SUMS IN TERMS OF POLYGAMMA FUNCTIONS

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ABSTRACT. We consider sums involving the product of reciprocal binomial coefficient and polynomial terms and develop some double integral identities. In particular cases it is possible to express the sums in closed form, give some general results, recover some known results in Coffey [8] and produce new identities.

1. INTRODUCTION

In a recent paper Coffey [8], considers summations over digamma and polygamma functions and develops many results, namely two of his propositions are respectively equations (58) and (66b)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)^{p+1}} = 1 + p - 2\ln 2 + \sum_{m=1}^{p} \left(2^{-m} - 1\right) \zeta(m+1) \qquad (1.1)$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)^{p+1}} = 1 + p - \sum_{m=1}^{p} \zeta(m+1). \qquad (1.2)$$

Coffey [8] also constructs new integral representations for these sums. The major aim of this paper is to investigate general binomial sums with various parameters that then enables one to give more general representations of (1.1) and (1.2), thereby generalizing the propositions of Coffey, both in closed form in terms of zeta functions and digamma functions at possible rational values of the argument, and in double integral form. The following definitions will be used throughout this paper. The generalized hypergeometric representation $_pF_q\ [\cdot,\cdot]$, is defined as

$$_pF_q \left[\begin{array}{c} a_1, a_2, ..., a_p \\ b_1, b_2, ..., b_q \end{array} \middle| z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n\ (a_2)_n ... (a_p)_n}{(b_1)_n\ (b_2)_n ... (b_q)_n} \frac{z^n}{n!}$$

$$\left(\begin{array}{c} p, q \in \{0, 1, 2, 3...\}\,;\ p \le q + 1;\ p \le q \text{ and } |z| < \infty; \\ p = q + 1 \text{ and } |z| < 1;\ p = q + 1,\ |z| = 1 \text{ and} \\ \text{Re}\left\{ \sum_{m=1}^{q} b_m - \sum_{m=1}^{p} a_m \right\} > 0,\ b_m \notin \{0, -1, -2, -3, ...\} \end{array} \right)$$

where $(w)_\alpha = \begin{cases} \frac{\Gamma(w+\alpha)}{\Gamma(w)}, & \text{for } \alpha > 0 \\ 1, & \text{for } \alpha = 0 \end{cases}$ is Pochhammer's symbol. The Gamma and Beta functions are defined respectively as

$$\Gamma(z) = \int_0^\infty w^{z-1} e^{-w} dw, \text{ for } \operatorname{Re}(z) > 0,$$

and

$$B(s, z) = \int_0^1 w^{s-1} (1-w)^{z-1} dw = \frac{\Gamma(s)\Gamma(w)}{\Gamma(s+w)}$$

for $\operatorname{Re}(s) > 0$ and $\operatorname{Re}(z) > 0$. The well known Riemann zeta function is defined as

$$\zeta(z) = \sum_{r=1}^\infty \frac{1}{r^z}, \text{ Re}(z) > 1,$$

and we have

$$\sum_{r=1}^\infty \frac{(-1)^r}{r^z} = \left(2^{1-z} - 1\right) \zeta(z), \text{ Re}(z) > 0, \ z \neq 1.$$

The generalized harmonic numbers of order α are given by

$$H_n^{(\alpha)} = \sum_{r=1}^n \frac{1}{r^\alpha} \text{ for } (\alpha, n) \in \mathbb{N} := \{1, 2, 3, ...\}.$$

In the case of non integer values we may write the generalized harmonic numbers in terms of polygamma functions

$$H_z^{(\alpha)} = \zeta(\alpha) - \frac{(-1)^\alpha}{\Gamma(\alpha)} \psi^{(\alpha-1)}(z+1), \ z \neq \{-1, -2, -3, ...\} \tag{1.3}$$

and for $\alpha = 1$,

$$H_n^{(1)} = H_n = \int_0^1 \frac{1-t^n}{1-t} dt = \sum_{r=1}^n \frac{1}{r} = \gamma + \psi(n+1),$$

where γ denotes the Euler-Mascheroni constant, defined by

$$\gamma = \lim_{n\to\infty} \left(\sum_{r=1}^n \frac{1}{r} - \log(n) \right) = -\psi(1) \approx 0.5772156649015.....$$

and where $\psi(z)$ denotes the Psi, or digamma function, defined by

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \sum_{n=0}^\infty \left(\frac{1}{n+1} - \frac{1}{n+z} \right) - \gamma.$$

