Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 2 Issue 4(2010), Pages 16-24.

COUPLED FIXED POINT THEOREMS IN PARTIALLY ORDERED METRIC SPACES

(DEDICATED IN OCCASION OF THE 70-YEARS OF PROFESSOR HARI M. SRIVASTAVA)

NGUYEN VAN LUONG, NGUYEN XUAN THUAN

Abstract. The purpose of this paper is to prove some coupled fixed point theorems for mapping having the mixed monotone property in partially ordered metric sapces which are generalization of the main results of Bhaskar and Lakshmikantham [T.G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal.TMA 65(2006) 1379 - 1393]. We support our results by an example.

1. Introduction

Existence of a fixed point for contraction type mappings in partially ordered metric spaces and aplications have been considered recently by many authors (see, for details, [2], [3], [4], [5], [13], [14], [15], [16], [17], [18], [19], [21], [22],[23], [24], [25], [27], [28], [29]).

In [15], Bhaskar and Lakshmikantham have introduced notions of a mixed monotone mapping and a coupled fixed point and proved some couped fixed point theorems for mixed monotone mapping and discuss the existence and uniqueness of solution for periodic boundary value problem. The notions of a mixed monotone mapping and a coupled fixed point state as follows.

Definition 1.1. *([15]) Let $(X, \preceq)$ be a partially ordered set and $F : X\times X \to X$. The mapping F is said to has the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$,*

$$x_1, x_2 \in X, \quad x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2)$$

Definition 1.2. *([15]) An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if*

$$x = F(x, y) \ \text{and}\ y = F(y, x)$$

2000 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. Coupled fixed point, mixed monotone property, partially ordered set.

Submitted June 2, 2010. Published September 9, 2010.

The main results of Bhaskar and Lakshmikantham in [15] are the following coupled fixed point theorems

Theorem 1.3. *([15]) Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continous mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with*

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}[d(x,u) + d(y,v)]$$

for all $x \succeq u$ and $y \preceq v$. If there exist two elements $x_0, y_0 \in X$ with

$$x_0 \preceq F(x_0, y_0) \;\; \text{and} \;\; y_0 \succeq F(y_0, x_0)$$

then there exist $x, y \in X$ such that

$$x = F(x,y) \;\; \text{and} \;\; y = F(y,x).$$

Theorem 1.4. *([15]) Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Assume that X has the following property:*

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n.

Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}[d(x,u) + d(y,v)]$$

for all $x \succeq u$ and $y \preceq v$. If there exist two elements $x_0, y_0 \in X$ with

$$x_0 \preceq F(x_0, y_0) \;\; \text{and} \;\; y_0 \succeq F(y_0, x_0)$$

then there exist $x, y \in X$ such that

$$x = F(x,y) \;\; \text{and} \;\; y = F(y,x).$$

In this paper, we give some coupled fixed point theorems for mapping having the mixed monotone property in partially ordered metric sapces which are generalization of the main results of Bhaskar and Lakshmikantham [15].

2. The main results

Theorem 2.1. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \preceq F(x_0, y_0) \;\; \text{and} \;\; y_0 \succeq F(y_0, x_0)$$

Suppose there exist non-negative real numbers α, β and L with $\alpha + \beta < 1$ such that

$$\begin{aligned} d(F(x,y), F(u,v)) &\leq \alpha d(x,u) + \beta d(y,v) \\ &+ L \min \left\{ \begin{array}{l} d(F(x,y), u), d(F(u,v), x), \\ d(F(x,y), x), d(F(u,v), u) \end{array} \right\} \end{aligned} \tag{2.1}$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. Suppose either

(a) F is continuous or

(b) X has the following property:

(i) if a non-decreasing sequence $\{x_n\} \to x$ *, then* $x_n \preceq x$ *, for all* n*,*
(ii) if a non-increasing sequence $\{y_n\} \to y$ *, then* $y \preceq y_n$ *, for all* n*.*
then there exist $x, y \in X$ *such that*

$$x = F(x, y) \text{ and } y = F(y, x),$$

that is, F *has a coupled fixed point in* X*.*

Proof. Let $x_0, y_0 \in X$ be such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. We construct sequences $\{x_n\}$ and $\{y_n\}$ in X as follows

$$x_{n+1} = F(x_n, y_n) \text{ and } y_{n+1} = F(y_n, x_n) \quad \text{for all } n \geq 0 \tag{2.2}$$

We shall show that

$$x_n \preceq x_{n+1} \quad \text{for all } n \geq 0 \tag{2.3}$$

and

$$y_n \succeq y_{n+1} \quad \text{for all } n \geq 0 \tag{2.4}$$

We shall use the mathematical induction.
Let $n = 0$. Since $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$ and as $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$, we have $x_0 \preceq x_1$ and $y_0 \succeq y_1$. Thus (2.3) and (2.4) hold for $n = 0$.

