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L^1-CONVERGENCE OF THE $r-th$ DERIVATIVE OF CERTAIN COSINE SERIES WITH r-QUASI CONVEX COEFFICIENTS

(DEDICATED IN OCCASION OF THE 70-YEARS OF PROFESSOR HARI M. SRIVASTAVA)

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Abstract. We study L^1-convergence of $r-th$ derivative of modified cosine sums introduced in [3]. Exactly it is proved the L_1- convergence of $r-th$ derivative of modified cosine sums with r-quasi convex coefficients.

1. Introduction

It is well known that if a trigonometric series converges in L_1-metric to a function $f \in L_1$, then it is the Fourier series of the function f. Riesz [2] gave a counter example showing that in a metric space L_1 we cannot expect the converse of the above said result to hold true. This motivated the various authors to study L_1-convergence of the trigonometric series. During their investigations some authors introduced modified trigonometric sums as these sums approximate their limits better than the classical trigonometric series in the sense that they converge in L_1-metric to the sum of the trigonometric series whereas the classical series itself may not. In this contest we will show L_1-convergence of the $r-th$ derivative of certain cosine series with r-quasi convex coefficients by this modified cosine series(given in [3]) by relation

$$N_n^{(3)}(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^6}\sum_{k=1}^{n}\sum_{j=k}^{n}\left(\Delta^5 a_{j-3} - \Delta^5 a_{j-2}\right)\cos kx - \frac{a_1(15 - 6\cos x + \cos 2x)}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_2(6-\cos x)}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_3}{\left(2\sin\frac{x}{2}\right)^6} \tag{1.1}$$

In the sequel we will briefly describe the notation and definitions which are used throughout the paper. Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{1.2}$$

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be cosine trigonometric series with its partial sums denoted by

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx,$$

and let $g(x) = \lim_{n\to\infty} S_n(x)$.

Definition 1.1. *A sequence of scalars (a_n) is said to be quasi semi-convex if $a_n \to 0$ as $n \to \infty$, and*

$$\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1} - \Delta^2 a_n| < \infty, (a_0 = 0, a_{-1} = 0, a_{-2} = 0), \tag{1.3}$$

where $\Delta^2 a_n = \Delta a_n - \Delta a_{n+1}, \Delta a_n = a_n - a_{n+1}$.

Definition 1.2. *A sequence of scalars (a_n) is said to be r-quasi convex if $a_n \to 0$ as $n \to \infty$, and*

$$\sum_{n=1}^{\infty} n^{r+1}|\Delta a_{n-1} - \Delta a_n| < \infty, (a_0 = 0, a_{-1} = 0, a_{-2} = 0, r \geq 0). \tag{1.4}$$

As usually with $D_n(x)$ and $\tilde{D}_n(x)$ we will denote the Dirichlet and its conjugate kernels defined by

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx, \quad \tilde{D}_n(x) = \sum_{k=1}^{n} \sin kx.$$

In [3] is studied the L^1-convergence of modified cosine sums given by relation (1) with third semi-convex coefficients proving the following theorem.

Theorem 1.3. *Let (a_n) be a third semi-convex null sequence, then $N_n^{(3)}(x)$ converges to $g(x)$ in L^1-norm.*

The main goal of the present work is to study the L^1-convergence of $r-th$ derivative of these new modified cosine sums with r-quasi convex null coefficients and to deduce the sufficient condition of Theorem 1.3 as corollaries. We point out here that a lot of authors investigated the L_1-convergence of the series (2), see for example [1],[8],[4],[5],[6],[7].

Everywhere in this paper the constants in the O-expression denote positive constants and they may be different in different relations. All other notations are like as in [9], [2].

2. Preliminaries

To prove the main results we need the following lemmas:

Lemma 2.1. *If $x \in [\epsilon, \pi], \epsilon > 0$ and $m \in \mathbb{N}$, then the following estimate holds*

$$\left| \left(\frac{\tilde{D}_m(x)}{2\sin^6 \frac{x}{2}} \right)^{(r)} \right| = O_{r,\epsilon}\left(m^{r+1}\right), \quad (r = 0, 1, 2 \dots)$$

where $O_{r,\epsilon}$ depends only on r and ϵ.