Furthermore

$$\psi(z+1) = \sum_{n=1}^\infty \left(\frac{1}{n} - \frac{1}{n+z} \right) - \gamma, \tag{1.4}$$

$$2\psi(2z) = \psi(z) + \psi\left(z + \frac{1}{2}\right) + 2\ln 2, \tag{1.5}$$

and the polygamma functions

$$\psi^{(k)}(z+1) = \sum_{n=1}^\infty \frac{(-1)^{k+1} k!}{(n+z)^{k+1}} = \int_0^\infty \frac{(-1)^{k+1} t^k e^{-(z+1)t}}{1 - e^{-t}} dt, \tag{1.6}$$

see [3]. The Lerch transcedent $\Phi(z, s, b)$ is defined as

$$\Phi(z, s, b) = \sum_{n=0}^{\infty} \frac{z^n}{(n+b)^s}, \ |z| < 1, \ b \neq 0, -1, -2, ...$$

and the Hurwitz zeta function

$$\zeta(s, b) = \Phi(1, s, b) = \sum_{n=0}^{\infty} \frac{1}{(n+b)^s}, \ \mathrm{Re}(s) > 1.$$

The Polylogarithmic function, see [25],

$$Li_k(z) = PolyLog(k, z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$$

Sums of reciprocals of binomial coefficients appear in the calculation of massive Feynman diagrams [12] within several different approaches: for instance, as solutions of differential equations for Feynman amplitudes, through a naive ε-expansion of hypergeometric functions within Mellin-Barnes technique or in the framework of recently proposed algebraic approach [11]. There has recently been renewed interest in the study of series involving binomial coefficients and a number of authors have obtained either closed form representation or integral representation for some particular cases of these series. The interested reader is referred to ([1, 2, 4, 5, 6, 9, 10, 15, 16, 17, 18, 19, 20, 21, 22, 23, 25, 26, 27]).

The following Lemma and Theorem are the main results presented in this paper.

2. THE MAIN RESULTS

The following Lemma will be useful in the proof of the main theorem.

Lemma 2.1. *Let $|z| \leq 1$, $m > 1$ and $q \geq 0$. Then*

$$\sum_{r=1}^{\infty} \frac{z^r}{(q+r)^m} = \frac{(-1)^{m-1}}{(m-1)!} \int_0^1 \frac{z \ y^q \ (\ln(y))^{m-1}}{1 - zy} dy. \tag{2.1}$$

Proof. First we expand $\frac{1}{1-zy}$ as a power series about $y = 0$, then interchange the order of integration and summation so that

$$\frac{(-1)^{m-1}}{(m-1)!} \int_0^1 \frac{z \ y^q \ (\ln(y))^{m-1}}{1 - zy} dy = \frac{(-1)^{m-1}}{(m-1)!} \sum_{s=0}^{\infty} z^{s+1} \int_0^1 y^{q+s} \ (\ln(y))^{m-1} dy.$$

Now we successively integrate $(m-1)$ times so that

$$\int_0^1 y^{q+s} \ (\ln(y))^{m-1} dy = \frac{(-1)^{m-1} (m-1)!}{(q+s+1)^m}$$

and replacing the counter $s + 1 = r$ we obtain (2.1). $\square$

The next Lemma deals with four infinite sums.

Lemma 2.2. *Let a and r be positive real numbers. Then*

$$\sum_{n=1}^{\infty} \frac{1}{n(an+r)} = \frac{H_{\frac{r}{a}}^{(1)}}{r} \quad , \tag{2.2}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(an+r)} = \frac{1}{r}\left\{ H_{\frac{r}{2a}}^{(1)} - H_{\frac{r}{a}}^{(1)} \right\}, \tag{2.3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(an+b)^{m+1}} = \frac{1}{b^m}\left(\sum_{s=1}^{m+1} \left(\frac{b}{a}\right)^{s-1} H_{\frac{b}{a}}^{(s)} - \sum_{s=2}^{m+1} \left(\frac{b}{a}\right)^{s-1} \zeta(s) \right) \quad and \tag{2.4}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(an+b)^{m+1}}$$

$$= \frac{1}{b^{m+1}}\left\{ H_{\frac{b}{2a}}^{(1)} - H_{\frac{b}{a}}^{(1)} \right\} + \sum_{p=2}^{m+1} \frac{a}{b^{m+2}} \left(\frac{b}{2a}\right)^p \left(H_{\frac{b}{2a}}^{(p)} - H_{\frac{b}{2a}-\frac{1}{2}}^{(p)} \right) \tag{2.5}$$

