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POLYNOMIAL APPROXIMATION AND INTERPOLATION OF ENTIRE FUNCTIONS OF SLOW GROWTH IN SEVERAL COMPLEX VARIABLES

(DEDICATED IN OCCASION OF THE 70-YEARS OF PROFESSOR HARI M. SRIVASTAVA)

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Abstract. In the present paper a characterization of the generalized order and type of entire functions of several complex variables by means of polynomial approximation and interpolation have been obtained. Our results improve and generalize various results of S.M. Shah [5], M.N. Seremeta [4], Kapoor and Nautiyal [2], Vakarchuk [9], Vakarchuk and Zhir [10], Winiarski ([11],[12]). In this way we summarize and unify the work which has been done on this subject to-date.

1. Introduction

Let C be the complex plane and let C^N be the N-dimensional complex Euclidean space.We have

$$D = \{z \in C : |z| < 1\}$$

to denote the open unit disc in C,and use

$$D^N = \{D \times ... \times D\} = \left\{z = z_1, ..., z_N \in C^N : |z_k| < 1, 1 \leq k \leq N\right\}$$

to denote the polydisc in C^N.Let g be an entire transcendental function in $C^N, N \geq 1$ and

$$S(r, g) = \sup\left\{|g(z)| : |z_1|^2 + \cdots + |z_N|^2 = r^2\right\}, r > 0.$$

be its maximum modulus. The growth of g is measured in terms of its order ρ and type T defined as under

$$\limsup_{r\to+\infty} \frac{\log\log S(r, g)}{\log r} = \rho, \tag{1.1}$$

$$\limsup_{r\to+\infty} \frac{\log S(r, g)}{r^\rho} = T, \tag{1.2}$$

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for $0 < \rho < \infty$. Various authors have given different characterizations for entire functions of fast growth ($\rho = \infty$). M.N.Seremeta[4] and S.M.Shah[5] defined the generalized order and generalized type with the help of general functions as follows.

Let L^0 denote the class of functions h satisfying the following conditions

(i) $h(x)$ is defined on $[d, +\infty)$ strictly increasing, differentiable and tends to ∞ as $x \to \infty$,

(ii) $\lim_{x\to+\infty} \frac{h\{(1+1/\varphi(x))x\}}{h(x)} = 1$

for every function $\varphi(x)$ such that $\varphi(x)$ such that $\varphi(x) \to +\infty$ as $x \to +\infty$.

Let Δ denote the class of functions h satisfying condition (i) and

$$\lim_{x\to+\infty} \frac{h(cx)}{h(x)} = 1 \ \ provided \ \ c > 0.$$

For entire transcendental function $f(z) = \sum_{n=0}^{\infty} \ a_n \ z^n$ and functions $\alpha(x) \in \Delta, \beta(x) \in L^0$, Seremeta [4,Th.1] and Shah[5] proved in a single complex variable that

$$\rho(\alpha, \beta, f) = \limsup_{r\to+\infty} \frac{\alpha[\log S(r,f)]}{\beta(\log r)} = \limsup_{n\to+\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\log|a_n|\right)} \tag{1.3}$$

Further, for $\alpha(x) \in L^0, \beta^{-1}(x) \in L^0, \gamma(x) \in L^0$,

$$T(\alpha, \beta, f) = \limsup_{r\to+\infty} \frac{\alpha[\log S(r,f)]}{\beta[(\gamma(r))^\rho]}$$

$$= \limsup_{n\to+\infty} \frac{\alpha(n/\rho)}{\beta\left\{[\gamma(e^{1/\rho})|a_n|^{-1/n}]^\rho\right\}}. \tag{1.4}$$

where $0 < \rho < \infty$ is a fixed number. If $\alpha(x) = \log x, \beta(x) = x$, we get classical definitions of order and type of an entire function.

Above relations were obtained under the condition

$$\frac{d(\beta^{-1}(c\alpha(x)))}{d(\log x)} \le b, \quad x \ge a, \tag{1.5}$$

where a and b are positive constants. Surprisingly the condition (1.5) does not hold for $\alpha = \beta$. To remove this problem, G.P. Kapoor and A. Nautiyal [2] defined generalized order $\rho(\alpha, \alpha, f)$ of slow growth with the help of general functions as follows.