Proof. By Leibniz formula we have

$$\begin{aligned}\frac{\tilde{D}_m(x)}{2\sin^6\frac{x}{2}}\Bigg)^{(r)} &= \sum_{i=0}^{r}\binom{r}{i}\quad\frac{1}{2\sin^6\frac{x}{2}}\Bigg)^{(r-i)}\left(\tilde{D}_m(x)\right)^{(i)}\\ &= \sum_{i=0}^{r}\binom{r}{i}\quad\frac{1}{2\sin^6\frac{x}{2}}\Bigg)^{(r-i)}\sum_{j=1}^{m}j^i\sin\left(jx+\frac{i\pi}{2}\right)\\ &= O(1)m^{r+1}\sum_{i=0}^{r}\binom{r}{i}\quad\frac{1}{2\sin^6\frac{x}{2}}\Bigg)^{(r-i)}. \qquad (2.1)\end{aligned}$$

Using mathematical induction we will prove the equality $\left(\frac{1}{2\sin^6\frac{x}{2}}\right)^{(r)}=\frac{P_r(\cos\frac{x}{2})}{\sin^{r+6}\frac{x}{2}}$, where P_r is a cosine polynomial of degree r.

Namely, we have $\left(\frac{1}{2\sin^6\frac{x}{2}}\right)^{'}=-\frac{1}{2}\frac{3\cdot\sin^5\frac{x}{2}\cos\frac{x}{2}}{\sin^{12}\frac{x}{2}}=\frac{P_1(\cos\frac{x}{2})}{\sin^7\frac{x}{2}}$, so that for $r=1$ the above equality is true.

Assume that the equality $F(x):=\left(\frac{1}{2\sin^6\frac{x}{2}}\right)^{(r)}=\frac{P_r(\cos\frac{x}{2})}{\sin^{r+6}\frac{x}{2}}$ holds. For the $(r+1)-th$ derivative of $\frac{1}{2\sin^6\frac{x}{2}}$ we get

$$\begin{aligned}&F^{'}(x):=\\ =\ &\frac{P_r^{'}(\cos\frac{x}{2})\left(-\frac{1}{2}\sin^{r+7}\frac{x}{2}\right)-P_r(\cos\frac{x}{2})(r+6)\sin^{r+5}\frac{x}{2}\cdot\frac{1}{2}\cdot\cos\frac{x}{2}}{\sin^{2r+12}\frac{x}{2}}\\ =\ &\frac{-\frac{1}{2}\sin^{r+5}\frac{x}{2}\left[P_r^{'}(\cos\frac{x}{2})\sin^2\frac{x}{2}+P_r(\cos\frac{x}{2})(r+6)\cos\frac{x}{2}\right]}{\sin^{r+5}\frac{x}{2}\cdot\sin^{r+7}\frac{x}{2}}\\ =\ &\frac{1}{2}\frac{-P_r^{'}(\cos\frac{x}{2})\frac{1}{2}(1-\cos x)-P_r(\cos\frac{x}{2})(r+6)\cos\frac{x}{2}}{\sin^{r+7}\frac{x}{2}}\\ =\ &\frac{1}{2}\frac{-\frac{1}{2}P_r^{'}(\cos\frac{x}{2})+\cos x\cdot\frac{1}{2}P_r^{'}(\cos\frac{x}{2})-P_r(\cos\frac{x}{2})(r+6)\cos\frac{x}{2}}{\sin^{r+7}\frac{x}{2}}\\ =\ &\frac{1}{2}\frac{(-1/2)H_{r-1}(\cos\frac{x}{2})+(1/2)H_{r-1}(\cos\frac{x}{2})\cos x-(r+6)P_r(\cos\frac{x}{2})\cos\frac{x}{2}}{\sin^{r+7}\frac{x}{2}}\\ =\ &\frac{1}{2}\frac{Q_{r+1}(\cos\frac{x}{2})-(r+6)R_{r+1}(\cos\frac{x}{2})}{\sin^{r+7}\frac{x}{2}}=\frac{T_{r+1}(\cos\frac{x}{2})}{\sin^{r+7}\frac{x}{2}}, \qquad (2.2)\end{aligned}$$