Proof.

$$\sum_{n=1}^{\infty} \frac{1}{n(an+r)} = \frac{1}{r}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+\frac{r}{a}}\right)$$

$$= \frac{1}{r}\left[\gamma + \psi\left(\frac{r}{a}+1\right)\right], \text{ using (1.4)}$$

$$= \frac{H_{\frac{r}{a}}^{(1)}}{r} \text{ , hence (2.2) is attained.}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(an+r)} = \frac{1}{2r}\left[\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+\frac{r}{2a}}\right) - \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+\frac{r-a}{2a}}\right) - \sum_{n=1}^{\infty}\frac{1}{n(2n-1)}\right]$$

$$= \frac{1}{2r}\left[-\gamma + \psi\left(\frac{r}{2a}+1\right) + \gamma - \psi\left(\frac{r}{2a}+\frac{1}{2}\right) - 2\ln 2\right],$$

from (1.5) we may write after substituting for $\psi\left(\frac{r}{2a}+\frac{1}{2}\right)$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(an+r)} = \frac{1}{2r}\left[2\psi\left(\frac{r}{2a}+1\right) - 2\psi\left(\frac{r}{a}+1\right)\right]$$

$$= \frac{1}{r}\left[H_{\frac{r}{2a}}^{(1)} - \gamma - H_{\frac{r}{a}}^{(1)} + \gamma\right], \text{ therefore (2.3) follows.}$$

The partial fraction decomposition

$$\frac{1}{n(an+b)^{m+1}} = \frac{1}{b^{m+1}}\left(\frac{1}{n} - \frac{1}{n+\frac{b}{a}} - \sum_{s=2}^{m+1}\left(\frac{b}{a}\right)^{s-1}\frac{1}{\left(n+\frac{b}{a}\right)^s}\right)$$

will be useful in the expansion of (2.4). Consider the sum (2.4)

$$\sum_{n=1}^{\infty} \frac{1}{n\,(an+b)^{m+1}} = \sum_{n=1}^{\infty} \left(\frac{1}{b^{m+1}} \left(\frac{1}{n} - \frac{1}{n+\frac{b}{a}} \right) - \sum_{s=2}^{m+1} \frac{a}{b^{m+2}} \left(\frac{b}{a} \right)^{s} \frac{1}{\left(n+\frac{b}{a} \right)^{s}} \right)$$

$$= \frac{1}{b^{m+1}} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+\frac{b}{a}} \right) - \sum_{s=2}^{m+1} \frac{a}{b^{m+2}} \left(\frac{b}{a} \right)^{s} \sum_{n=1}^{\infty} \frac{1}{\left(n+\frac{b}{a} \right)^{s}}$$

using (1.6) and (2.3) we have

$$\sum_{n=1}^{\infty} \frac{1}{n\,(an+b)^{m+1}} = \frac{1}{b^{m+1}} H_{\frac{b}{a}}^{(1)} - \sum_{s=2}^{m+1} \frac{a\,(-1)^{s}}{b^{m+2}\,(s-1)!} \left(\frac{b}{a} \right)^{s} \psi^{(s-1)} \left(\frac{b}{a} + 1 \right).$$

From (1.3)

$$\sum_{n=1}^{\infty} \frac{1}{n\,(an+b)^{m+1}} = \frac{1}{b^{m+1}} H_{\frac{b}{a}}^{(1)} - \sum_{s=2}^{m+1} \frac{a\,(-1)^{s}}{b^{m+2}\,(s-1)!} \left(\frac{b}{a} \right)^{s} \left(H_{\frac{b}{a}}^{(s)} - \zeta\,(s) \right)$$

$$= \sum_{s=1}^{m+1} \frac{a}{b^{m+2}} \left(\frac{b}{a} \right)^{s} H_{\frac{b}{a}}^{(s)} - \sum_{s=2}^{m+1} \frac{a}{b^{m+2}} \left(\frac{b}{a} \right)^{s} \zeta\,(s)$$

and rearranging we obtain (2.4). For the last identity (2.5) let

$$\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n\,(an+b)^{m+1}} = \sum_{n=1}^{\infty} (-1)^{n} \left(\frac{1}{b^{m+1}} \left(\frac{1}{n} - \frac{1}{n+\frac{b}{a}} \right) - \sum_{p=2}^{m+1} \frac{a}{b^{m+2}} \left(\frac{b}{a} \right)^{p} \frac{1}{\left(n+\frac{b}{a} \right)^{p}} \right)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n}}{b^{m+1}} \left(\frac{1}{n} - \frac{1}{n+\frac{b}{a}} \right) - \sum_{p=2}^{m+1} \frac{a}{b^{m+2}} \left(\frac{b}{a} \right)^{p} \sum_{n=1}^{\infty} \frac{(-1)^{n}}{\left(n+\frac{b}{a} \right)^{p}}$$