Let Ω be the class of functions $h(x)$ satisfying (i) and

(iii) there exists a $\delta(x) \in \Omega$ and x_0, K_1 and K_2 such that

$$0 < K_1 \le \frac{d(h(x))}{d(\delta(logx))} \le K_2 < \infty \ \ for \ \ all \ \ x > x_0.$$

Let $\overline{\Omega}$ be the class of functions $h(x)$ satisfying (i) and

(iv) $\lim_{x\to+\infty} \frac{d(h(x))}{d(\log x)} = K, 0 < K < \infty.$

Kapoor and Nautiyal [2] showed that classes Ω and $\overline{\Omega}$ are contained in Δ. Further, $\Omega \bigcap \overline{\Omega} = \phi$ and they defined the generalized order $\rho(\alpha, \alpha, f)$ for entire functions $f(z)$ of slow growth as

$$\rho(\alpha, \alpha, f) = \limsup_{r \to +\infty} \frac{\alpha(\log S(r, f))}{\alpha(\log r)},$$

where $\alpha(x)$ either belongs to Ω or to $\overline{\Omega}$.

The characterization for the generalized order and type by means of polynomial approximation and interpolation to g on a compact subsets of $C^N, N > 1$ for slow growth have been studied by Srivastava and Susheel Kumar [8].It has been noticed that the characterization for approximating entire function g in certain Banach spaces by generalized order and type of slow growth on a compact subsets of C^N have not been studied so far.

The paper is organized as follows. First we define the generalized order and type by means of polynomial approximation and interpolation to g on a compact subsets of C^N for slow growth. Next we obtain necessary and sufficient conditions of generalized order and type of slow growth in certain Banach spaces ($B(p, q, m)$ space, Hardy space and Bergman spaces) introduced by Vakarchuk and Zhir [10]and W.Rudin [3].

We shall assume throughout that function $\alpha \in \overline{\Omega}$.

Let K be a compact set in C^N and let $\|\|_K$ denote the supremum norm on K. The function

$$\phi_K(z) = \sup\{|p(z)|^{1/n}; p - polynomial, \quad deg \quad p \leq n, \|p\|_K \leq 1, n \in N\},$$

$z \in C^N$, is called the Siciak extremal function of the compact $K([6], [7])$.

Given a function f, defined and bounded on K, we set for $n \in N$

$$\begin{aligned} E_n^{(1)}(f, K) &= \inf\{\|f - t_n\|_K\}; \\ E_n^{(2)}(f, K) &= \inf\{\|f - l_n\|_K\}; \\ E_{n+1}^{(3)}(f, K) &= \inf\{\|l_{n+1} - l_n\|_K\}, \end{aligned}$$

where t_n denotes the $n - th$ Chebyshev polynomial of the best approximation to f on K and l_n denotes the $n - th$ Lagrange interpolation polynomial for f with nodes at extremal points of K (see[6],[7]).

Recently Vakarchuk and Zhir [10] studied the approximation of entire functions in Banach spaces in a single complex variable z.Let $H(D^N)$ denote the space of all holomorphic functions in D^N.For any $0 < q < \infty$ the Hardy space $H_q(D^N)$ contains of all $g \in H(D^N)$ such that

$$S_q(r, g) = \left\{ \int_{T^N} |g(r\zeta)|^q \, d\sigma(\zeta) : 0 < r < 1 \right\}^{1/q}, q > 0.$$

where

$$T^N = \left\{ \zeta = \zeta_1, ..., \zeta_N \in C^N : |\zeta_k| = 1, 1 \leq k \leq N \right\}$$

is the distinguished boundary of D^N and

$$d\sigma(\zeta) = \left\{ \frac{|d\zeta_1|...|d\zeta_N|}{(2\pi)^N} \right\}$$

is the normalized Haar measure on T^N.It is well known that for every function $g \in H_q(D^N)$,the radial limit

$$g(\zeta) = \left\{ \lim_{r \to 1^-} g(r\zeta) \right\}$$

exist for allmost every $\zeta \in T^N$.Furthermore,

$$\|g\|_{H_q} = \left\{ \int_{T^N} |g(\zeta)|^q \, d\sigma(\zeta) \right\}^{1/q}.$$

See [3] for more information about Hardy spaces of the polydisc.For $0 < q < \infty$ the Bergman space $H_q^{'}(D^N)$ consists of all functions $g \in H(D^N)$ such that

$$\|g\|_{H_q^{'}} = \left\{ \int_{T^N} |g(z)|^q dA(z_1)...dA(z_N) \right\}^{1/q} < \infty.$$

where

$$dA(z) = \left\{ \frac{dxdy}{\pi} \right\}$$

being normalized area measure on D.