Suppose now that (2.3) and (2.4) hold for some fixed $n \geq 0$. Then, since $x_n \preceq x_{n+1}$ and $y_n \succeq y_{n+1}$, and by the mixed monotone property of F, we have

$$x_{n+2} = F(x_{n+1}, y_{n+1}) \succeq F(x_n, y_{n+1}) \succeq F(x_n, y_n) = x_{n+1} \tag{2.5}$$

and

$$y_{n+2} = F(y_{n+1}, x_{n+1}) \preceq F(y_n, x_{n+1}) \preceq F(y_n, x_n) = y_{n+1} \tag{2.6}$$

Thus by the mathematical induction we conclude that (2.3) and (2.4) hold for all $n \geq 0$.
Therefore,

$$x_0 \preceq x_1 \preceq x_2 \preceq ... \preceq x_n \preceq x_{n+1} \preceq ... \tag{2.7}$$

and

$$y_0 \succeq y_1 \succeq y_2 \succeq ... \succeq y_n \succeq y_{n+1} \succeq ... \tag{2.8}$$

Since $x_n \succeq x_{n-1}$ and $y_n \preceq y_{n-1}$, from (2.1) and (2.2), we have

$$d\left(F\left(x_n, y_n\right), F\left(x_{n-1}, y_{n-1}\right)\right) \leq \alpha d\left(x_n, x_{n-1}\right) + \beta d\left(y_n, y_{n-1}\right)$$

$$+ \; L \min\left\{ \begin{array}{c} d\left(F\left(x_n, y_n\right), x_{n-1}\right), d\left(F\left(x_{n-1}, y_{n-1}\right), x_n\right), \\ d\left(F\left(x_n, y_n\right), x_n\right), d\left(F\left(x_{n-1}, y_{n-1}\right), x_{n-1}\right) \end{array} \right\}$$

or

$$d(x_{n+1}, x_n) \leq \alpha d(x_n, x_{n-1}) + \beta d(y_n, y_{n-1}) \tag{2.9}$$

Similarly, since $y_{n-1} \succeq y_n$ and $x_{n-1} \preceq x_n$, we have

$$d\left(F\left(y_{n-1}, x_{n-1}\right), F\left(y_n, x_n\right)\right) \leq \alpha d\left(y_{n-1}, y_n\right) + \beta d\left(x_{n-1}, x_n\right)$$

$$+ \; L \min\left\{ \begin{array}{c} d\left(F\left(y_{n-1}, x_{n-1}\right), y_n\right), d\left(F\left(y_n, x_n\right), y_{n-1}\right), \\ d\left(F\left(y_{n-1}, x_{n-1}\right), y_{n-1}\right), d\left(F\left(y_n, x_n\right), y_n\right) \end{array} \right\}$$

or

$$d(y_n, y_{n+1}) \leq \alpha d(y_{n-1}, y_n) + \beta d(x_{n-1}, x_n) \tag{2.10}$$

Adding (2.9) and (2.10), we get

$$d(x_{n+1}, x_n) + d(y_{n+1}, y_n) \leq (\alpha + \beta)\left[d(x_n, x_{n-1}) + d(y_n, y_{n-1})\right] \tag{2.11}$$

Set $d_n = d(x_{n+1}, x_n) + d(y_{n+1}, y_n)$ and $\delta = \alpha + \beta < 1$, we have

$$0 \leq d_n \leq \delta d_{n-1} \leq \delta^2 d_{n-2} \leq ... \leq \delta^n d_0$$

which implies

$$\lim_{n\to\infty} [d(x_{n+1}, x_n) + d(y_{n+1}, y_n)] = \lim_{n\to\infty} d_n = 0$$

Thus,

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = \lim_{n\to\infty} d(y_{n+1}, y_n) = 0$$

For each $m \geq n$ we have

$$d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + ... + d(x_{n+1}, x_n)$$

and

$$d(y_m, y_n) \leq d(y_m, y_{m-1}) + d(y_{m-1}, y_{m-2}) + ... + d(y_{n+1}, y_n)$$