from (2.3), and (1.6)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n\,(an+b)^{m+1}} = \frac{1}{b^{m+1}} \left(H_{\frac{b}{2a}}^{(1)} - H_{\frac{b}{a}}^{(1)} \right)$$

$$- \sum_{p=2}^{m+1} \frac{a}{b^{m+2}} \left(\frac{b}{2a} \right)^{p} \sum_{n=1}^{\infty} \left(\frac{1}{\left(n+\frac{b}{2a} \right)^{p}} - \frac{1}{\left(n+\frac{b}{2a} - \frac{1}{2} \right)^{p}} \right)$$

$$= \frac{1}{b^{m+1}} \left(H_{\frac{b}{2a}}^{(1)} - H_{\frac{b}{a}}^{(1)} \right)$$

$$- \sum_{p=2}^{m+1} \frac{(-1)^{p}\,a}{(p-1)!\,b^{m+2}} \left(\frac{b}{2a} \right)^{p} \left(\psi^{(p-1)} \left(\frac{b}{2a} + 1 \right) - \psi^{(p-1)} \left(\frac{b}{2a} + \frac{1}{2} \right) \right).$$

Applying (1.3)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n\,(an+b)^{m+1}} = \frac{1}{b^{m+1}}\left(H_{\frac{b}{2a}}^{(1)} - H_{\frac{b}{a}}^{(1)}\right) - \sum_{p=2}^{m+1} \frac{a}{b^{m+2}}\left(\frac{b}{2a}\right)^p\left(-H_{\frac{b}{2a}}^{(p)} + H_{\frac{b}{2a}-\frac{1}{2}}^{(p)}\right)$$

and rearranging we obtain (2.5).   □

Remark. *In the following Corollaries and remarks we encounter harmonic numbers at possible rational values of the argument, of the form $H_{\frac{r}{a}}^{(\alpha)}$ where $r = 1, 2, 3, ..., k$, $\alpha = 1, 2, 3, ...$ and $k \in \mathbb{N}$. The polygamma function $\psi^{(\alpha)}(z)$ is defined as:*

$$\psi^{(\alpha)}(z) = \frac{d^{\alpha+1}}{dz^{\alpha+1}}\left[\log\Gamma(z)\right] = \frac{d^{\alpha}}{dz^{\alpha}}\left[\psi(z)\right],\ z \neq \{0, -1, -2, -3, ...\}.$$

To evaluate $H_{\frac{r}{a}}^{(\alpha)}$ we have available a relation in terms of the polygamma function $\psi^{(\alpha)}(z)$, (1.3), or, for rational arguments $z = \frac{r}{a}$,

$$H_{\frac{r}{a}}^{(\alpha+1)} = \zeta(\alpha+1) + \frac{(-1)^{\alpha}}{\alpha!}\psi^{(\alpha)}\left(\frac{r}{a}+1\right)$$

we also define

$$H_{\frac{r}{a}}^{(1)} = \gamma + \psi\left(\frac{r}{a}+1\right),\ \text{and}\ H_0^{(\alpha)} = 0.$$

The evaluation of the polygamma function $\psi^{(\alpha)}\left(\frac{r}{a}\right)$ at rational values of the argument can be explicitly done via a formula as given by Kölbig [14], (see also [13]), or Choi and Cvijovic [7] in terms of the Polylogarithmic or other special functions. Some specific values are given as, many others are listed in the book [24]:

$$\psi^{(n)}\left(\frac{1}{2}\right) = (-1)^n\,n!\,\left(2^{n+1}-1\right)\zeta(n+1)$$

$$H_{\frac{1}{2}}^{(4)} = 16 - 4\zeta(2),\quad H_{\frac{3}{4}}^{(2)} = \frac{16}{9} + 8G - 5\zeta(2),$$

$$H_{\frac{1}{3}}^{(1)} = \frac{3}{2} - \frac{\pi}{2\sqrt{3}} - \frac{3\ln 3}{2},\ \text{and}\ H_{\frac{5}{6}}^{(1)} = \frac{6}{5} + \frac{\sqrt{3}\pi}{2} - \frac{3\ln(3)}{2} - 2\ln(2).$$

The main result of this paper is embodied in the following theorem.