For $q = \infty$, let $\|g\|_{H_\infty^{'}} = \|g\|_{H_\infty} = \sup\{|g(z)|, z \in (D^N)\}$. Then H_q and $H_q^{'}$ are Banach spaces for $q \geq 1$. Following [3,chapter-III], we say that a function $g \in H(D^N)$ belongs to the space $B(p,q,m)$ if

$$\|g\|_{p,q,m} = \sup \left\{ \int_{T^N} (1 - r\zeta)^{m(1/p - 1/q) - 1} S_q^m(r,g) d\sigma(\zeta) : 0 < r < 1 \right\}^{1/m} < \infty,$$

$0 < p < q \leq \infty, 0 < m < \infty$ and

$$\|g\|_{p,q,\infty} = \sup \left\{ (1 - r\zeta)^{1/p - 1/q} S_q(r,g) d\sigma(\zeta) : 0 < r < 1 \right\} < \infty.$$

It can be seen [1] that $B(p,q,m)$ is a Banach space for $p > 0$ and $q, m > 1$, otherwise it is a Frechet space. Further [9],

$$H_q \leq H_q^{'} = B(q/2, q, q), 1 \leq q < \infty.$$

Let χ denote one of the Banach spaces defined above and let for $n \in N$

$$\begin{aligned} E_n^{(1)}(g,\chi) &= \inf \{ \|g - t_n\|_\chi \} \\ E_n^{(2)}(g,\chi) &= \inf \{ \|g - l_n\|_\chi \} \\ E_n^{(3)}(g,\chi) &= \inf \{ \|l_{n+1} - l_n\|_\chi \}. \end{aligned}$$

One form of the generalized order in terms of the errors $E_n^{(1)}(f,\chi)$ was studied in the work Vakarchuk and Zhir [10] ,when $N = 1$ and $\alpha = \beta(\alpha \in \Delta)$. These results do not hold good for $\alpha = \beta = \gamma$. So our results improve and generalize the results of [10] for several complex variables.

Now we define the generalized order $\rho(\alpha,\alpha,g)$ and generalized type $T(\alpha,\alpha,g)$ of an trancedental entire function $g(z), z \in C^N$ as

$$\rho(\alpha,\alpha,g) = \limsup_{r\to+\infty} \frac{\alpha(\log S(r,g))}{\alpha(\log r)}; T(\alpha,\alpha,g) = \limsup_{r\to+\infty} \frac{\alpha(\log S(r,g))}{[\alpha(\log r)]^{\rho}}$$

where $\alpha(x)$ either belongs to Ω or $\overline{\Omega}$.

2. Main Results

Theorem 2.1. Let $\alpha(x) \in \overline{\Omega}$, and $f(z) \in B(p,q,m)$ then $f(z)$is an entire function of generalized order $\rho(\alpha,\alpha,f)$ if, and only if

$$\rho(\alpha,\alpha,f) \equiv \rho = \limsup_{n\to+\infty} \frac{\alpha(n)}{\alpha\left(\log\left(\left|E_n^{(s)}(B(p,q,m):f)\right|\right)^{-1/n}\right)}, s = 1,2,3. \tag{2.1}$$

Proof. First we prove the result for $q = 2, 0 < p < 2, m \geq 1$ and $s = 1$. Let $f \in B(p,q,m)$ be of generalized order ρ. Then in the consequence of[8 ,Th. 2.1] we can easily get

$$\limsup_{n\to+\infty} \frac{\alpha(n)}{\alpha\left(\log[\|p_n\|_K]^{-1/n}\right)} = \rho. \tag{2.2}$$

Then for given $\varepsilon > 0$, and all $n > m = m(\varepsilon)$, we have

$$\|p_n\|_K \leq \exp\left\{-n\left(\alpha^{-1}\left(\frac{1}{\rho+\varepsilon}\alpha(n)\right)\right)\right\}. \tag{2.3}$$

