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NEW IMPLICIT METHOD FOR GENERAL NONCONVEX VARIATIONAL INEQUALITIES

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Abstract. In this paper, we suggest and analyze a new implicit iterative projection method for solving general nonconvex variational inequalities. We prove that the new implicit method is equivalent to the modified projection method. Using this alternative equivalence between the implicit method and the modified projection, we prove that the convergence of the new implicit method requires only the partially relaxed strongly monotonicity, which is a weaker condition that the monotonicity. We are also discuss several special cases. Our method of proof is very simple.

1. Introduction

General variational inequalities involving two operators were introduced and studied by Noor [8] in 1988. It turned out that a wide class of nonsymmetric and odd-order unrelated problems, which arise in various branches of pure and applied sciences can be studied in the unified framework of general variational inequalities. General variational inequalities can be considered a significant and novel generalization of the variational inequalities, which were introduced and studied by Stampacchia [30] in 1964. For applications, physical formulation, numerical methods and other aspects of variational inequalities, see [1-30] and the references therein. However, all the work carried out in this direction assumed that the underlying set is a convex set. In many practical situations, a choice set may not be a convex set so that the existing results may not be applicable. To handle such situations, Noor [22] has introduced and considered a new class of variational inequalities, called the general nonconvex variational inequality on the uniformly prox-regular sets. It is well-known that uniformly prox-regular sets are nonconvex and include the convex sets as special cases, see [4,5,29]. Using the projection operator, Noor [22] has established the equivalence between the general nonconvex variational inequalities and the fixed point problem. The main of this paper is to suggest and analyze an implicit extragradient method for solving the general nonconvex variational inequalities. It is well-known that the convergence of the extragradient method requires that the operator must be monotone and Lipschitz continuous. It is known that the evaluation of the Lipschitz continuity is itself a very difficult problems, To overcome these drawbacks, several modifications have been suggested, see, for, example, [1-3,

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11,12] and the references therein. The main motivation of this paper is to improve this criteria. We show that the implicit method is equivalent to the modified extragradient method. We use this equivalence between the extragradient method and the implicit method to show that the convergence of the implicit projection method only requires only the partially relaxed strongly monotonicity, which is a weaker condition than monotonicity. It is worth mention that we do not need the Lipschitz continuity of the operator. In this sense, our result represents an improvement and refinement of the known results.

2. Basic Concepts

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$ respectively. Let K be a nonempty closed convex set in H. The basic concepts and definitions used in this paper are exactly the same as in Noor [17]. We now recall some basic concepts and results from nonsmotth analysis [5,29].

Definition 2.1 [5,29]. The proximal normal cone of K at $u \in H$ is given by

$$N_K^P(u) := \{\xi \in H : u \in P_K[u + \alpha\xi]\},$$

where $\alpha > 0$ is a constant and

$$P_K[u] = \{u^* \in K : d_K(u) = \|u - u^*\|\}.$$

Here $d_K(.)$ is the usual distance function to the subset K, that is

$$d_K(u) = \inf_{v \in K} \|v - u\|.$$

The proximal normal cone $N_K^P(u)$ has the following characterization.

Lemma 2.1 [5,29]. Let K be a nonempty, closed and convex exists a constant $\alpha > 0$ such that

$$\langle \zeta, v - u \rangle \le \alpha \|v - u\|^2, \quad \forall v \in K.$$

Definition 2.2. The Clarke normal cone, denoted by $N_K^C(u)$, is defined as

$$N_K^C(u) = \overline{co}[N_K^P(u)],$$

where $\overline{co}$ means the closure of the convex hull. Clearly $N_K^P(u) \subset N_K^C(u)$, but the converse is not true. Note that $N_K^P(u)$ is always closed and convex, whereas $N_K^C(u)$ is convex, but may not be closed (Ref. 29].

Definition 2.3[29]. For a given $r \in (0, \infty]$, a subset K_r is said to be normalized uniformly r-prox-regular if and only if every nonzero proximal normal to K_r can be realized by an r-ball, that is,$\forall u \in K_r$ and $0 \neq \xi \in N_{K_r}^P(u)$, one has

$$\langle (\xi)/\|\xi\|, v - u \rangle \le (1/2r)\|v - u\|^2, \quad \forall v \in K_r.$$

It is clear that the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets; see [5,29]. Obviously, for $r = \infty$, the uniformly prox-regularity of K_r is equivalent to the convexity of K. This class of uniformly prox-regular sets have played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions. It is known that if K_r is a uniformly prox-regular set, then the proximal normal cone $N_{K_r}^P(u)$ is closed as a set-valued mapping.