Theorem 2.3. *Let a be a positive real number, $|t| \leq 1$, $j \geq 0$, and $k \in \mathbb{N} \cup \{0\}$. Then*

$$S_{k+1}(a,\ j,\ t) = \sum_{n=1}^{\infty} \frac{t^n}{n\,(an+j+1)^k\left(\begin{array}{c} an+j+1 \\ j+1 \end{array}\right)} \tag{2.6}$$

$$= \begin{cases} \frac{(j+1)t(-1)^k}{a^k\,k!} \int_0^1 \int_0^1 \frac{(1-x)^j\ x^{a-1}y^{\frac{j+1}{a}}\,(\ln(y))^k}{1-tx^a y}dxdy,\ \text{for}\ k \geq 1 \\[2mm] at\int_0^1 \frac{(1-x)^{j+1}\ x^{a-1}}{1-tx^a}dx,\qquad\qquad\ \text{for}\ k = 0 \end{cases}.$$

$$= T_0 \;_{a+k+1}F_{a+k} \left[\begin{array}{c} \overbrace{1,1, \dfrac{a+j+1}{a}, ..., \dfrac{a+j+1}{a}}^{k-terms}, \overbrace{\dfrac{a+1}{a}, ..., \dfrac{2a-1}{a}}^{(a-1)-terms} \\ \underbrace{\dfrac{2a+j+1}{a}, ..., \dfrac{2a+j+1}{a}}_{(k+1)-terms}, \underbrace{\dfrac{a+j+2}{a}, ..., \dfrac{a+j+a}{a}}_{(a-1)-terms} \end{array} \middle| t \right]$$

(2.7)

where

$$T_0 = \frac{t\,(j+1)\;B\,(j+1,a+1)}{(a+j+1)^k}.$$

Proof. Consider

$$\sum_{n=1}^{\infty} \frac{t^n}{n\,(an+j+1)^k \left(\begin{array}{c} an+j+1 \\ j+1 \end{array} \right)} = \sum_{n=1}^{\infty} \frac{(j+1)\;t^n\;\Gamma\,(j+1)\;an\;\Gamma\,(an)}{n\,(an+j+1)^{k+1}\;\Gamma\,(an+j+1)}$$

$$= a\,(j+1) \sum_{n=1}^{\infty} \frac{t^n}{(an+j+1)^{k+1}} B\,(an,j+1)$$

now replacing the Beta function with its integral representation, we have

$$a\,(j+1) \sum_{n=1}^{\infty} \frac{t^n}{(an+j+1)^{k+1}} B\,(an,j+1) = \sum_{n=1}^{\infty} \frac{a\,(j+1)\;t^n}{(an+j+1)^{k+1}} \int_0^1 x^{an-1}\,(1-x)^j\,dx$$

$$= \frac{a\,(j+1)}{a^{k+1}} \int_0^1 \frac{(1-x)^j}{x}\,dx \sum_{n=1}^{\infty} \frac{(tx^a)^n}{\left(n+\frac{j+1}{a} \right)^{k+1}}.$$

By a justified changing the order of integration and summation, by the dominated convergence theorem, we have,

$$\sum_{n=1}^{\infty} \frac{t^n}{n\,(an+j+1)^k \left(\begin{array}{c} an+j+1 \\ j+1 \end{array} \right)} = \frac{a\,(j+1)}{a^{k+1}} \sum_{n=1}^{\infty} \frac{(tx^a)^n}{\left(n+\frac{j+1}{a} \right)^{k+1}} \int_0^1 \frac{(1-x)^j}{x}\,dx$$

$$= \frac{(j+1)\;t\;(-1)^k}{a^k\;k!} \int_0^1 \int_0^1 \frac{(1-x)^j\;x^{a-1}\;y^{\frac{j+1}{a}}\;(\ln(y))^k}{1-tx^a y}\,dx\,dy, \text{ for } k \geq 1$$

upon utilizing Lemma 2.1. The case of $k=0$ follows in a similar way so that

$$S_1\,(a,\;j,\;t) = \sum_{n=1}^{\infty} \frac{t^n}{n \left(\begin{array}{c} an+j+1 \\ j+1 \end{array} \right)}$$

$$= at \int_0^1 \frac{(1-x)^{j+1}\;x^{a-1}}{1-tx^a}\,dx,$$

hence the integrals in (2.6) are attained. By the consideration of the ratio of successive terms $\frac{U_{n+1}}{U_n}$ where

$$U_n = \frac{t^n}{n\left(an + j + 1\right)^k \begin{pmatrix} an + j + 1 \\ j + 1 \end{pmatrix}}$$

we obtain the result (2.7). □

The following interesting corollaries follow from Theorem 2.3.

