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SOME SUFFICIENT CONDITIONS FOR STARLIKENESS USING SUBORDINATION CRITERIA

(DEDICATED IN OCCASION OF THE 70-YEARS OF PROFESSOR HARI M. SRIVASTAVA)

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Abstract. In this paper we derive certain sufficient conditions for the starlikeness of functions analytic in the open unit disk. Our main result generalizes some previously established results. We also present some useful consequences of the main result and point out relevances with some of the known results.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and univalent in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. A function $f(z)$ in $\mathcal{A}$ is said to be in class $\mathcal{S}^*(\alpha)$ of starlike functions of order $\alpha\,(0 \leqq \alpha < 1)$ in $\mathbb{U}$, if it satisfies the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \ (z \in \mathbb{U};\ 0 \leqq \alpha < 1). \tag{1.2}$$

A function $f(z) \in \mathcal{A}$ is said to be strongly starlike of order α in $\mathbb{U}$, if it satisfies

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2} \ (z \in \mathbb{U};\ 0 < \alpha \leqq 1), \tag{1.3}$$

for some $\alpha\,(0 < \alpha \leqq 1)$. We denote by $\widetilde{\mathcal{S}}^*(\alpha)$ $(0 < \alpha \leqq 1)$ the subclass of $\mathcal{A}$ consisting of all functions which are strongly starlike of order α in $\mathbb{U}$. Also, we denote by $\mathcal{S}^*(0) = \widetilde{\mathcal{S}}^*(1) = \mathcal{S}^*$.

If the functions f and g are analytic in $\mathbb{U}$, then we say that f is subordinate to g in $\mathbb{U}$, and write $f \prec g$, if there exists a function $w(z)$ analytic in $\mathbb{U}$ such that $|w(z)| < 1,\ z \in \mathbb{U}$, and $w(0) = 0$ with $f(z) = g(w(z))$ in $\mathbb{U}$. If g is univalent in $\mathbb{U}$,

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then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

For $-1 \leqq b < a \leqq 1$, the function $f(z)$ in $\mathcal{A}$ is said to be in the class $\mathcal{S}^*[a, b]$, if it satisfies

$$\frac{zf'(z)}{f(z)} \prec \frac{1+az}{1+bz} \ (z \in \mathbb{U}). \tag{1.4}$$

The class $\mathcal{S}^*[a, b]$ $(-1 \leqq b < a \leqq 1)$ can be identified with a known class of starlike functions by giving particular values to the parameters a and b. Indeed, we have

$$\mathcal{S}^*[1-2\alpha, -1] \equiv \mathcal{S}^*(\alpha) \ (0 \leqq \alpha < 1).$$

Throughout this paper, the power functions, wherever occurring, are interpreted to be in terms of their principal values. In order to prove our main result (Theorem 2.1 below), we need the following well known result due to Miller and Mocanu [2] see also [3, p. 132].

Lemma 1.1. *Let the function $q(z)$ be univalent in $\mathbb{U}$, and let the functions θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$, with $\phi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set*

$$Q(z) = zq'(z)\phi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + Q(z)$$

and suppose that

(i) $Q(z)$ is univalent and starlike in $\mathbb{U}$

(ii) $\Re\left(\frac{zh'(z)}{Q(z)}\right) = \Re\left(\frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right) > 0 \ (z \in \mathbb{U})$.

If $p(z)$ is analytic in $\mathbb{U}$, with $p(0) = q(0)$, $p(\mathbb{U}) \subset \mathbb{D}$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z), \tag{1.5}$$

then $p(z) \prec q(z) \ \ (z \in \mathbb{U})$, and $q(z)$ is the best dominant.

2. MAIN RESULT

Theorem 2.1. *Let $f(z) \in \mathcal{A}$ satisfy $f(z) \neq 0$ $(z \in \mathbb{U})$. Also, let the function $q(z)$ be univalent in $\mathbb{U}$, with $q(0) = 1$ and $q(z) \neq 0$, such that*

$$\Re\left(1 + (\beta - 1)\frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right) > 0 \ (z \in \mathbb{U}) \tag{2.1}$$

and

$$\Re\left([1 + (\lambda - 1)\beta] + (\beta + 1)q(z) + (\beta - 1)\frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right) > 0 \ (z \in \mathbb{U}) \tag{2.2}$$

for $\lambda > 0$ and $|\beta| \leqq 1$.

If

$$\left(\frac{zf'(z)}{f(z)}\right)^{\beta}\left(\lambda + \frac{zf''(z)}{f'(z)}\right) \prec h(z) \ \ (z \in \mathbb{U}), \tag{2.3}$$

where

$$h(z) = [q(z)]^{\beta+1} + (\lambda - 1)[q(z)]^{\beta} + zq'(z)[q(z)]^{\beta-1}, \tag{2.4}$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z) \ (z \in \mathbb{U}),$$

and $q(z)$ is the best dominant of (2.3).