Let $g_n(f) = \sum_{j=0}^{n} p_j$ be the n^{th} partial sum of sequence of polynomials. Following [10,p.1396] with $|p_n(z)| \leq \|p_n\|_K \phi_K^n(z) z \in C^N$, we obtain

$$E_n^{(1)}(B(p,2,m);f) \leq B^{1/m}[(n+1)m+1; m(1/p-1/2)]\left\{\sum_{j=n+1}^{\infty}[\|p_j\|_K]^{1/2}\right\} \tag{2.4}$$

where $B(a,b)(a,b > 0)$ denotes the beta function. By using (2.3), we have

$$E_n^{(1)}(B(p,2,m);f) \leq \frac{B^{1/m}[(n+1)m+1; m(1/p-1/2)]}{\exp\left\{(n+1)\left(\alpha^{-1}\left(\frac{\alpha(n+1)}{\rho}\right)\right)\right\}}\left\{\sum_{j=n+1}^{\infty}\varphi_j^2(\alpha)\right\}^{1/2}, \tag{2.5}$$

where

$$\varphi_j(\alpha) \cong \frac{\exp\left\{(n+1)\left(\alpha^{-1}\left(\frac{\alpha(n+1)}{\rho+\varepsilon}\right)\right)\right\}}{\exp\left\{j\left(\alpha^{-1}\left(\frac{\alpha(j)}{\rho+\varepsilon}\right)\right)\right\}}.$$

Set

$$\varphi(\alpha) \cong \exp\left\{-\alpha^{-1}\left(\frac{\alpha(1)}{\rho+\varepsilon}\right)\right\}.$$

Since $\alpha(x)$ is increasing and $j \geq n+1$, we get

$$\varphi_j(\alpha) \leq \exp\left\{((n+1)-j)\left[\alpha^{-1}\left(\frac{\alpha(n+1)}{\rho+\varepsilon}\right)\right]\right\} \leq \varphi_{j-(n+1)}(\alpha). \tag{2.6}$$

Since $\varphi(\alpha) < 1$, we get from (2.5) and (2.6),

$$E_n^{(1)}(B(p,2,m);f) \leq \frac{B^{1/m}[(n+1)m+1;m(1/p-1/2)]}{(1-\varphi^2(\alpha))^{1/2}\left[\exp\left\{n\alpha^{-1}\left(\frac{\alpha(n+1)}{\rho+\varepsilon}\right)\right\}\right]} \tag{2.7}$$

For $n \geq n_0$, (2.7) gives

$$\rho+\varepsilon \geq \frac{\alpha(n+1)}{\alpha\left((1+1/n)^{-1}\left\{\log\left(|E_n^{(1)}(B;f)\Big|\right)^{-1/n} + \log\left(\frac{B^{1/m}[(n+1)m+1;m(1/p-1/2)]}{(1-\varphi^2(\alpha))^{1/2}}\right)^{1/n}\right\}\right)}.$$

Now

$$B[(n+1)m+1;m(1/p-1/2)] = \frac{\Gamma((n+1)m+1)\Gamma(m(1/p-1/2))}{\Gamma((n+1/2+1/p)m+1)}.$$

Hence

$$B[(n+1)m+1;m(1/p-1/2)] \simeq \frac{e^{-[(n+1)m+1]}[(n+1)m+1]^{(n+1)m+3}\Gamma(1/p-1/2)}{e^{[(n+1/2+1/p)m+1]}[(n+1/2+1/p)m+1]^{(n+1/2+1/p)k+3/2}}.$$

Thus

$$B[(n+1)m+1;m(1/p-1/2)]^{1/n+1} \cong 1. \tag{2.8}$$

Proceeding to limits we get

$$\rho \geq \limsup_{n\to+\infty} \frac{\alpha(n)}{\alpha\left(-\frac{1}{n}\log E_n^{(1)}(B(p,2,m);f)\right)}. \tag{2.9}$$

For reverse inequality we may use

$$\|p_{n+1}\|_K B^{1/m}[(n+1)m+1;m(1/p-1/2)] \leq E_n^{(1)}(B(p,2,m);f).$$

Then for sufficiently large n, we have

$$\begin{aligned}\frac{\alpha(n)}{\alpha\left(\log\left(|E_n^{(1)}(B;f)\Big|\right)^{-1/n}\right)} &\geq \frac{\alpha(n)}{\alpha\left\{\log[\|p_{n+1}\|_K]^{-1/n} + \log(B^{-1/nm}[(n+1)m+1;m(1/p-1/2)])\right\}}\\ &\geq \frac{\alpha(n)}{\alpha\left\{\log[\|p_n\|_K]^{1/n} + \log\left(B^{-1/nm}[(n+1)m+1;m(1/p-1/2)]\right)\right\}}.\end{aligned}$$