Corollary 2.4. *Let $t = 1$ and $a > 0$. Also let $j \geq 0$ and $k \geq 1$ be integers. Then*

$$S_{k+1}\left(a,\ j,\ 1\right) = \sum_{n=1}^{\infty} \frac{1}{n\left(an + j + 1\right)^k \begin{pmatrix} an + j + 1 \\ j + 1 \end{pmatrix}}$$

$$= \frac{(j+1)\left(-1\right)^k}{a^k\ k!} \int_0^1 \int_0^1 \frac{(1-x)^j\ x^{a-1} y^{\frac{j+1}{a}} \left(\ln\left(y\right)\right)^k}{1 - x^a y} dx dy, \quad for\ k \geq 1 \tag{2.8}$$

$$= \sum_{s=0}^{k} A_s\, (j+1)! \sum_{p=1}^{k+1-s} \frac{H_{\frac{j+1}{a}}^{(p)}}{a^{s+p-1}\, (j+1)^{k+2-s-p}} \tag{2.9}$$

$$+ \sum_{r=1}^{j} \frac{(-1)^{r+1}\, (j+1)}{(j+1-r)^{k+1}} \begin{pmatrix} j \\ r \end{pmatrix} H_{\frac{r}{a}}^{(1)} - \sum_{s=0}^{k} A_s\, (j+1)! \sum_{p=2}^{k+1-s} \frac{\zeta\left(p\right)}{a^{s+p-1}\, (j+1)^{k+2-s-p}}$$

where

$$A_s = \lim_{n \to \left(-\frac{j+1}{a}\right)} \left[\frac{1}{s!}\frac{d^s}{dn^s} \left\{ \frac{(an + j + 1)^{k+1}}{(an + j + 1)^{k+1}\, \prod_{r=1}^{j} (an + r)} \right\} \right], \quad s = 0, 1, 2, ...k. \tag{2.10}$$

Proof. By expansion,

$$\sum_{n=1}^{\infty} \frac{1}{n\left(an + j + 1\right)^k \begin{pmatrix} an + j + 1 \\ j + 1 \end{pmatrix}} = \sum_{n=1}^{\infty} \frac{(j+1)!}{n\left(an + j + 1\right)^k (an + 1)_{j+1}}$$

$$= \sum_{n=1}^{\infty} \frac{(j+1)!}{n\left(an + j + 1\right)^k \prod_{r=1}^{j+1} (an + r)}$$

$$= \sum_{n=1}^{\infty} \frac{(j+1)!}{n\left(an + j + 1\right)^{k+1} \prod_{r=1}^{j} (an + r)}$$

$$= \sum_{n=1}^{\infty} \frac{(j+1)!}{n} \left[\sum_{r=1}^{j} \frac{B_r}{an + r} + \sum_{s=0}^{k} \frac{A_s}{(an + j + 1)^{k+1-s}} \right],$$

where

$$B_r = \lim_{n \to \left(-\frac{r}{a}\right)} \left\{ \frac{an + r}{(an + j + 1)^{k+1}\, \prod_{r=1}^{j} (an + r)} \right\} = \frac{(-1)^{r+1}}{j!} \frac{r}{(j+1-r)^{k+1}} \begin{pmatrix} j \\ r \end{pmatrix},$$

A_s is defined by (2.10). Hence, after interchanging the sums, we have

$$\sum_{n=1}^{\infty} \frac{1}{n\,(an+j+1)^k \begin{pmatrix} an+j+1 \\ j+1 \end{pmatrix}}$$

$$= \sum_{r=1}^{j} (j+1)!\,B_r \sum_{n=1}^{\infty} \frac{1}{n\,(an+r)} + \sum_{s=0}^{k} (j+1)!\,A_s \sum_{n=1}^{\infty} \frac{1}{n\,(an+j+1)^{k+1-s}}$$

$$= \sum_{r=1}^{j} (-1)^{r+1} \binom{j}{r} \frac{j+1}{(j+1-r)^{k+1}} H_{\frac{r}{a}}^{(1)}$$