Proof. We first choose

$$p(z)=\frac{zf'(z)}{f(z)},\ \theta(w)=w^{\beta}(\lambda-1+w)\text{ and }\phi(w)=w^{\beta-1}, \tag{2.5}$$

then $\theta(w)$ and $\phi(w)$ are analytic inside the domain $\mathbb{D}^*=\mathbb{C}\setminus\{0\}$, which contains $q(\mathbb{U})$, $q(0)=1$, and $\phi(w)\neq 0$ when $w\in q(\mathbb{U})$.

Now, if we define the functions $Q(z)$ and $h(z)$ by

$$Q(z)=zq'(z)\phi(q(z))=zq'(z)[q(z)]^{\beta-1}$$

and

$$h(z)=\theta(q(z))+Q(z)=[q(z)]^{\beta+1}+(\lambda-1)[q(z)]^{\beta}+zq'(z)[q(z)]^{\beta-1},$$

then it follows from (2.1) and (2.2) that $Q(z)$ is starlike in $\mathbb{U}$ and

$$\Re\left(\frac{zh'(z)}{Q(z)}\right)>0\ (z\in\mathbb{U}).$$

We also note that the function $p(z)$ is analytic in $\mathbb{U}$, with $p(0)=q(0)=1$. Since $0\notin p(\mathbb{U})$, therefore, $p(\mathbb{U})\subset\mathbb{D}^*$. and hence, the hypothesis of Lemma 1 are satisfied. Applying Lemma 1.1, we find that

$$\begin{aligned}
&\left(\tfrac{zf'(z)}{f(z)}\right)^{\beta}\left(\lambda+\tfrac{zf''(z)}{f'(z)}\right)\\
&\qquad=[p(z)]^{\beta+1}+(\lambda-1)[p(z)]^{\beta}+zp'(z)[p(z)]^{\beta-1}\\
&\qquad=\theta(p(z))+zp'(z)\phi(p(z))\prec h(z)\\
&\qquad=[q(z)]^{\beta+1}+(\lambda-1)[q(z)]^{\beta}+zq'(z)[q(z)]^{\beta-1}\\
&\qquad=\theta(q(z))+zq'(z)\phi(q(z))\quad(z\in\mathbb{U}),
\end{aligned}$$

which implies that $\frac{zf'(z)}{f(z)}\prec q(z)$ $(z\in\mathbb{U})$ and $q(z)$ is the best dominant of (2.3). □

3. SOME CONSEQUENCES OF THEOREM 2.1

In this section we present some useful corollaries of Theorem 2.1 and also mention their relevances with some of the known results.

Let us set

$$\lambda=\frac{1}{\gamma}(\gamma>0)\text{ and }q(z)=\left(\frac{1+az}{1+bz}\right)^{\alpha}\ \ (-1\leqq b<a\leqq 1,\ 0<\alpha\leqq 1).$$

It is easily verified that the above function $q(z)$ being analytic and convex in $\mathbb{U}$ is such that $q(\mathbb{U})$ is symmetric with respect to the real axis and satisfies the inequality (see [5, p. 16]):

$$0\le\left(\frac{1-a}{1-b}\right)^{\alpha}<\Re(q(z))\quad(z\in\mathbb{U};-1\le a<b\le 1;0<\alpha\le 1)\,.$$

Thus, (2.1) is equivalent to

$$\begin{aligned}
\Re\left[1+(\alpha\beta-1)\frac{az}{1+az}-(\alpha\beta+1)\frac{bz}{1+bz}\right]&=\Re\left[-1+(1-\alpha\beta)\frac{1}{1+az}+(1+\alpha\beta)\frac{1}{1+bz}\right]\\
&>-1+(1-\alpha\beta)\frac{1}{1+|a|}+(1+\alpha\beta)\frac{1}{1+|b|}\geqq 0,
\end{aligned}$$

provided that

$$(1-\alpha\beta)\frac{1}{1+|a|}+(1+\alpha\beta)\frac{1}{1+|b|}\geqq 1.$$

Similarly, (2.2) is satisfied if

$$\beta\left(\frac{1-\gamma}{\gamma}\right)+(1+\beta)\left(\frac{1-a}{1-b}\right)^{\alpha}+(1-\alpha\beta)\frac{1}{1+|a|}+(1+\alpha\beta)\frac{1}{1+|b|}-1\geqq 0.$$

With the above specialization of $q(z)$ and the corresponding related conditions, Theorem 2.1 gives the following result:

Corollary 3.1. *Let $f(z)\in\mathcal{A}$ satisfy $f(z)\neq 0$ $(z\in\mathbb{U})$. Also, let $\gamma>0$, $|\beta|\leqq 1$; $-1\leqq b<a\leqq 1$ and $0<\alpha\leqq 1$ be so chosen that*

$$(1-\alpha\beta)\frac{1}{1+|a|}+(1+\alpha\beta)\frac{1}{1+|b|}-1\geqq 0 \tag{3.1}$$

and

$$\beta\left(\frac{1-\gamma}{\gamma}\right)+(1+\beta)\left(\frac{1-a}{1-b}\right)^{\alpha}+(1-\alpha\beta)\frac{1}{1+|a|}+(1+\alpha\beta)\frac{1}{1+|b|}-1\geqq 0. \tag{3.2}$$

If

$$\left(\frac{zf'(z)}{f(z)}\right)^{\beta}\left(1+\gamma\frac{zf''(z)}{f'(z)}\right)\prec h(z)\quad(z\in\mathbb{U}), \tag{3.3}$$

where

$$h(z)=\left(\frac{1+az}{1+bz}\right)^{\alpha\beta-1}\left[(1-\gamma)\frac{1+az}{1+bz}+\gamma\left(\frac{1+az}{1+bz}\right)^{\alpha+1}+\frac{\alpha\gamma z(a-b)}{(1+bz)^2}\right], \tag{3.4}$$

then $\frac{zf'(z)}{f(z)}\prec\left(\frac{1+az}{1+bz}\right)^{\alpha}$ $(z\in\mathbb{U})$.