Applying limits with (2.2) we get

$$\limsup_{n\to+\infty} \frac{\alpha(n)}{\alpha\left(\log\left(|E_n^{(1)}(B(p,2,m);f)\Big|\right)^{-1/n}\right)} \geq \rho. \tag{2.10}$$

Inequalities (2.9) and (2.10) together yields

$$\rho = \limsup_{n\to+\infty} \frac{\alpha(n)}{\alpha\left(\log\left(|E_n^{(1)}(B(p,2,m);f)\Big|\right)^{-1/n}\right)}. \tag{2.11}$$

Now we consider the spaces $B(p,q,m)$ for $0 < p < q, q \neq 2$, and $q,m \geq 1$. Gvaradze [1] showed that, for $p \geq p_1, q \leq q_1$, and $m \leq m_1$, if at least one of the inequalities is strict, them inclusion $B(p,q,m) \subset B(p_1,q_1,m_1)$ holds and the following relation is true;

$$\|f\|_{p_1,q_1,m_1} \leq 2^{1/q-1/q_1}[m(1/p-1/q)]^{1/m-1/m_1}\|f\|_{p,q,m}.$$

For any function f defined and bounded on K and in $B(p,q,m)$ the last inequalities gives

$$E_n^{(1)}(B(p,q,m);f) \leq 2^{1/q-1/q_1}[m(1/p-1/q)]^{1/m-1/m_1}E_n^{(1)}(B(p,q,m);f). \quad (2.12)$$

For general case $B(p,q,m), q \neq 2$, we prove the necessity of the condition (2.3).

Let $f(z)$ be defined and bounded on K (compact set in C^N), $z \in C^N$ and $f(z) \in B(p,q,m)$ be an entire transcendental function having finite generalized order defined by (2.2). Using the relation (2.3), for $n > n_0$ we estimate the value of the best polynomial approximation by using the fact that if $K \subset D^N = \{z \in C^N : |z_1|^2 + \cdots + |z_N|^2 \leq 1\}$ then $E_n^{(1)} \leq E_n^{(1)}(g, D^N)$. Hence

$$E_n^{(1)}(B(p,q,m);f) = \|f - g_n(f)\|_{p,q,m} \leq \sup\left(\int_{T^N}(1-r\zeta)^{(m(1/p-1/q)-1)}S_q^m d\sigma(\zeta)\right)^{1/m}.$$

Now

$$|f|^q = \left|\sum_{n=0}^{\infty} p_n\right|^q \leq \left(\sum_{n=0}^{\infty}|p_n|\right)^q \leq \left(\sum_{k=n+1}^{\infty}\|p_k\|r^{k+1}\right)^q.$$

Thus

$$\begin{aligned} E_n^{(1)}(B(p,q,m);f) &\leq B^{1/m}[(n+1)m+1;m(1/p-1/q)]\sum_{k=n+1}^{\infty}\|p_k\| \\ &\leq \frac{B^{1/m}[(n+1)m+1;m(1/p-1/q)]}{(1-\varphi(\alpha))\left[\exp\left\{n\alpha^{-1}\left(\frac{\alpha(n+1)}{\rho+\varepsilon}\right)\right\}\right]}. \end{aligned} \quad (2.13)$$

For $n > n_0$, (2.13) yields

$$\rho + \varepsilon \geq \frac{\alpha(n+1)}{\alpha\left((1+1/m)^{-1}\left\{\log\left(\left|E_n^{(1)}(B;f)\right|\right)^{-1/n} + \log\left(\frac{B^{1/m}[(n+1)m+1;m(1/p-1/2)]}{1-\varphi(\alpha)}\right)^{1/n}\right\}\right)}.$$

Since $\varphi(\alpha) < 1$, and $\alpha \in \overline{\Omega}$, proceeding to limits and using (2.8) we get

$$\rho \geq \limsup_{n\to+\infty}\frac{\alpha(n)}{\alpha\left(\log\left(\left|E_n^{(1)}(B(p,q,m);f)\right|\right)^{-1/n}\right)}. \quad (2.14)$$