Therefore

$$\begin{aligned} d(x_m, x_n) + d(y_m, y_n) &\leq d_{m-1} + d_{m-2} + ... + d_n \\ &\leq \left(\delta^{m-1} + \delta^{m-2} + ... + \delta^n\right) d_0 \\ &\leq \frac{\delta^n}{1-\delta} d_0 \end{aligned} \tag{2.12}$$

which implies that

$$\lim_{n,m\to\infty} [d(x_m, x_n) + d(y_m, y_n)] = 0$$

Therefore, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since X is a complete metric space, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y \tag{2.13}$$

Now, suppose that assumption (a) holds. Taking the limit as $n \to \infty$ in (2.2) and by (2.13), we get

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} F(x_{n-1}, y_{n-1}) = F\left(\lim_{n\to\infty} x_{n-1}, \lim_{n\to\infty} y_{n-1}\right) = F(x, y)$$

and

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} F(y_{n-1}, x_{n-1}) = F\left(\lim_{n\to\infty} y_{n-1}, \lim_{n\to\infty} x_{n-1}\right) = F(y, x)$$

Thus we proved that $x = F(x, y)$ and $y = F(y, x)$.

Finally, suppose that (b) holds. Since $\{x_n\}$ is non-decreasing sequence and $x_n \to x$ and as $\{y_n\}$ is non-increasing sequence and $y_n \to y$, by assumption (b), we have $x_n \succeq x$ and $y_n \preceq y$ for all n. We have

$$\begin{aligned} d(F(x_n, y_n), F(x, y)) &\leq \alpha d(x_n, x) + \beta d(y_n, y) \\ &+ L \min \left\{ \begin{array}{c} d(F(x_n, y_n), x), d(F(x, y), x_n), \\ d(F(x_n, y_n), x_n), d(F(x, y), x) \end{array} \right\} \end{aligned} \tag{2.14}$$

Taking $n \to \infty$ in (2.14) we get $d(x, F(x, y)) \leq 0$ which implies $F(x, y) = x$. Similarly, we can show that $F(y, x) = y$.
Therefore, we proved that F has a coupled fixed point.$\diamondsuit$

Corollary 2.2. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \preceq F(x_0, y_0) \quad \text{and} \quad y_0 \succeq F(y_0, x_0)$$

Suppose there exist non-negative real numbers α, β with $\alpha + \beta < 1$ such that

$$d\left(F\left(x,y\right), F\left(u,v\right)\right) \le \alpha d\left(x,u\right) + \beta d\left(y,v\right)$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. Suppose either
(a) F is continuous or
(b) X has the following property:
(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$, for all n,
(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$, for all n.
then F has a coupled fixed point in X.

Proof. Taking $L = 0$ in Theorem (2.1), we obtain Corollary (2.2).$\Diamond$

Remark 2.3. *In Corollary (2.2), taking $\alpha = \beta$, we get the main results of Bhaskar and Lakshmikantham* [15].

Now we shall prove the uniqueness of coupled fixed point. Note that if $(X, \preceq)$ is a partially ordered set , then we endow the product $X \times X$ with the following partial order relation:

$$\text{for all } (x,y),(u,v) \in (X \times X), \quad (x,y) \preceq (u,v) \Leftrightarrow x \preceq u, y \succeq v.$$

Theorem 2.4. *In addition to hypotheses of Theorem (2.1), suppose that for every $(x, y), (z, t) \in X \times X$, there exists a $(u, v) \in X \times X$ that is comparable to (x, y) and (z, t), then F has a unique coupled fixed point .*

Proof. From Theorem (2.1) the set of coupled fixed points of F is non-empty. Suppose (x, y) and (z, t) are coupled fixed points of F, that is, $x = F(x, y), y = F(y, x), z = F(z, t)$ and $t = F(t, z)$. We shall show that $x = z$ and $y = t$.
By assumption, there exists $(u, v) \in X \times X$ that is comparable to (x, y) and (z, t) .
We define sequences $\{u_n\}, \{v_n\}$ as follows

$$u_0 = u, v_0 = v, u_{n+1} = F(u_n, v_n) \text{ and } v_{n+1} = F(v_n, u_n) \text{ for all } n.$$