Remark. It may be observed that if we set $b=-1$ and $\alpha=1$, then Corollary 3.1 corresponds to the known result of Aghalary and Jahangiri [1, p. 2054, Theorem 2.2]. On the other hand, if we set $b=0$ and $\alpha=1$, then Corollary 3.1 readily gives another known result [1, p. 2056, Theorem 2.5]. Also, by setting $\beta=1$, $\gamma=\frac{1}{\lambda}$ $(\lambda>0)$ and $\alpha=1$ in Corollary 3.1, and simplyfying the conditions (3.1) and (3.2), we get the result which was earlier obtained by Xu and Yang [6, p. 581, Theorem 1].

A much simpler consequence of Corollary 3.1 occurs when we assign in it the values $\beta=1$, $\gamma=\frac{1}{\lambda}$ $(\lambda>0)$ $a=1$ and $b=-1$, so that the conditions (3.1) and (3.2) are both satisfied provided that $\lambda\geqq 1$. Also, we note that (see [6, p. 589])

$$\frac{zf'(z)}{f(z)}\prec\left(\frac{1+z}{1-z}\right)^{\alpha}$$

$$\Rightarrow arg\left(\frac{zf'(z)}{f(z)}\right)=\alpha\ arg\left(\frac{1+w(z)}{1-w(z)}\right)\ (\text{where } w(0)=0 \text{ and } |w(z)|\leqq|z|<1\ (z\in\mathbb{U}))$$

$$\Rightarrow\quad\left|arg\left(\frac{zf'(z)}{f(z)}\right)\right|=\alpha\left|arg\left(\frac{1+w(z)}{1-w(z)}\right)\right|\to\frac{\alpha\pi}{2}\ as\ w\to i.$$

Corollary 3.1 then eventually gives the following result:

Corollary 3.2. *Let $f(z) \in \mathcal{A}$ satisfy $f(z) \neq 0$ $(z \in \mathbb{U})$ and*

$$\frac{z^2 f''(z)}{f(z)} + \lambda \frac{zf'(z)}{f(z)} \prec h(z) \quad (z \in \mathbb{U}; \lambda \geq 1), \tag{3.5}$$

where

$$h(z) = (\lambda - 1)\left(\frac{1+z}{1-z}\right)^{\alpha} + \left(\frac{1+z}{1-z}\right)^{2\alpha} + \frac{2\alpha z(1+z)^{\alpha-1}}{(1-z)^{\alpha+1}}, \tag{3.6}$$

then $\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}$ $(z \in \mathbb{U})$ $\Rightarrow f(z) \in \widetilde{\mathcal{S}}^*(\alpha)$.

If we put $\beta = 0$ and $\lambda = 1$ in Theorem 2.1, then we get the following interesting corollary:

Corollary 3.3. *Let $f(z) \in \mathcal{A}$ satisfy $f(z) \neq 0$ $(z \in \mathbb{U})$. Also let that the function $q(z)$ be univalent in $\mathbb{U}$, with $q(0) = 1$ and $q(z) \neq 0$ $(z \in \mathbb{U})$, satisfying each of the following inequlities:*

$$\Re\left(1 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right) > 0 \tag{3.7}$$

and

$$\Re\left(1 + q(z) - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right) > 0. \tag{3.8}$$

If

$$1 + \frac{zf''(z)}{f'(z)} \prec q(z) + \frac{zq'(z)}{q(z)} \ (z \in \mathbb{U}), \tag{3.9}$$

then $\frac{zf'(z)}{f(z)} \prec q(z)$ $(z \in \mathbb{U})$.

Remark. Corollary 3.3 can also be derived from a known result due to Srivastava and Attiya [4, p. 1154, Theorem 1 (when $p = \lambda = 1$)].

By putiing $\beta = -1$, $\lambda = 1$ and $q(z) = \frac{1+az}{1+bz}$ in Theorem 2.1, we get the following result.

Corollary 3.4. *Let $f(z) \in \mathcal{A}$ satisfy $f(z) \neq 0$ $(z \in \mathbb{U})$. If $-1 \leqq b < a \leqq 1$ and*

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 + \frac{z(a-b)}{(1+az)^2} \quad (z \in \mathbb{U}), \tag{3.10}$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+az}{1+bz} \quad (z \in \mathbb{U}). \tag{3.11}$$

Remark. For $b = -1$, Corollary 3.4 corresponds to the result [1, p. 2055, Corollary 2.3].

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