We now recall the well known proposition which summarizes some important properties of the uniformly prox-regular sets K_r.

Lemma 2.2. Let K be a nonempty closed subset of H, $\quad r \in (0, \infty]$ and set $K_r = \{u \in H : d(u, K) < r\}$. If K_r is uniformly prox-regular, then

(i) $\quad \forall u \in K_r, P_{K_r}(u) \neq \emptyset$.

(ii) $\quad \forall r^{'} \in (0, r)$, P_{K_r} is Lipschitz continuous with constant $\frac{r}{r-r'}$ on $K_{r'}$.

For given nonlinear operators T, g, we consider the problem of finding $u \in H$: $g(u) \in K_r$ such that

$$\langle Tu, g(v) - g(u) \rangle \geq 0, \quad \forall v \in H : g(v) \in K_r, \tag{2.1}$$

which is called the *general nonconvex variational inequality,* introduced and studied by Noor [22].

It is well-known [5,29] that the union of two disjoint intervals [a,b] and [c,d] is a prox-regular set with $r = \frac{c-b}{2}$. We also consider the following simple example to give an idea of the importance of the nonconvex sets.

Example 2.1 [25]. Let $u = \langle x, y \rangle$ and $\langle t, z \rangle$ belong to the real Euclidean plane and consider $Tu = \langle 2x, 2(y-1) \rangle$. Let $K = t^2 + (z-2)^2 \geq 4, \quad -2 \leq t \leq 2, \quad z \geq -2$ be a subset of the Euclidean plane. Then one can easily show that the set K is a prox-regular set K_r.

We note that, if $K_r \equiv K$, the convex set in H, then problem (2.1) is equivalent to finding $u \in H : g(u) \in K$ such that

$$\langle Tu, g(v) - g(u) \rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{2.2}$$

which is known as the general variational inequality, introduced and studied by Noor [8] in 1988. For the applications, numerical methods, formulation and other aspects of the general variational inequalities (2.2), see [1-3,9-22,25] and the references therein.

If $g \equiv I$, the identity operator, then problem (2.1) is equivalent to finding $u \in K_r$ such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall v \in K_r, \tag{2.3}$$

which is called the nonconvex variational inequality. For the formulation and numerical methods for the nonconvex variational inequalities, see Noor [16-18,20-22,25].

We note that, if $K_r \equiv K$, the convex set in H, and $g \equiv I$, the identity operator, then problem (2.1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u \rangle \geq 0, \qquad \forall v \in K. \tag{2.4}$$

Inequality of type (2.4) is called the *variational inequality*, which was introduced and studied by Stampacchia [30] in 1964. It turned out that a number of unrelated obstacle, free, moving, unilateral and equilibrium problems arising in various branches of pure and applied sciences can be studied via variational inequalities, see [1-30] and the references therein.

If K_r is a nonconvex (uniformly prox-regular) set, then problem (1) is equivalent to finding $u \in K_r$ such that

$$0 \in \rho Tu + g(u) - g(u) + \rho N^P_{K_r}(g(u)) \tag{2.5}$$

where $N^P_{K_r}(g(u))$ denotes the normal cone of K_r at $g(u)$ in the sense of nonconvex analysis. Problem (2.5) is called the general nonconvex variational inclusion problem associated with general nonconvex variational inequality (2.1). This equivalent formulation plays a crucial and basic part in this paper. We would like to point out this equivalent formulation allows us to use the projection operator technique for solving the general nonconvex variational inequalities of the type (2.1).

Definition 2.4. An operator $T : H \to H$ with respect to an arbitrary operator g is said to be:

(i) *g-monotone* iff

$$\langle Tu - Tv, g(u) - g(v)\rangle \geq 0, \quad \forall u, v \in H.$$

(ii) *partially relaxed strongly g-pseudomonotone,* iff, there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, g(z) - g(v)\rangle \geq -\alpha\|g(u) - g(z)\|^2, \quad \forall u, v, z \in H.$$

Note that, for $z = u$, the partially relaxed strongly g-monotonicity reduces to g-monotonicity.

3. Main Results

It is known that the general nonconvex variational inequalities (2.1) are equivalent to the fixed point problem, which is the following.