$$+ \sum_{s=0}^{k} (j+1)!\,A_s \left(\sum_{p=1}^{k+1-s} \frac{H_{\frac{j+1}{a}}^{(p)}}{a^{s+p-1}\,(j+1)^{k+2-s-p}} - \sum_{p=2}^{k+1-s} \frac{\zeta(p)}{a^{s+p-1}\,(j+1)^{k+2-s-p}} \right)$$

$$= \sum_{r=1}^{j} (-1)^{r+1} \binom{j}{r} \frac{j+1}{(j+1-r)^{k+1}} H_{\frac{r}{a}}^{(1)}$$

$$+ \sum_{s=0}^{k} (j+1)!\,A_s \sum_{p=1}^{k+1-s} \frac{H_{\frac{j+1}{a}}^{(p)}}{a^{s+p-1}\,(j+1)^{k+2-s-p}} - \sum_{s=0}^{k} (j+1)!\,A_s \sum_{p=2}^{k+1-s} \frac{\zeta(p)}{a^{s+p-1}\,(j+1)^{k+2-s-p}}$$

upon utilizing Lemma 2.2, which is the result (2.9). The degenerate case, for $j = -1$, gives the known result

$$\sum_{n=1}^{\infty} \frac{1}{a^k n^{k+1}} = \frac{1}{a^k} \zeta(k+1).$$

The integral (2.8) follows from the integral in (2.6). □

A similar result is evident for the case $t = -1$.

Corollary 2.5. *Let $t = -1$ and $a > 0$. Also let $j \geq 0$ and $k \geq 1$ be integers. Then*

$$S_{k+1}(a,\,j,\,-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n\,(an+j+1)^k \begin{pmatrix} an+j+1 \\ j+1 \end{pmatrix}}$$

$$= \frac{(j+1)\,(-1)^{k+1}}{a^k\,k!} \int_0^1 \int_0^1 \frac{(1-x)^j\; x^{a-1} y^{\frac{j+1}{a}}\,(\ln(y))^k}{1+x^a y} dx\,dy,\ \ for\ k \geq 1$$

$$= \sum_{r=1}^{j} \frac{(-1)^{r+1}\,(j+1)}{(j+1-r)^{k+1}} \binom{j}{r} \left(H_{\frac{r}{2a}}^{(1)} - H_{\frac{r}{a}}^{(1)} \right) \tag{2.11}$$

$$+ \sum_{s=0}^{k} \frac{A_s\,(j+1)!}{a^s\,(j+1)^{k+1-s}} \left(H_{\frac{j+1}{2a}}^{(1)} - H_{\frac{j+1}{a}}^{(1)} \right)$$

$$+ \sum_{s=0}^{k} \frac{A_s\;(j+1)!\,a^{1-s}}{(j+1)^{k+2-s}} \sum_{p=2}^{k+1-s} \left(\frac{j+1}{2a} \right)^p \left(H_{\frac{j+1}{2a}}^{(p)} - H_{\frac{j+1}{2a}-\frac{1}{2}}^{(p)} \right)$$

Proof. The proof, uses (2.3) and (2.5) and follows the same details as that of Corollary 2.4, and will not be given here.

The degenerate case, for $j = -1$, gives the known result

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{a^k n^{k+1}} = \frac{1}{(2a)^k} \left(1 - 2^k\right) \zeta\left(k+1\right).$$

□

We give the following example to illustrate some of the above identities.

Example. *From (2.11)*

$$S_3\left(6, 2, \ -1\right) \quad = \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n\left(6n+3\right)^2 \dbinom{6n+3}{3}} = \frac{5}{3} - \frac{11}{9}G - \frac{\pi}{8}\left(8\sqrt{3} - \frac{139}{9}\right)$$

$$-\frac{\pi^3}{144} + \left(\frac{26}{9} + \frac{3\sqrt{3}}{4}\right)\ln 2 - \frac{3\sqrt{3}}{2}\ln\left(\sqrt{3}+1\right),$$

here G is Catalan's constant, defined by

$$G = \sum_{r=0}^{\infty} \frac{(-1)^r}{\left(2r+1\right)^2} \approx 0.915965....$$

Remark. *The very special case of $a = 1$ and $j = 0$ allows one to evaluate (1.1) and (1.2).*

A recurrence relation for a degenerate case, $j = 0$, of Theorem 2.3 is embodied in the following corollary.