Since (u, v) is comparable with (x, y), we may assume that $(x, y) \succeq (u, v) = (u_0, v_0)$.
By using the mathematical induction, it is easy to prove that

$$(x,y) \succeq (u_n, v_n) \text{ for all } n. \tag{2.15}$$

From (2.1) and (2.15), we have

$$\begin{aligned} d(F(x,y), F(u_n,v_n)) \quad &\le \quad \alpha d(x,u_n) + \beta d(y,v_n) \\ &\quad + L \min \left\{ \begin{array}{c} d(F(x,y),u_n), d(F(u_n,v_n),x)), \\ d(F(x,y),x), d(F(u_n,v_n),u_n) \end{array} \right\} \end{aligned}$$

or

$$d(x, u_{n+1}) \le \alpha d(x, u_n) + \beta d(y, v_n) \tag{2.16}$$

Similarly, we also have

$$d(v_{n+1}, y) \le \alpha d(v_n, y) + \beta d(u_n, x) \tag{2.17}$$

Adding (2.16) and (2.17), we get

$$\begin{aligned} d(x,u_{n+1})+d(y,v_{n+1}) &\leq (\alpha+\beta)[d(x,u_n)+d(y,v_n)] \\ &\leq (\alpha+\beta)^2[d(x,u_{n-1})+d(y,v_{n-1})] \\ &\quad ... \\ &\leq (\alpha+\beta)^{n+1}[d(x,u_0)+d(y,v_0)] \end{aligned} \tag{2.18}$$

Taking the limit as $n\to\infty$ in (2.18), we get

$$\lim_{n\to\infty}\left[d\left(x,u_{n+1}\right)+d\left(y,v_{n+1}\right)\right]=0$$

Thus,

$$\lim_{n\to\infty} d\left(x,u_{n+1}\right)=\lim_{n\to\infty} d\left(y,v_{n+1}\right)=0 \tag{2.19}$$

Similarly, one can show that

$$\lim_{n\to\infty} d\left(z,u_{n+1}\right)=\lim_{n\to\infty} d\left(t,v_{n+1}\right)=0 \tag{2.20}$$

From (2.19) and (2.20), we obtain $x=z$ and $y=t$. $\Diamond$

Theorem 2.5. *In addition to hypotheses of Theorem (2.1), if x_0 and y_0 are comparable then F has a fixed point, that is, there exists $x\in X$ such that $x=F(x,x)$.*

Proof. Following the proof of Theorem (2.1), F has a coupled fixed point (x,y) . We only have to show that $x=y$. Since x_0 and y_0 are comparable, we may assume that $x_0\succeq y_0$. By using the mathematical induction, one can show that

$$x_n\succeq y_n \text{ for all } n\geq 0 \tag{2.21}$$

where$\{x_n\}$ and $\{y_n\}$ be defined by (2.2).
From (2.1) and (2.21), we have

$$\begin{aligned} d(F(x_n,y_n),F(y_n,x_n)) &\leq \alpha d(x_n,y_n)+\beta d(y_n,x_n) \\ &\quad +L\min\left\{\begin{array}{c} d(F(x_n,y_n),y_n),d(F(y_n,x_n),x_n)), \\ d(F(x_n,y_n),x_n),d(F(y_n,x_n),y_n) \end{array}\right\} \end{aligned}$$

or

$$\begin{aligned} d(x_{n+1},y_{n+1}) &\leq \alpha d(x_n,y_n)+\beta d(y_n,x_n) \\ &+ L\min\left\{d(x_{n+1},y_n),d(y_{n+1},x_n),d(x_{n+1},x_n),d(y_{n+1},y_n)\right\} \end{aligned}$$

Taking $n\to\infty$ in the above inequality, we have

$$d(x,y)\leq(\alpha+\beta)d(x,y)+L\min\{d(x,y),0\}=(\alpha+\beta)d(x,y)$$

which implies $d(x,y)=0$ (since $\alpha+\beta<1$). Therefore $x=y$, that is, F has a fixed point.$\Diamond$

The following example shows that Theorem 2.1 is indeed a proper extension on Theorem 1.3 and Theorem 1.4.

Example 2.6. *Let $X=[0,1]$ with metric $d(x,y)=|x-y|$, for all $x,y\in X$. On the set X, we consider the following relation:*

$$for\, x,y\in X,\;\; x\preceq y\Leftrightarrow x,y\in\{0,1\} \text{ and } x\leq y,$$

where $\leq$ be usual ordering. Clearly, (X,d) be a complete metric space and $(X,\preceq)$ be a partially orderd set. Moreover, X has the property:

(i) if a non-decreasing sequence $\{x_n\} \to x$, *then* $x_n \preceq x$ *for all* n,

(ii) if a non-increasing sequence $\{y_n\} \to y$, *then* $y \preceq y_n$ *for all* n.