where H_{r-1}, Q_{r+1}, R_{r+1}, T_{r+1} are cosine polynomials of degree $r-1$ and $r+1$ respectively. Therefore for $x\in[\epsilon,\pi]$, $\epsilon>0$, from (5) and (6) we obtain

$$\left|\quad\frac{\tilde{D}_m(x)}{2\sin^6\frac{x}{2}}\Bigg)^{(r)}\right| = O(1)m^{r+1}\sum_{i=0}^{r}\binom{r}{i}\frac{\left|P_{r-i}(\cos\frac{x}{2})\right|}{\sin^{r-i+6}\frac{x}{2}}=O_{r,\epsilon}\left(m^{r+1}\right).$$

□

Lemma 2.2. *If $x\in[\epsilon,\pi],\epsilon>0$ and $m\in\mathbb{N}$, then the following estimate holds*

$$\left|\quad\frac{D_m(x)}{2\sin^6\frac{x}{2}}\Bigg)^{(r)}\right|=O_{r,\epsilon}\left(m^{r+1}\right),\quad (r=0,1,2\ldots),$$

where $O_{r,\epsilon}$ depends only on r and ϵ.

Proof is similar to the Lemma 2.1.

3. Results

In what follows we prove the main result of the paper:

Theorem 3.1. *Let (a_n) be a r-quasi convex null sequence, then $\left(N_n^{(3)}(x)\right)^{(r)}$ converges to $g^{(r)}(x)$ in L^1-norm.*

Proof. From definition of $S_n(x)$ we have:

$$S_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^6}\sum_{k=1}^{n}\Delta^6 a_{k-3}\cos kx + \frac{a_{-2}\cos x}{\left(2\sin\frac{x}{2}\right)^6} +$$
$$\frac{a_{-1}\cos 2x}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_0}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_{n-2}\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_{n-1}\cos(n+2)x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_n\cos(n+3)x}{\left(2\sin\frac{x}{2}\right)^6} -$$
$$\frac{6a_{-1}\cos x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{6a_0\cos 2x}{\left(2\sin\frac{x}{2}\right)^6} + \frac{6a_{n-1}\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^6} + \frac{6a_n\cos(n+2)x}{\left(2\sin\frac{x}{2}\right)^6} + \frac{15a_0}{\left(2\sin\frac{x}{2}\right)^6} -$$
$$\frac{15a_n\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{15a_1}{\left(2\sin\frac{x}{2}\right)^6} + \frac{15a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^6} + \frac{6a_1\cos x}{\left(2\sin\frac{x}{2}\right)^6} + \frac{6a_2}{\left(2\sin\frac{x}{2}\right)^6} -$$
$$\frac{6a_{n+1}\cos(n-1)x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{6a_{n+2}\cos nx}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_1\cos 2x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_2\cos x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_3}{\left(2\sin\frac{x}{2}\right)^6} +$$
$$\frac{a_{n+1}\cos(n-2)x}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_{n+2}\cos(n-1)x}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_{n+3}\cos nx}{\left(2\sin\frac{x}{2}\right)^6}$$