Lemma 3.1[19]. $u \in H : g(u) \in K_r$ is a solution of the general nonconvex variational inequality (2.1) if and only if $u \in K_r$ satisfies the relation

$$g(u) = P_{K_r}[g(u) - \rho Tu], \tag{3.1}$$

where P_{K_r} is the projection of H onto the uniformly prox-regular set K_r.

Lemma 3.1 implies that the general nonconvex variational inequality (2.1) is equivalent to the fixed point problem (3.1). This alternative equivalent formulation is very useful from the numerical and theoretical points of view. Using the fixed point formulation (3.1), we suggest and analyze the following iterative methods for solving the general nonconvex variational inequality (2.1).

Algorithm 3.1. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$g(u_{n+1}) = P_{K_r}[g(u_n) - \rho Tu_n], \quad n = 0, 1, \ldots,$$

which is called the explicit iterative method. For the convergence analysis of Algorithm 3.1, see Noor [19].

Using the idea of Noor [25], we now suggest and analyze the following iterative method for solving the general nonlinear variational inequality 92.1).

Algorithm 3.2. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$g(u_{n+1}) = P_{K_r}[g(u_{n+1}) - \rho Tu_{n+1}], \quad n = 0, 1, \ldots.$$

Algorithm 3.2 is an implicit iterative method for solving the general nonconvex variational inequalities (2.1).

To implement the Algorithm 3.2, we use the predictor-corrector technique. We use Algorithm 3.1 as predictor and Algorithm 3.2 as a corrector to obtain the following predictor-corrector method for solving the general nonconvex variational inequality (2.1).

Algorithm 3.3. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$g(w_n) = P_{K_r}[g(u_n) - \rho T u_n] \tag{3.2}$$
$$g(u_{n+1}) = P_{K_r}[g(w_n) - \rho T w_n], \quad n = 0, 1, \ldots, \tag{3.3}$$

Algorithm 3.3 is known as the modified extragradient method, which was suggested and studied by Noor [22]. We would like to remark that this modified extragradient method is quite different than the extragradient method, which was suggested by Korpelevich [7]. Here we would like to point out that the implicit method (Algorithm 3.2) and the extragradient method (Algorithm 3.3) are equivalent. We use this equivalent to prove the convergence of the implicit projection method (Algorithm 3.2) requires only the partially relaxed strongly monotonicity, which is the main motivation of this paper.

If $K_r \equiv K$, then Algorithm 3.2 reduces to the following algorithm for solving the general variational inequality (2.2), which was suggested and analyzed by Noor [25].

Algorithm 3.4. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$g(u_{n+1}) = P_K[g(u_{n+1}) - \rho T u_{n+1}], \quad n = 0, 1, \ldots,$$

We would like to mention that one can obtain several new and previously known iterative methods for solving the general nonconvex variational inequalities by selecting the appropriate choice of the operators, and subspaces. For more details, see Noor [12,22].

We now consider the convergence analysis of Algorithm 3.3 using the technique of Noor [25] and this is the main motivation of our next result.

Theorem 3.1. Let $u \in H : g(u) \in K_r$ be a solution of (2.1) and let u_{n+1} be the approximate solution obtained from Algorithm 3.3. If the operator T is partially relaxed strongly g-monotone with constant $\alpha > 0$, then

$$\|g(u_{n+1}) - g(u)\|^2 \leq \|g(w_n) - g(u)\|^2 - (1 - 2\alpha\rho)\|g(u_{n+1}) - g(w_n)\|^2 \tag{3.4}$$
$$\|g(w_n) - g(u)\|^2 \leq \|g(u_n) - g(u)\|^2 - (1 - 2\alpha\rho)\|g(w_n) - g(u_n)\|^2. \tag{3.5}$$

Proof. Let $u \in H : g(u) \in K_r$ be solution of (2.1). Then

$$\langle Tu, g(v) - g(u)\rangle \geq 0, \quad \forall v \in H : g(v) \in K_r. \tag{3.6}$$

Take $v = w_n$ in (3.6), we have

$$\langle Tu, g(w_n) - g(u)\rangle \geq 0. \tag{3.7}$$

Using Lemma 3.1, equation (3.2) can be written as

$$\langle \rho T u_n + g(w_n) - g(u_n), g(v) - g(w_n)\rangle \geq 0, \quad \forall v \in H : g(v) \in K_r. \tag{3.8}$$

Taking $v = u$ in (3.8) and using (3.7), we have

$$\begin{aligned} \langle g(w_n) - g(u_n), g(u) - g(w_n)\rangle &\geq \rho\langle T u_n - Tu, g(w_n) - g(u)\rangle \\ &\geq -\alpha\rho\|g(u_n) - g(w_n)\|^2, \end{aligned} \tag{3.9}$$

since T is partially relaxed strongly g-monotone with constant $\alpha > 0$.