To prove the reverse inequality, let us suppose that $0 < p < q < 2$ and $m, q \geq 1$. By (2.11), where $p_1 = p, q_1 = 2$, and $m_1 = m$, and the condition (2.3) is already proved for the space $B(p,2,m)$, we get

$$\limsup_{n\to+\infty}\frac{\alpha(n)}{\alpha\left(\log\left(\left|E_n^{(n)}(B(p,2,m);f)\right|\right)^{-1/n}\right)} \geq \limsup_{n\to\infty}\frac{\alpha(n)}{\alpha\left(\log\left(\left|E_n^{(1)}(B(p,2,m);f)\right|\right)^{-1/n}\right)} = \rho.$$

Now for $0 < p \leq 2 < q$ we have

$$S_2(r,f) \leq S_q(r,f), 0 < r < 1,$$

therefore

$$E_n^{(1)}(B(p,q,m);f) \geq \|p_{n+1}\|B^{1/m}[(n+1)m+1;m(1/p-1/q)]. \quad (2.15)$$

Then for sufficiently large n, we have

$$\frac{\alpha(n)}{\alpha\left(\log\left|E_n^{(1)}(B(p,q,m);f)\right)\right|^{-1/n}} \geq \frac{\alpha(n)}{\alpha\left(\log\left(\|p_{n+1}\|^{-1/n}+\log\left(B^{-1/mn}\left[(n+1)m+1;m\left(\frac{1}{p}-\frac{1}{q}\right)\right]\right)\right)\right)}$$

By applying limits and from (2.2), we get

$$\limsup_{n\to+\infty}\frac{\alpha(n)}{\alpha\left(\log\left(|E_n^{(1)}(B(p,q,m);f)\right|\right))^{-1/n}} \geq \limsup_{n\to+\infty}\frac{\alpha(n)}{\alpha\left(\log\left[\|p_n\|\right]^{-1/n}\right)}=\rho.$$

Now we assume that $2\leq p<q$. Set $q_1=q, m_1=m$, and $0<p_1<2$ in the inequality (2.11), where p_1 is an arbitrary fixed number. Substituting p_1 for p n (2.14), we get

$$E_n^{(1)}(B(p,q,m);f)\geq\|p_{n+1}\|B^{1/m}[(n+1)m+1;m(1/p_1-1/q)]. \tag{2.16}$$

Using (2.16) and applying the same method as in the previous case $0<p\leq 2<q$, for sufficiently large n, we obtain

$$\begin{aligned}
&\frac{\alpha(n)}{\alpha\left(\log\left(|E_n^{(1)}(B(p,q,m);f)\right|\right)^{-1/n}\right)}\\
\geq\ &\frac{\alpha(n)}{\alpha\left(\log[\|p_{n+1}\|]^{-1/n}+\log\left(B^{-1/nm}\left[(n+1)m+1;m(1/p_1-1/q)\right]\right)\right)}\\
\geq\ &\frac{\alpha(n)}{\alpha\left(\log[\|p_n\|]^{-1/n}+\log\left(B^{-1/nm}\left[(n+1)m+1;m(1/p_1-1/q)\right]\right)\right)}.
\end{aligned}$$

By taking limits and using (2.2), we get

$$\limsup_{n\to+\infty}\frac{\alpha(n)}{\alpha\left(\log\left(|E_n^{(1)}(B(p,q,m);f)\right|\right)^{-1/n}\right)}\geq\rho.$$

From inequalities (2.9),(2.10) with above inequality, the required relation (2.11) can be obtained. Hence the proof is completed for $s=1$. For $s=2,3$ the theorem can be easily proved using the same technique as[8 ,Th. 2.1].

Theorem 2.2. Let $\alpha(x)\in\overline{\Omega}$, then a necessary and sufficient condition for an entire function $f(z)\in B(p,q,m)$. Then $f(z)$ is an entire function of generalized type T having finite generalized order $\rho, 1<\rho<\infty$ if,and only if

$$T=\limsup_{n\to+\infty}\frac{\alpha(n/\rho)}{\left[\alpha\left\{\frac{\rho}{\rho-1}\log\left(|\left(E_n^{(s)}(B(p,q,m);f)\right)\right|\right)^{-1/n}\right\}\right]^{(\rho-1)}}. \tag{2.17}$$

Proof. The proof can be easily obtained following the lines of Theorem 2.1 after some mechanical work.