Applying Abel's transformation, we obtain

$$S_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^6}\cdot\sum_{k=1}^{n-1}(\Delta^5 a_{k-3} - \Delta^5 a_{k-2})\widetilde{D}_k(x) + \frac{(\Delta^5 a_{n-3} - \Delta^5 a_{n-2})\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^4} -$$
$$\frac{a_{n-2}\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_{n-1}\cos(n+2)x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_n\cos(n+3)x}{\left(2\sin\frac{x}{2}\right)^6} + \frac{6a_{n-1}\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^6} +$$
$$\frac{6a_n\cos(n+2)x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{15a_n\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{15a_1}{\left(2\sin\frac{x}{2}\right)^6} + \frac{15a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^6} + \frac{6a_1\cos x}{\left(2\sin\frac{x}{2}\right)^6} + \frac{6a_2}{\left(2\sin\frac{x}{2}\right)^6} -$$
$$\frac{6a_{n+1}\cos(n-1)x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{6a_{n+2}\cos nx}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_1\cos 2x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_2\cos x}{\left(2\sin\frac{x}{2}\right)^6} - \frac{a_3}{\left(2\sin\frac{x}{2}\right)^6} +$$
$$\frac{a_{n+1}\cos(n-2)x}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_{n+2}\cos(n-1)x}{\left(2\sin\frac{x}{2}\right)^6} + \frac{a_{n+3}\cos nx}{\left(2\sin\frac{x}{2}\right)^6}$$

Therefore

$$S_n^{(r)}(x) = -\frac{1}{\left(2\sin\frac{x}{4}\right)^6}\cdot\sum_{k=1}^{n-1}(\Delta^5 a_{k-3} - \Delta^5 a_{k-2})\widetilde{D}_k^{(r)}(x) + (\Delta^5 a_{n-3} - \Delta^5 a_{n-2})\cdot \left.\frac{\widetilde{D}_n(x)}{2\sin^6\frac{x}{2}}\right)^{(r)}$$

$$-a_{n-2}\left.\frac{\cos(n+1)x}{2\sin^6\frac{x}{2}}\right)^{(r)}-a_{n-1}\left.\frac{\cos(n+2)x}{2\sin^6\frac{x}{2}}\right)^{(r)}-a_n\left.\frac{\cos(n+3)x}{2\sin^6\frac{x}{2}}\right)^{(r)}+6a_{n-1}\left.\frac{\cos(n+1)x}{2\sin^6\frac{x}{2}}\right)^{(r)}+$$

$$6a_n\left.\frac{\cos(n+2)x}{2\sin^6\frac{x}{2}}\right)^{(r)}-15a_n\left.\frac{\cos(n+1)x}{2\sin^6\frac{x}{2}}\right)^{(r)}-15a_1\left.\frac{1}{2\sin^6\frac{x}{2}}\right)^{(r)}+$$

$$+15a_{n+1}\left.\frac{\cos nx}{2\sin^6\frac{x}{2}}\right)^{(r)}+6a_1\left.\frac{\cos x}{2\sin^6\frac{x}{2}}\right)^{(r)}+6a_2\left.\frac{1}{2\sin^6\frac{x}{2}}\right)^{(r)}-6a_{n+1}\left.\frac{\cos(n-1)x}{2\sin^6\frac{x}{2}}\right)^{(r)}$$

$$-6a_{n+2}\left.\frac{\cos nx}{2\sin^6\frac{x}{2}}\right)^{(r)}-a_1\left.\frac{\cos 2x}{2\sin^6\frac{x}{2}}\right)^{(r)}-a_2\left.\frac{\cos x}{2\sin^6\frac{x}{2}}\right)^{(r)}-a_3\left.\frac{1}{2\sin^6\frac{x}{2}}\right)^{(r)}+$$

$$a_{n+1}\left.\frac{\cos(n-2)x}{2\sin^6\frac{x}{2}}\right)^{(r)}+a_{n+2}\left.\frac{\cos(n-1)x}{2\sin^6\frac{x}{2}}\right)^{(r)}+a_{n+3}\left.\frac{\cos nx}{2\sin^6\frac{x}{2}}\right)^{(r)}. \quad (3.1)$$

