{z(J_{\lambda,\mu}^{c,d}f)'(z)}{(J_{\lambda,\mu}^{c,d}f)(z)} - \alpha\right) \quad (z \in D),$$

for $f \in S_s^*(c,d+1;\lambda,\mu;\alpha,\beta)$.

Now using (1.17), we get

$$d+\alpha-1+(1-\alpha)\,p(z) = d\frac{\left(J_{\lambda,\mu}^{c,d+1}f\right)(z)}{\left(J_{\lambda,\mu}^{c,d}f\right)(z)}.$$

Following the steps similar to that of Theorem 1, we can show that $f \in S_s^* (c, d; \lambda, \mu; \alpha, \beta)$, which establishes the desired inclusion relation.

The following results Theorems 6-8 can be established by following the methods given in [11] and [12]. Their proof-details can well be omitted here.

Theorem 6. *Let* $0 \le \alpha < 1, 0 < \beta \le 1, \lambda \in C, \mu \in C \backslash Z_0^-$ *and* $\min\{c+1,\ d,\ d+\alpha-1\} > 0$*, then*

$$\mathcal{K}_c (c, d+1; \lambda, \mu; \alpha, \beta) \subset \mathcal{K}_c (c, d; \lambda, \mu; \alpha, \beta).$$

Theorem 7. *Let* $f \in A,\ 0 \le \alpha < 1, 0 < \beta \le 1, \lambda \in C, \mu \in C \backslash Z_0^-$ *and* $\min\{b+1,\ b+\alpha,\ c+1,\ d\} > 0$*, and also let* $\left(\frac{z(J_{\lambda,\mu}^{c,d}F(f))'(z)}{(J_{\lambda,\mu}^{c,d}F(f))(z)}\right) \neq \alpha\ \ (z \in D)$ *, then*

$$f(z) \in S_s^* (c, d; \lambda, \mu; \alpha, \beta) \Longrightarrow F(f(z)) \in S_s^* (c, d; \lambda, \mu; \alpha, \beta),$$

where the operator F *is defined by* (1.13).

Theorem 8. *Let* $f \in A$ *and let* $\left(\frac{(z(J_{\lambda,\mu}^{c,d}F(f))')'(z)}{(J_{\lambda,\mu}^{c,d}F(f))'(z)}\right) \neq \alpha\ \ (z \in D)$ *under the restrictions to the parameters given in Theorem 7, then*

$$f(z) \in \mathcal{K}_c (c, d; \lambda, \mu; \alpha, \beta) \Longrightarrow F(f(z)) \in \mathcal{K}_c (c, d; \lambda, \mu; \alpha, \beta).$$

3. Special cases of main results

First we note the following consequences of Theorems 1 and 2 when $\mu = 1$.

Corollary 1. *Let* $f \in A$ *and* $z\left(J_{\lambda+1,\mu}^{c,d} I^\gamma (f(z))\right)' \neq \alpha \left(J_{\lambda+1,\mu}^{c,d} I^\gamma (f)\right)(z),\ z \in D$*. If*

$$\left|\arg\left(\frac{z\left(J_{\lambda+\gamma,\mu}^{c,d} I^\gamma f(z)\right)'}{J_{\lambda+\gamma,\mu}^{c,d} I^\gamma f(z)} - \alpha\right)\right| < \frac{\pi}{2}\beta, \quad (0 \le \alpha < 1,\ \ 0 < \beta \le 1, \gamma > 0)$$

then $I^\gamma f(z) \in S_s^* (c, d; \lambda+1, \mu; \alpha, \beta)$*, where* I^γ *is the operator* (1.14).

Corollary 2. *Let* $f \in A$ *and* $\left(z\left(J_{\lambda+1,\mu}^{c,d} I^\gamma (f(z))\right)'\right)' \neq \alpha\left(J_{\lambda+1,\mu}^{c,d} I^\gamma (f)\right)'(z),\ (z \in D)$*. If*

$$\left|\arg\left(\frac{\left(z\left(J_{\lambda+\gamma,\mu}^{c,d} I^\gamma f(z)\right)'\right)'}{\left(J_{\lambda+\gamma,\mu}^{c,d} I^\gamma f(z)\right)'} - \alpha\right)\right| < \frac{\pi}{2}\beta, \quad (0 \le \alpha < 1,\ \ 0 < \beta \le 1, \gamma > 0)$$

then $I^\gamma f(z) \in \mathcal{K}_c (c, d; \lambda+1, \mu; \alpha, \beta)$*, where* I^γ *is the operator* (1.14).

If we put $\lambda = \gamma = 1$ and $c = 1, d = 2$ in the above Corollaries 1 and 2, we obtain Corollaries 2 and 4 of [11] .

Upon setting $\lambda = \mu = 1$, $c = 1, d = 2$, and $\beta = 1$ in Theorems 1 and 2 we obtain the following:

Corollary 3. *Let* $f \in A$ *and* $zf(z) \neq (\alpha+1)\int\limits_0^z f(t)dt;$ $(z \in D)$*. If* $f(z)$ *satisfies following condition* $\operatorname{Re}(zf(z)) \neq (\alpha+1)\operatorname{Re}(\int\limits_0^z f(t)dt) \quad (0 \leq \alpha < 1)$*, then*

$$\frac{4}{z}\int\limits_0^z \frac{1}{u}\int\limits_0^u f(t)dt\,du \in S^*(\alpha), \qquad (z,u \in D).$$

Corollary 4. *Let* $f \in A$ *and* $z^2 f'(z) \neq (\alpha+1)\left(zf(z) - \int\limits_0^z f(t)dt\right)$*;* $z \in D$*. If* $f(z)$ *satisfies the condition,*

$$\operatorname{Re}\left(z^2 f'(z)\right) \neq (\alpha+1)\operatorname{Re}\left((\alpha+1)\left(zf(z) - \int\limits_0^z f(t)dt\right)\right) \quad (0 \leq \alpha < 1),$$

then

$$\frac{4}{z}\int\limits_0^z \frac{1}{u}\int\limits_0^u f(t)dt\,du \in \mathcal{K}(\alpha), \qquad (z,u \in D).$$

Remark 2. If we put $c = 1, d = 2$ Theorems 1, 2 , 7 and 8 would reduce to the corresponding results due to Prajapat and Goyal [11]. Further by setting the parameter $\mu = 1$, the results due to Liu [7] follow for $\lambda > 0$.
If $\lambda = 0$ in our results the corresponding results for Choi-Saigo-Srivastava operator [2] can be easily obtained.

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