Corollary 2.6. *Let the conditions of Theorem 2.3 hold with $j = 0$ and put*

$$S_{k+1}^0 \quad : \quad = S_{k+1}^0\left(a, t\right) = \sum_{n \geq 1} \frac{t^n}{n\left(an+1\right)^{k+1}},$$

$$= \quad \frac{t}{a+1} \quad {}_{k+3}F_{k+2}\left[\begin{array}{c} \overbrace{1, 1, \dots\dots, 1}^{(k+2)-terms}, \frac{a+1}{a} \\ \underbrace{2, 2, \dots\dots, 2}_{(k+1)-terms}, \frac{2a+1}{a} \end{array} \middle| t\right],$$

then

$$S_{k+1}^0 - S_k^0 + \frac{1}{a^k}\Phi\left(t, k+1, \frac{1}{a}\right) = a, \ for \ k \geq 1$$

with solution

$$S_{k+1}^0 \quad = \quad S_1^0 + ak - \sum_{r=1}^{k} \frac{1}{a^r}\Phi\left(t, r+1, \frac{1}{a}\right)$$

where

$$S_1^0 = \sum_{n \geq 1} \frac{t^n}{n\left(an+1\right)} = \frac{t}{a+1} \ {}_3F_2\left[\begin{array}{c} 1, 1, \frac{a+1}{a} \\ 2, \frac{2a+1}{a} \end{array} \middle| t\right]$$

and Φ is the Lerch transcendent.

Proof. We notice that

$$S_{k+1}^0 = S_k^0 - a \left[\sum_{n \geq 0} \frac{t^n}{(an+1)^{k+1}} - 1 \right]$$

hence the solution follows by iteration.   □

Some examples are:
- For $t = -1$

$$S_{k+1}^0 (a, -1) = S_1^0 + ak - \sum_{r=1}^k \frac{1}{a^r} \Phi \left(-1, r+1, \frac{1}{a} \right), \text{ for } k \geq 1$$

$$= ak - \ln(2) + \frac{1}{2} \left[\psi \left(\frac{1}{2a} \right) - \psi \left(\frac{a+1}{2a} \right) \right] + \sum_{r=1}^k \frac{\zeta \left(r+1, \frac{a+1}{2a} \right) - \zeta \left(r+1, \frac{2a+1}{2a} \right)}{2^{r+1} a^r},$$

$$= ak + \psi \left(\frac{1}{2a} \right) - \psi \left(\frac{1}{a} \right) + \sum_{r=1}^k \frac{\zeta \left(r+1, \frac{a+1}{2a} \right) - \zeta \left(r+1, \frac{2a+1}{2a} \right)}{2^{r+1} a^r}, \text{ using } (1.5).$$

When $a = 1$, we obtain Coffey's [8], result (1.1)

$$S_{k+1}^0 (1, -1) = 1 - 2 \ln(2) + k + \sum_{r=1}^k \left(2^{-r} - 1 \right) \zeta (r+1).$$

When $a = 2$,

$$S_{k+1}^0 (2, -1) = 2 - \frac{\pi}{2} - \ln(2) + 2k + \sum_{r=1}^k \frac{\zeta \left(r+1, \frac{3}{4} \right) - \zeta \left(r+1, \frac{1}{4} \right)}{2^{2r+1}}.$$

- For $t = 1$

$$S_{k+1}^0 (a, 1) = S_1^0 + ak - \sum_{r=1}^k \frac{1}{a^r} \Phi \left(1, r+1, \frac{1}{a} \right), \text{ for } k \geq 1$$

$$= \gamma + a(k+1) - \psi \left(\frac{1}{a} \right) - \sum_{r=1}^k \frac{\zeta \left(r+1, \frac{1}{a} \right)}{a^r}.$$

When $a = 1$, we obtain Coffey's [8] result (1.2), by noting that $\psi(1) = -\gamma$

$$S_{k+1}^0 (1, 1) = 1 + k - \sum_{r=1}^k \zeta (r+1).$$

When $a = 2$,

$$S_{k+1}^0 (2, 1) = 2 - 2 \ln(2) + 2k - \sum_{r=1}^k \left(\frac{2^{r+1} - 1}{2^r} \right) \zeta (r+1),$$

similarly for $a = 8$,

$$S_{k+1}^0 (8, 1) = 8(1+k) - \frac{\pi}{2} \sqrt{3 + 2\sqrt{2}} - 4 \ln(2) + \sqrt{2} \ln \left(3 - 2\sqrt{2} \right) - \sum_{r=1}^k 2^{2r} \zeta \left(r+1, \frac{1}{8} \right).$$

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