On the other hand we have:

$$N_n^{(3)}(x)=-\frac{1}{\left(2\sin\frac{x}{2}\right)^6}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta^5 a_{j-3}-\Delta^5 a_{j-2})\cos kx-$$
$$\frac{a_1(15-6\cos x+\cos 2x)}{\left(2\sin\frac{x}{2}\right)^6}+\frac{a_2(6-\cos x)}{\left(2\sin\frac{x}{2}\right)^6}-\frac{a_3}{\left(2\sin\frac{x}{2}\right)^6},$$

respectively

$$N_n^{(3)}(x)=-\frac{1}{\left(2\sin\frac{x}{2}\right)^6}\sum_{k=1}^{n}\Delta^5 a_{k-3}\cos kx+\frac{\Delta^5 a_{n-2}\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^6}-$$
$$\frac{a_1(15-6\cos x+\cos 2x)}{\left(2\sin\frac{x}{2}\right)^6}+\frac{a_2(6-\cos x)}{\left(2\sin\frac{x}{2}\right)^6}-\frac{a_3}{\left(2\sin\frac{x}{2}\right)^6}.$$

Now applying Abel's transformation we get the following relation:

$$N_n^{(3)}(x)=-\frac{1}{\left(2\sin\frac{x}{2}\right)^6}\sum_{k=1}^{n-1}(\Delta^5 a_{k-3}-\Delta^5 a_{k-2})\widetilde{D}_k(x)+\frac{\Delta^5 a_{n-3}\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^6}+\frac{\Delta^5 a_{n-2}\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^6}-$$
$$\frac{a_1(15-6\cos x+\cos 2x)}{\left(2\sin\frac{x}{2}\right)^6}+\frac{a_2(6-\cos x)}{\left(2\sin\frac{x}{2}\right)^6}-\frac{a_3}{\left(2\sin\frac{x}{2}\right)^6}, \quad (3.2)$$

and the r-derivative of the relation (8) is the following relation:

$$\left(N_n^{(3)}(x)\right)^{(r)}=-\frac{1}{\left(2\sin\frac{x}{2}\right)^6}\sum_{k=1}^{n-1}(\Delta^5 a_{k-3}-\Delta^5 a_{k-2})\widetilde{D}_k^{(r)}(x)+\Delta^5 a_{n-3}\left.\frac{\widetilde{D}_n(x)}{2\sin^6\frac{x}{2}}\right)^{(r)}+$$

$$\Delta^5 a_{n-2}\left.\frac{\widetilde{D}_n(x)}{2\sin^6\frac{x}{2}}\right)^{(r)}-a_1\left.\frac{15-6\cos x+\cos 2x}{2\sin^6\frac{x}{2}}\right)^{(r)}+a_2\left.\frac{6-\cos x}{2\sin^6\frac{x}{2}}\right)^{(r)}-a_3\left.\frac{1}{2\sin^6\frac{x}{2}}\right)^{(r)}. \quad (3.3)$$

By Lemma 2.1 and since (a_n) are r-quasi-convex null sequences, we obtain:

$$2\left|\left(\Delta^5 a_{n-2}\right) \quad \frac{\tilde{D}_n(x)}{2\sin^6\frac{x}{2}}\right)^{(r)}\right| = O_{r,\epsilon}\left(\left|n^{r+1}\left(\Delta^5 a_{n-2}\right)\right|\right) = O_{r,\epsilon} \quad \left|n^{r+1}\sum_{k=n}^{\infty}\left(\Delta^5 a_{k-2} - \Delta^5 a_{k-1}\right)\right|\right)$$

$$= O_{r,\epsilon} \quad \sum_{k=n}^{\infty} k^{r+1}|\Delta^5 a_{k-2} - \Delta^5 a_{k-1}|\right) = O_{r,\epsilon} \quad \sum_{k=n}^{\infty} k^{r+1}|\Delta a_{k-2} - \Delta a_{k-1}|\right) +$$

$$4O_{r,\epsilon} \quad \sum_{k=n}^{\infty} k^{r+1}|\Delta a_{k-1} - \Delta a_{k}|\right) + 6O_{r,\epsilon} \quad \sum_{k=n}^{\infty} k^{r+1}|\Delta a_{k} - \Delta a_{k+1}|\right) +$$

$$+4O_{r,\epsilon} \quad \sum_{k=n}^{\infty} k^{r+1}|\Delta a_{k+1} - \Delta a_{k+2}|\right) + O_{r,\epsilon} \quad \sum_{k=n}^{\infty} k^{r+1}|\Delta a_{k+2} - \Delta a_{k+3}|\right) = o(1), n \to \infty. \tag{3.4}$$