From (3.9), we have

$$\|g(u) - g(w_n)\|^2 \leq \|g(u) - g(u_n)\|^2 - (1 - 2\alpha\rho)\|g(w_n) - g(u_n)\|^2,$$

the required result (3.5).

Now taking $v = u_{n+1}$ in (3.6), we have

$$\langle Tu, g(u_{n+1}) - g(u)\rangle \geq 0. \tag{3.10}$$

Using Lemma 3.1, we rewrite (3.3) in the equivalent form as:

$$\langle \rho Tw_n + g(u_{n+1}) - g(w_n), g(v) - g(u_{n+1})\rangle \geq 0, \quad \forall v \in H : g(v) \in K_r. \tag{3.11}$$

Taking $v = u$ in (3.11), we have

$$\langle \rho Tw_n + g(u_{n+1}) - g(w_n), g(u) - g(u_{n+1})\rangle \geq 0. \tag{3.12}$$

From (3.10), (3.12) and using the partially relaxed strongly g-monotonicity of T with constant $\alpha > 0$, we have

$$\|g(u_{n+1}) - g(u)\|^2 \leq \|g(w_n) - g(u)\|^2 - (1 - 2\alpha\rho)\|g(u_{n+1}) - g(w)_n\|^2,$$

the required result (3.4). $\square$

Theorem 3.2. Let $u \in H : g(u) \in K_r$ be a solution of (1) and let u_{n+1} be the approximate solution obtained from Algorithm 3.3. Let H be a finite dimensional space and g^{-1} exist. If $0 < \rho < \frac{1}{2\alpha}$, then $\lim_{n\to\infty} u_n = u$.

Proof. Let $\bar{u} \in H : g(\bar{u}) \in K_r$ be a solution of (3.1). Then, the sequences $\{\|g(u_n) - g(\bar{u})\|\}$ is nonincreasing and bounded and

$$\begin{aligned} \sum_{n=0}^{\infty}(1 - 2\alpha\rho)\|g(u_{n+1}) - g(w_n)\|^2 &\leq \|g(w_0) - g(u)\|^2 \\ \sum_{n=0}^{\infty}(1 - 2\alpha\rho)\|g(w_n) - g(u_n)\|^2 &\leq \|g(u_0) - g(u)\|^2, \end{aligned}$$

which implies

$$\begin{aligned} \lim_{n\to\infty} \|g(u_{n+1}) - g(w_n)\| &= 0 \\ \lim_{n\to\infty} \|g(w_n) - g(u_n)\| &= 0. \end{aligned}$$

Thus

$$\begin{aligned} \lim_{n\to\infty} \|g(u_{n+1}) - g(u_n)\| &= |\lim_{n\to\infty} \|g(u_{n+1}) - g(w_n)\| \\ &\quad + \lim_{n\to\infty} \|g(w_n) - g(u)\| = 0. \end{aligned} \tag{3.13}$$

Since g^{-1} exists, it follows that

$$\lim_{n\to\infty} \|u_{n+1}) - u_n\| = 0.$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$; there exists a subsequence $\{u_{n_i}\}$ such that $\{u_{n_i}\}$ converges to $\hat{u}$. Replacing u_{n+1} by u_{n_i} in (3.8), w_n by u_{n_i} in (3.12) and taking the limits and using (3.13), we have

$$\langle T\hat{u}, g(v) - g(\hat{u})\rangle \geq 0, \quad \forall v \in H : g(v) \in K_r.$$

This shows that $\hat{u} \in K_r$ solves the general nonconvex variational inequality (2.1) and

$$\|g(u_{n+1}) - g(\hat{u})\|^2 \leq \|g(u_n) - g(\hat{u})\|^2,$$

which implies that the sequence $\{u_n\}$ has a unique cluster point and $\lim_{n\to\infty} u_n = \hat{u}$, is the solution of (2.1), the required result. $\Box$.

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