Now we prove

Theorem 2.3. Assuming that the conditions of Theorem 2.1 are satisfied and $\xi(\alpha)$ is a positive number.Then $f(z)\in H_q$,is an entire function of generalized order ρ

if,and only if

$$\limsup_{n\to+\infty} \frac{\alpha(n)}{\alpha\left(\log\left(|E_n^{(s)}(H_q,f)\Big|)^{-1/n}\right)} = \xi(\alpha,\alpha,g). \tag{2.18}$$

Proof. Let $f(z) = \sum_{n=0}^{\infty} p_n$ be an entire function having finite generalized order. $f(z) \in B(p,q,m)$, where $0 < p < q \le \infty$ and $q,m \ge 1$. From [9] we get $E_n((q/2,q,q);f) \le \zeta_q E_n(h_q;f), 1 \le q < \infty$, where ζ_q is a constant independent of n and f. In the case of Hardy space H_∞

$$E_n^{(s)}(B(p,\infty,\infty);f) \le E_n^{(s)}(H_\infty;f), 1 < p < \infty. \tag{2.19}$$

Since

$$\begin{aligned}
\xi(\alpha,\alpha,g) &= \limsup_{n\to+\infty} \frac{\alpha(n)}{\alpha\left(\log(|E_n^{(s)}(H_q;f)|)^{-1/n}\right)} \\
&\ge \limsup_{n\to+\infty} \frac{\alpha(n)}{\alpha\left(\log(|E_n^{(s)}(B(q/2,q,q);f)|)^{-1/n}\right)} \\
&\ge \rho, 1 \le q < \infty. \qquad (2.20)
\end{aligned}$$

The inequality (2.20) can be proved for $q = \infty$ by (2.19). For the reverse inequality

$$\xi(\alpha,\alpha,g) \le \rho, \tag{2.21}$$

we use the relation (2.3), which is valid for $n > n_0$, and we can estimate from above, the generalized order ρ as follows.

$$\begin{aligned}
E_n^{(1)}(H_q;f) &\le \|f - g_n\|_{H_q} \\
&\le \sum_{j=n+1}^{\infty} |p_j|
\end{aligned}$$

$$\le \exp\left(-(n+1)\alpha^{-1}\left(\frac{\alpha(n+1)}{\rho+\varepsilon}\right)\right) \sum_{j=n+1}^{\infty} \varphi_j(\alpha).$$

Using (2.6),

$$\begin{aligned}
E_n^{(1)}(H_q;f) &\le \|f-g_n\|_{H_q} \\
&\le \frac{1}{(1-\varphi(alpha))\left[\exp\left((n+1)\alpha^{-1}\left(\frac{\alpha(n+1)}{|\rho+\varepsilon}\right)\right)\right]}
\end{aligned}$$

or

$$\frac{1}{E_n^{(1)}(H_q;f)} \ge (1-\varphi(\alpha))\exp\left\{(n+1)\alpha^{-1}\left(\frac{\alpha(n+1)}{\rho+\varepsilon}\right)\right\}.$$

This yields

$$\rho+\varepsilon \ge \frac{\alpha(n+1)}{\alpha\left(\log(|E_n^{(1)}(H_q;f)|)^{-1/n+1} + \log\left((1-\varphi(alpha))^{-1/n+1}\right)\right)}.$$

Using the fact that $\varphi(\alpha) < 1$ and passing to the limits as $n \to +\infty$ with the properties of α, we get the inequality (2.21). Thus we have

$$\xi(\alpha,\alpha,g) = \rho$$

for $s = 1$. The theorem can be proved easily for $s = 2, 3$. So we omit the proof.

Theorem 2.4. Let $\alpha(x) \in \overline{\Omega}$, and $f \in H_q$.Then $f(z)$ is an entire function of generalized type T^* if,and only if

$$\limsup_{n\to+\infty} \frac{\alpha(n/\rho)}{\left[\alpha\left\{\frac{\rho}{\rho-1}\log\left(\left|\left(E_n^{(s)}(H_q;f)\right)\right|\right)^{-1/n}\right\}\right]^{\rho-1}} = T^*(\alpha,\alpha,f).$$

Proof. This theorem also can be proved by using some inequalities of Theorem 2.2 after a simple calculation.

Remark 1: An analog of Theorem 2.4 for the Bergmann spaces for $N = 1$ follows from [5] $1 \leq q < \infty$ and from Theorem 2.1 for $q = \infty$.

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