After some calculations and by virtue of the Lemma 2.1 we obtain

$$a_n \cdot \quad \frac{\cos(n+3)(x)}{2\sin^6\frac{x}{2}}\right)^{(r)} - a_{n+3} \cdot \quad \frac{\cos nx}{2\sin^6\frac{x}{2}}\right)^{(r)} =$$

$$= a_n\left[\quad \frac{D_{n+3}(x)}{2\sin^6\frac{x}{2}}\right)^{(r)} - \quad \frac{D_{n+2}(x)}{2\sin^6\frac{x}{2}}\right)^{(r)}\right] - a_{n+3}\left[\quad \frac{D_n(x)}{2\sin^6\frac{x}{2}}\right)^{(r)} - \quad \frac{D_{(n-1)}(x)}{2\sin^6\frac{x}{2}}\right)^{(r)}\right]$$

$$= a_n O_{r,\epsilon}\left((n+3)^{r+1} - (n+2)^{r+1}\right) - a_{n+3} O_{r,\epsilon}\left(n^{r+1} - (n-1)^{r+1}\right) =$$

$$(a_n - a_{n+3})O_{r,\epsilon}(n^r) = O_{r,\epsilon} \quad n^r\sum_{k=n}^{\infty}\left((\Delta^2 a_k - \Delta^2 a_{k+1}) - (\Delta^2 a_{k+1} - \Delta^2 a_{k+2})\right) + 3\sum_{k=n}^{\infty}\left(\Delta a_{k+1} - \Delta a_{k+2}\right)\right)$$

$$= O_{r,\epsilon} \quad \sum_{k=n}^{\infty} k^{r+1}|\Delta a_{k} - \Delta a_{k+1}|\right) + 2O_{r,\epsilon} \quad \sum_{k=n}^{\infty} k^{r+1}|\Delta a_{k+1} - \Delta a_{k+2}|\right) +$$

$$O_{r,\epsilon} \quad \sum_{k=n}^{\infty} k^{r+1}|\Delta a_{k+2} - \Delta a_{k+3}|\right) + 3O_{r,\epsilon} \quad \sum_{k=n}^{\infty} k^{r+1}|\Delta a_{k+1} - \Delta a_{k+2}|\right)$$

$$= o(1) + 2o(1) + o(1) + 3o(1) = o(1), n \to \infty. \tag{3.5}$$

In the same way we can conclude that two other expressions

$$a_{n-1} \cdot \quad \frac{\cos(n+2)(x)}{2\sin^6\frac{x}{2}}\right)^{(r)} - a_{n+2} \cdot \quad \frac{\cos(n-1)x}{2\sin^6\frac{x}{2}}\right)^{(r)},$$

$$a_{n-2} \cdot \quad \frac{\cos(n+1)(x)}{2\sin^6\frac{x}{2}}\right)^{(r)} - a_{n+1} \cdot \quad \frac{\cos(n-2)x}{2\sin^6\frac{x}{2}}\right)^{(r)}$$

are $o(1)$, where $n \to \infty$. In what follows we will estimate this expressions:

$$6a_{n-1} \cdot \quad \frac{\cos(n+1)(x)}{2\sin^6\frac{x}{2}}\right)^{(r)} - 6a_{n+1} \cdot \quad \frac{\cos(n-1)x}{2\sin^6\frac{x}{2}}\right)^{(r)},$$

$$6a_n\cdot\left(\frac{\cos{(n+2)(x)}}{2\sin^6\frac{x}{2}}\right)^{(r)}-6a_{n+2}\cdot\left(\frac{\cos nx}{2\sin^6\frac{x}{2}}\right)^{(r)}$$

and

$$15a_n\cdot\left(\frac{\cos{(n+1)(x)}}{2\sin^6\frac{x}{2}}\right)^{(r)}-15a_{n+1}\cdot\left(\frac{\cos nx}{2\sin^6\frac{x}{2}}\right)^{(r)}.$$