Let $F : X \times X \to X$ *be given by*

$$F(x,y) = \begin{cases} (x-y)/2, & \text{if } x, y \in [0,1], x \geq y, \\ 0, & \text{if } x < y. \end{cases}$$

Obviously, F is continuous and has the mixed monotone property. Also, there are $x_0 = 0, y_0 = 0$ in X such that

$$x_0 = 0 \preceq F(0,0) = F(x_0, y_0) \text{ and } y_0 = 0 \succeq F(0,0) = F(y_0, x_0)$$

Clearly, F has a coupled fixed point that is $(0,0)$. But the condition

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}[d(x,u) + d(y,v)], \forall x \succeq u, y \preceq v,$$

in the Theorem 1.3 and Theorem 1.4 is not true for every $k \in [0,1)$. Indeed, for $x = 1, y = 0, u = 0, v = 0$ and for every $k \in [0,1)$, we have

$$\begin{aligned} \frac{k}{2}[d(x,u) + d(y,v)] &= \frac{k}{2}[d(1,0) + d(1,0)] = \frac{k}{2} \\ &< \frac{1}{2} = d(F(1,0), F(0,)) = d(F(x,y), F(u,v)) \end{aligned}$$

So we can not use the Theorem 1.3 or 1.4 for the mapping F.

Now, we verify the mapping F satisfies the condition (2.1) with $\alpha = 2/3, \beta = 0$ and $L = 2$. We take $x, y, u, v \in X$ such that $x \succeq u, y \preceq v$ or $(x,y) \succeq (u,v)$. We have the following cases:

Case 1. $(x,y) = (u,v)$ or $(x,y) = (0,0), (u,v) = (0,1)$ or $(x,y) = (1,1), (u,v) = (0,1)$, we have $d(F(x,y), F(u,v)) = 0$. Hence (2.1) holds.

Case 2. $(x,y) = (1,0), (u,v) = (0,0)$, we have

$$d(F(x,y), F(u,v)) = d(F(1,0), F(0,0)) = \frac{1}{2} < \frac{2}{3} = \frac{2}{3}.d(1,0) = \alpha d(x,u)$$

Hence (2.1) holds.

Case 3. $(x,y) = (1,0), (u,v) = (0,1)$, we have

$$d(F(x,y), F(u,v)) = d(F(1,0), F(0,1)) = \frac{1}{2} < \frac{2}{3} = \frac{2}{3}.d(1,0) = \alpha d(x,u)$$

Hence (2.1) holds.

Case 4. $(x,y) = (1,0), (u,v) = (1,1)$, we have

$$\begin{aligned} L \min \left\{ \begin{array}{l} d(F(x,y),u), d(F(u,v),x), \\ d(F(x,y),x), d(F(u,v),u) \end{array} \right\} &= 2 \min \left\{ \begin{array}{l} d(F(1,0),1), d(F(1,1),1), \\ d(F(1,0),1), d(F(1,1),1) \end{array} \right\} \\ &= 2 \min\{\frac{1}{2}, 1\} = 1 \\ &> \frac{1}{2} = d(F(1,0), F(1,1)) \\ &= d(F(x,y), F(u,v)) \end{aligned}$$

Hence (2.1) holds.

Therefore, all the conditions of Theorem 2.1 are satisfied. Applying Theorem 2.1 we can conclude that F has a coupled fixed point in X.

Acknowledgement. The authors would like to express their sincere appreciation to the referees for their very helpful suggestions and many kind comments.

References

[1] R.P. Agarwal, M. Meehan, D. ORegan, *Fixed Point Theory and Applications*, Cambridge University Press, 2001.

[2] R.P. Agarwal, M.A. El-Gebeily, D. ORegan, *Generalized contractions in partially ordered metric spaces*, Appl. Anal. 87 (2008) 1-8.

[3] I. Altun, H. Simsek, *Some fixed point theorems on ordered metric spaces and application*, Fixed Point Theory Appl. 2010 (2010) 17 pages. Article ID 621469.

