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ON SOME ACCURATE ESTIMATES OF π

(DEDICATED IN OCCASION OF THE 70-YEARS OF PROFESSOR HARI M. SRIVASTAVA)

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Abstract. The aim of this paper is to establish some inequalities related to an accurate approximation formula of π. Being practically difficult, the computations arising in this problem were made using computer softwares such as Maple.

1. Introduction

Maybe the best known example of infinite product for estimating the constant π is the Wallis product [4]

$$\frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots=\prod_{n=1}^{\infty}\frac{4n^2}{4n^2-1}, \tag{1.1}$$

which is related to Euler's gamma function Γ, since

$$\prod_{k=1}^{n}\frac{4k^2}{4k^2-1}=\frac{16^n\left(\Gamma\left(n+1\right)\right)^4}{\left(2n+1\right)\left(\Gamma\left(2n+1\right)\right)^2}. \tag{1.2}$$

Although it has a nice form, (1.1) is very slow, so it is not suitable for approximating the constant π.

A possible starting point for accelerating (1.1) is the work of Fields [1] who shown that

$$\begin{aligned}\frac{\Gamma\left(z+a\right)}{\Gamma\left(z+b\right)} &\sim \left(z+a-\rho\right)^{a-b}\sum_{k=0}^{N-1}\frac{B_{2k}^{(2\rho)}\left(\rho\right)\left(b-a\right)_{2k}\left(z+a-\rho\right)^{-2k}}{(2k)!} \qquad (1.3)\\ &\quad +\left(z+a-\rho\right)^{a-b}O\left(\left(z+a-\rho\right)^{-2N}\right),\\ 2\rho &= 1+a-b,\quad \left|\arg\left(z+a\right)\right|\leq\pi-\varepsilon,\quad \varepsilon>0.\end{aligned}$$

where the symbols $B_{2k}^{(2\rho)}$ stand for the generalized Bernoulli polynomials [2, 5].

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With $z = n$, $a = -x$, $b = 1$, and $\rho = -x/2$ in (1.3), we get

$$\left(\begin{array}{c} x \\ n \end{array}\right) \sim \frac{(-1)^n (n - x/2)^{-(x+1)}}{\Gamma(-x)} \sum_{k=0}^{\infty} \frac{B_{2k}^{(2\rho)}(\rho)(x+1)_{2k}}{(2k)!(n-x/2)^{2k}}, \quad \rho = -\frac{x}{2},$$

$$\left(\begin{array}{c} x \\ n \end{array}\right) \sim \frac{(-1)^n (n - x/2)^{-(x+1)}}{\Gamma(-x)} \left[1 + \frac{(x)_3}{24(n-x/2)^2} + \frac{(x)_5(5x-2)}{5760(n-x/2)^4} + \frac{(x)_7(35x^2-42x+16)}{2903040(n-x/2)^6} + \cdots\right]. \tag{1.4}$$

See also [3, p. 142], where the following identity is stated

$$\left(\begin{array}{c} x \\ n \end{array}\right) = \frac{(-1)^n}{\Gamma(-x)} \frac{\Gamma(n-x)}{\Gamma(n+1)}.$$

Further, with $x = -1/2$ in (1.4), we obtain the following formula

$$\begin{aligned} \pi &= \frac{4(\Gamma(n+1))^4 16^n}{(4n+1)(\Gamma(2n+1))^2} \left[1 - \frac{1}{4(4n+1)^2} + \frac{21}{32(4n+1)^4} \right. \\ &\quad - \frac{671}{128(4n+1)^6} + \frac{180323}{2048(4n+1)^8} - \frac{20898423}{8192(4n+1)^{10}} \\ &\quad \left. + \frac{7426362705}{65536(4n+1)^{12}} - \frac{1874409465055}{262144(4n+1)^{14}} + O\left(\frac{1}{n^{16}}\right)\right]^2. \end{aligned} \tag{1.5}$$

The idea of expressing π using the asymptotic expansion of the ratio $\frac{\Gamma(n+1/2)}{\Gamma(n+1)}$ was introduced by Tricomi and Erdélyi in [3, p. 142, Rel. 23]. Here we make use of the asymptotic expansion for $\frac{\Gamma(n+1/2)}{\Gamma(n+1)}$ given in [1] to improve the results of Tricomi and Erdélyi [3].

2. The results

By truncation of series (1.5), increasingly accurate approximations for π can be derived. As example, if $n = 10$, use of the first five terms in (1.5) gives π with an error of 1.1639×10^{-12}, while use of the first six terms in (1.5) gives π with an error of 3.0431×10^{-14}.

We prove the following

Theorem 2.1. *For every integer $n \geq 1$, we have*

$$\frac{4(\Gamma(n+1))^4 16^n}{(4n+1)(\Gamma(2n+1))^2} a(n) < \pi < \frac{4(\Gamma(n+1))^4 16^n}{(4n+1)(\Gamma(2n+1))^2} b(n),$$

where

$$a(n) = \left(1 - \frac{1}{4(4n+1)^2} + \frac{21}{32(4n+1)^4} - \frac{671}{128(4n+1)^6} + \frac{180323}{2048(4n+1)^8}\right)^2$$

and

$$b(n) = \left(1 - \frac{1}{4(4n+1)^2} + \frac{21}{32(4n+1)^4} - \frac{671}{128(4n+1)^6} + \frac{180323}{2048(4n+1)^8} - \frac{20898423}{8192(4n+1)^{10}}\right)^2.$$

Proof. The sequences

$$x_n = \frac{4\left(\Gamma\left(n+1\right)\right)^4 16^n}{\left(4n+1\right)\left(\Gamma\left(2n+1\right)\right)^2} a\left(n\right) \quad , \qquad y_n = \frac{4\left(\Gamma\left(n+1\right)\right)^4 16^n}{\left(4n+1\right)\left(\Gamma\left(2n+1\right)\right)^2} b\left(n\right)$$

converge to π and it suffices to show that x_n is strictly increasing and y_n is strictly decreasing. In this sense, we have

$$\begin{aligned} \frac{x_{n+1}}{x_n} - 1 &= \frac{4\left(4n+1\right)\left(n+1\right)^2}{\left(4n+5\right)\left(2n+1\right)^2} \frac{a\left(n+1\right)}{a\left(n\right)} - 1 \\ &= -\frac{P\left(n\right)}{\left(4n+5\right)^{17}\left(2n+1\right)^2\left(134217728n^8 + \cdots + 172467\right)^2}, \end{aligned}$$

where the polynomial

$$P\left(n\right) = 602356032226758420017971 20n^{24} + \cdots + 226910180447723367864 09$$

has all coefficients positive. In consequence, x_n is strictly increasing, convergent to π, so $x_n < \pi$.

Then

$$\begin{aligned} \frac{y_{n+1}}{y_n} - 1 &= \frac{4\left(4n+1\right)\left(n+1\right)^2}{\left(4n+5\right)\left(2n+1\right)^2} \frac{b\left(n+1\right)}{b\left(n\right)} - 1 \\ &= \frac{Q\left(n\right)}{\left(2n+1\right)^2\left(4n+5\right)^{12}\left(8589934592n^{10} + \cdots - 20208555\right)^2}, \end{aligned}$$

where the polynomial

$$\begin{aligned} Q\left(n+1\right) &= 210420037966350927549442377646080n^{30} \\ &\quad + \cdots + 190967265341557883363002243421711 2437351 \end{aligned}$$

has all coefficients positive. In consequence, y_n is strictly decreasing, convergent to π, so $y_n > \pi$. $\square$

Remark. The computations in this paper were made using Maple software.

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