Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 2 Issue 4(2010), Pages 83-86.

ON THE GENERALIZED ABSOLUTE CESÀRO SUMMABILITY

(DEDICATED IN OCCASION OF THE 70-YEARS OF PROFESSOR HARI M. SRIVASTAVA)

HÜSEYIN BOR, DANSHENG YU

Abstract. In this paper, a known theorem dealing with $\mid C,\alpha \mid$ summability factors, has been generalized for $\mid C,\alpha,\beta \mid_k$ summability factors. Some new results have also been obtained.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by $u_n^{\alpha,\beta}$ and $t_n^{\alpha,\beta}$ the n-th Cesàro means of order (α,β), with $\alpha+\beta>-1$, of the sequence (s_n) and (na_n), respectively, i.e., (see [2])

$$u_n^{\alpha,\beta}=\frac{1}{A_n^{\alpha+\beta}}\sum_{v=0}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}s_v \tag{1.1}$$

$$t_n^{\alpha,\beta}=\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v, \tag{1.2}$$

where

$$A_n^{\alpha+\beta}=O(n^{\alpha+\beta}),\quad A_0^{\alpha+\beta}=1\quad and\quad A_{-n}^{\alpha+\beta}=0\quad for\quad n>0. \tag{1.3}$$

The series $\sum a_n$ is said to be summable $\mid C,\alpha,\beta \mid_k$, $k\geq 1$ and $\alpha+\beta>-1$, if (see [4])

$$\sum_{n=1}^{\infty}n^{k-1}\mid u_n^{\alpha,\beta}-u_{n-1}^{\alpha,\beta}\mid<\infty. \tag{1.4}$$

Since $t_n^{\alpha,\beta}=n(u_n^{\alpha,\beta}-u_{n-1}^{\alpha,\beta})$ (see [4]), condition (4) can also be written as

$$\sum_{n=1}^{\infty}\frac{1}{n}\mid t_n^{\alpha,\beta}\mid^k<\infty. \tag{1.5}$$

If we take $\beta=0$, then $\mid C,\alpha,\beta \mid_k$ summability reduces to $\mid C,\alpha \mid_k$ summability (see [5]). It should be noted that obviously $(C,\alpha,0)$ mean is the same as (C,α)

2000 *Mathematics Subject Classification.* 40D15, 40F05, 40G05, 40G99.

Key words and phrases. Absolute Cesàro summability, infinite series, summability factors.
©2010 Universiteti i Prishtinës, Prishtinë, Kosovë.
Submitted September 2, 2010. Published October 23, 2010.

mean. A sequence (λ_n) is said to be convex if $\Delta^2\lambda_n \geq 0$, where $\Delta^2\lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}$.

Pati [6] has proved the following theorem dealing with $\mid C,\alpha \mid$ summability factors.

Theorem 1.1. *If (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent and the sequence (θ_n^α) defined by*

$$\theta_n^\alpha = \mid t_n^\alpha \mid,\ \alpha = 1, \tag{1.6}$$

$$\theta_n^\alpha = \max_{1\leq v\leq n} \mid t_v^\alpha \mid,\ 0<\alpha<1 \tag{1.7}$$

satisfies the condition

$$\theta_n^\alpha = O(1)(C,1), \tag{1.8}$$

then the series $\sum a_n\lambda_n$ is summable $\mid C,\alpha \mid$ for $0<\alpha\leq 1$.

2. The Main Result

The aim of this paper is to generalize Theorem 1.1 for $\mid C,\alpha,\beta \mid_k$ summability. We shall prove the following theorem.

Theorem 2.1. *If (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent and the sequence $(\theta_n^{\alpha,\beta})$ defined by*

$$\theta_n^{\alpha,\beta} = \mid t_n^{\alpha,\beta} \mid,\ \alpha = 1, \beta > -1 \tag{2.1}$$

$$\theta_n^{\alpha,\beta} = \max_{1\leq v\leq n} \mid t_v^{\alpha,\beta} \mid,\ 0<\alpha<1, \beta>-1 \tag{2.2}$$

satisfies the condition

$$(\theta_n^{\alpha,\beta})^k = O(1)(C,1), \tag{2.3}$$

then the series $\sum a_n\lambda_n$ is summable $\mid C,\alpha,\beta \mid_k$ for $0<\alpha\leq 1$, $\beta>-1$ and $k\geq 1$.

It should be noted that if we take $k=1$ and $\beta=0$, then we get Theorem 1.1.

We need the following lemmas for the proof of our theorem.

Lemma 2.2. ([3]) *If (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then*

$$n\Delta\lambda_n \to 0,$$

$$\sum_{n=1}^{\infty}(n+1)\Delta^2\lambda_n$$

is convergent.

Lemma 2.3. ([1]). *If $0<\alpha\leq 1$, $\beta>-1$ and $1\leq v\leq n$, then*

$$\mid \sum_{p=0}^{v} A_{n-p}^{\alpha-1}A_p^\beta a_p \mid \leq \max_{1\leq m\leq v} \mid \sum_{p=0}^{m} A_{m-p}^{\alpha-1}A_p^\beta a_p \mid. \tag{2.4}$$

Proof of the theorem. Let $(T_n^{\alpha,\beta})$ be the n-th (C,α,β) mean of the sequence $(na_n\lambda_n)$. Then, by (1.2), we have

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n} A_{n-v}^{\alpha-1}A_v^\beta v a_v \lambda_v.$$