[4] I. Altun, B. Damjanovic, D. Djoric, *Fixed point and common fixed point theorems on ordered cone metric spaces*, Appl. Math. Lett., 23 (2010) 310-316.

[5] A. Amini-Harandi, H. Emami, *A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations*, Nonlinear Analysis, 72 (2010) 2238-2242.

[6] G. V. R . Babu, M. L. Sandhya, and M. V. R, Kameswari, *A note on a fixed point theorem of Berinde on weak contractions*, Carpathian J. Math. 24 (2008), No. 1, 8-12.

[7] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. 3 (1922) 133-181.

[8] I. Beg, A. R. Butt, *Fixed point for set-valued mappings satisfying an implicit relation in partially ordered metric spaces, Nonlinear Analysis*, 71 (2009) 3699-3704.

[9] V. Berinde. *A priori and a posteriori error estimates for a class of ϕ-contractions* Bull. Appl. Computing Math., 90, B (1999), 183-192.

[10] V. Berinde. *Iterative Approximation of Fixed Points*. Editura Efemeride, Baia Mare 2002.

[11] V. Berinde. *Approximating fixed points of weak contractions using Picard iteration*, Nonlinear Analysis Forum 9, 1 (2004), 43-53.

[12] A. Cabada, J.J. Nieto, *Fixed points and approximate solutions for nonlinear operator equations*, J. Comput. Appl. Math. 113 (2000) 17-25.

[13] L. Ciric, N. Caki, M. Rajovi, J.S. Ume, *Monotone generalized nonlinear contractions in partially ordered metric spaces*, Fixed Point Theory Appl. 2008 (2008) 11 pages. Article ID 131294.

[14] Z. Dricia, F. A. McRaeb, J. Vasundhara Devi, *Fixed point theorems in partially ordered metric spaces for operators with PPF dependence*, Nonlinear Analysis, 67 (2007)641-647.

[15] T. Gnana Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. 65 (2006) 1379-1393.

[16] S. Hong, *Fixed points of multivalued operators in ordered metric spaces with applications*, Nonlinear Analysis, 72 (2010) 3929-3942.

[17] J. Harjani, K. Sadarangani, *Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations*, Nonlinear Anal. 72 (2010) 1188-1197.

[18] Z. Kadelburg, M. Pavlovic, S. Radenovic, *Common xed point theorems for ordered contractions and quasicontractions in ordered cone metric spaces*, Computers and Mathematics with Applications (2010), doi:10.1016/j.camwa.2010.02.039.

[19] V. Lakshmikantham, L. Ciric, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal. 70 (2009) 4341-4349.

[20] B. Monjardet, *Metrics on partially ordered sets -A survey*, Discrete Mathematics, 35 (1981) 173-184.

[21] J.J. Nieto, R. Rodriguez-Lopez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equation*, Order 22 (2005) 223-239.

[22] J.J. Nieto, R. Rodriguez-Lopez, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta Math. Sinica, Engl. Ser. 23 (12) (2007) 2205-2212.

[23] A.C.M. Ran, M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc. 132 (2004) 1435-1443.

[24] D. ORegan, A. Petrusel, *Fixed point theorems for generalized contractions in ordered metric spaces*, J. Math. Anal. Appl. 341 (2008) 1241-1252.

[25] Sh. Rezapour, M. Derafshpour, N. Shahzad, *Best proximity points of cyclic φ-contractions in ordered metric spaces*, Topological Methods in Nonlinear Anal., (2010), In Press.

[26] F. Sabetghadam, H. P. Masiha, and A.H. Sanatpour , *Some coupled fixed point theorems in cone metric spaces*, Fixed Point Theory and Applications, Vol 2009 (2009), Article ID 125426, 8 pages.

[27] B. Samet, *Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces*, Nonlinear Anal., 72(2010), 4508-4517.
[28] E. S. Wolk, *Continuous convergence in partially ordered sets*, General Topology and its Appl., 5 (1975) 22l-234.
[29] X. Zhang, *Fixed point theorems of multivalued monotone mappings in ordered metric spaces*, Appl. Math. Lett., 23 (2010) 235-240.

Nguyen Van Luong
Department of Natural Sciences, Hong Duc University, Thanh Hoa, Viet Nam
E-mail address: luongk6ahd04@yahoo.com

Nguyen Xuan Thuan
Department of Natural Sciences, Hong Duc University, Thanh Hoa, Viet Nam
E-mail address: thuannx7@gmail.com