First, applying Abel's transformation and then using Lemma 2.3, we have that

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}\Delta\lambda_v\sum_{p=1}^{v}A_{n-p}^{\alpha-1}A_p^{\beta}pa_p+\frac{\lambda_n}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v,$$

thus,

$$\begin{aligned}
\mid T_n^{\alpha,\beta}\mid &\leq \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}\mid\Delta\lambda_v\mid\mid\sum_{p=1}^{v}A_{n-p}^{\alpha-1}A_p^{\beta}pa_p\mid+\frac{\mid\lambda_n\mid}{A_n^{\alpha+\beta}}\mid\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v\mid\\
&\leq \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}A_v^{\alpha}A_v^{\beta}\theta_v^{\alpha,\beta}\mid\Delta\lambda_v\mid+\mid\lambda_n\mid\theta_n^{\alpha,\beta}\\
&= T_{n,1}^{\alpha,\beta}+T_{n,2}^{\alpha,\beta}, \quad say.
\end{aligned}$$

Since

$$\mid T_{n,1}^{\alpha,\beta}+T_{n,2}^{\alpha,\beta}\mid^k\leq 2^k(\mid T_{n,1}^{\alpha,\beta}\mid^k+\mid T_{n,2}^{\alpha,\beta}\mid^k),$$

in order to complete the proof of the theorem, by (5), it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{1}{n}\mid T_{n,r}^{\alpha,\beta}\mid^k<\infty \quad for \quad r=1,2.$$

Whenever $k>1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, we get that

$$\begin{aligned}
\sum_{n=2}^{m+1}\frac{1}{n}\left|T_{n,1}^{\alpha,\beta}\right|^k &\leq \sum_{n=2}^{m+1}\frac{1}{n}\left|\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}A_v^{\alpha}A_v^{\beta}\theta_v^{\alpha,\beta}\Delta\lambda_v\right|^k\\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{1+(\alpha+\beta)k}}\left\{\sum_{v=1}^{n-1}v^{\alpha k}v^{\beta k}\Delta\lambda_v(\theta_v^{\alpha,\beta})^k\right\}\\
&\quad\times\left\{\sum_{v=1}^{n-1}\Delta\lambda_v\right\}^{k-1}\\
&= O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}\Delta\lambda_v(\theta_v^{\alpha,\beta})^k\sum_{n=v+1}^{m+1}\frac{1}{n^{1+(\alpha+\beta)k}}\\
&= O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}\Delta\lambda_v(\theta_v^{\alpha,\beta})^k\int_v^{\infty}\frac{dx}{x^{1+(\alpha+\beta)k}}\\
&= O(1)\sum_{v=1}^{m}\Delta\lambda_v(\theta_v^{\alpha,\beta})^k\\
&= O(1)\sum_{v=1}^{m-1}\Delta(\Delta\lambda_v)\sum_{p=1}^{v}(\theta_p^{\alpha,\beta})^k+O(1)\Delta\lambda_m\sum_{v=1}^{m}(\theta_v^{\alpha,\beta})^k\\
&= O(1)\sum_{v=1}^{m}v\Delta^2\lambda_v+O(1)m\Delta\lambda_m\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}$$

in view of hypotheses of the theorem and Lemma 2.2.

Similarly , we have that

$$\begin{aligned}
\sum_{n=1}^{m}\frac{1}{n}\mid\lambda_n\theta_n^{\alpha,\beta}\mid^k &= O(1)\sum_{n=1}^{m}\frac{\lambda_n}{n}(\theta_n^{\alpha,\beta})^k \\
&= O(1)\sum_{n=1}^{m-1}\Delta(n^{-1}\lambda_n)\sum_{v=1}^{n}(\theta_v^{\alpha,\beta})^k \\
&+ O(1)\frac{\lambda_m}{m}\sum_{v=1}^{m}(\theta_v^{\alpha,\beta})^k \\
&= O(1)\sum_{n=1}^{m-1}\Delta\lambda_n + O(1)\sum_{n=1}^{m-1}\frac{\lambda_{n+1}}{n+1} + O(1)\lambda_m \\
&= O(1)(\lambda_1-\lambda_m) + O(1)\sum_{n=1}^{m-1}\frac{\lambda_{n+1}}{n+1} + O(1)\lambda_m \\
&= O(1) \quad as \quad m\to\infty.
\end{aligned}$$

Therefore, by (1.5), we get that

$$\sum_{n=1}^{\infty}\frac{1}{n}\mid T_{n,r}^{\alpha,\beta}\mid^k<\infty \quad for \quad r=1,2.$$

This completes the proof of the theorem. If we take $\beta=0$, then we get a new result for $\mid C,\alpha\mid_k$ summability factors. Also, if we take $\beta=0$ and $\alpha=1$, then we get another new result for $\mid C,1\mid_k$ summability factors.

References

[1] H. Bor, On a new application of quasi power increasing sequences, *Proc. Estonian Acad. Sci. Phys. Math.* **57**, (2008), 205-209.

[2] D. Borwein, Theorems on some methods of summability, *Quart. J. Math., Oxford*, **Ser.9**, (1958), 310-316.

[3] H. C. Chow , On the summability factors of Fourier Series, *J. London Math. Soc.* **16**, (1941), 215-220.

[4] G. Das, A Tauberian theorem for absolute summability, *Proc. Camb. Phil. Soc.* **67**, (1970), 321-326.

[5] T. M. Flett, On an extension of absolute summability and some theorems of Littlewood and Paley, *Proc. London Math. Soc.* **7**, (1957), 113-141.

[6] T. Pati, The summability factors of infinite series, *Duke Math. J.* **21**, (1954), 271-284.

Hüseyin Bor
Department of Mathematics, Bahçelievler, P.O.Box 121 , 06502,
Ankara, Turkey
E-mail address: hbor33@gmail.com

Dansheng Yu
Department of Mathematics, Hangzhou Normal University,
Hangzhou Zhejiang 310036 , China
E-mail address: anshengyu@yahoo.com.cn