Let us start from the first expression

$$6a_{n-1}\cdot\left(\frac{\cos{(n+1)(x)}}{2\sin^6\frac{x}{2}}\right)^{(r)}-6a_{n+1}\cdot\left(\frac{\cos{(n-1)x}}{2\sin^6\frac{x}{2}}\right)^{(r)}=$$

$$=6(a_{n-1}-a_{n+1})O_{r,\epsilon}\left(n^r\right)=6O_{r,\epsilon}\left(n^r\sum_{k=n}^{\infty}\left((\Delta a_{k-1}+\Delta a_k)-(\Delta a_k+\Delta a_{k+1})\right)\right)$$

$$=6O_{r,\epsilon}\left(\sum_{k=n}^{\infty}k^{r+1}|\Delta a_{k-1}-\Delta a_k|\right)+6O_{r,\epsilon}\left(\sum_{k=n}^{\infty}k^{r+1}|\Delta a_k-\Delta a_{k+1}|\right)$$

$$=o(1)+o(1)=o(1),n\to\infty. \tag{3.6}$$

In the same way we can conclude that

$$6a_n\cdot\left(\frac{\cos{(n+2)(x)}}{2\sin^6\frac{x}{2}}\right)^{(r)}-6a_{n+2}\cdot\left(\frac{\cos nx}{2\sin^6\frac{x}{2}}\right)^{(r)}=o(1),n\to\infty$$

And now we are able to estimate the last expression

$$15a_n\cdot\left(\frac{\cos{(n+1)(x)}}{2\sin^6\frac{x}{2}}\right)^{(r)}-15a_{n+1}\cdot\left(\frac{\cos nx}{2\sin^6\frac{x}{2}}\right)^{(r)}=$$

$$=15a_n\left[\left(\frac{D_{n+1}(x)}{2\sin^6\frac{x}{2}}\right)^{(r)}-\left(\frac{D_n(x)}{2\sin^6\frac{x}{2}}\right)^{(r)}\right]-15$$

$$a_{n+1}\left[\left(\frac{D_n(x)}{2\sin^6\frac{x}{2}}\right)^{(r)}-\left(\frac{D_{n-1}(x)}{2\sin^6\frac{x}{2}}\right)^{(r)}\right]$$

$$=15a_n\cdot O_{r,\epsilon}\left((n+1)^{r+1}-n^{r+1}\right)-15a_{n+1}\cdot O_{r,\epsilon}\left(n^{r+1}-(n-1)^{r+1}\right)$$

$$=15O_{r,\epsilon}\left(n^r\left(a_n-a_{n+1}\right)\right)=15O_{r,\epsilon}\left(n^r\sum_{k=n}^{\infty}\left(\Delta a_k-\Delta a_{k+1}\right)\right)$$

$$=15O_{r,\epsilon}\left(\sum_{k=n}^{\infty}k^r|\Delta a_k-\Delta a_{k+1}|\right)=o(1),n\to\infty, \tag{3.7}$$

Therefore

$$g^{(r)}(x)=\lim_{n\to\infty}\left(N_n^{(3)}(x)\right)^{(r)}=\lim_{n\to\infty}S_n^{(r)}(x)=$$

$$= -\frac{1}{\left(2\sin\frac{x}{2}\right)^6}\sum_{k=1}^{\infty}(\Delta^5 a_{k-3}-\Delta^5 a_{k-2})\widetilde{D}_k^{(r)}(x)$$

$$- a_1\left(\frac{15-6\cos x+\cos 2x}{2\sin^6\frac{x}{2}}\right)^{(r)} + a_2\left(\frac{6-\cos x}{2\sin^6\frac{x}{2}}\right)^{(r)} - a_3\left(\frac{1}{2\sin^6\frac{x}{2}}\right)^{(r)}. \quad (3.8)$$

Using Lemma 2.1, relations (10), (11), (12), (13) and (14) we get the following estimation:

$$\begin{aligned}
&\int_{-\pi}^{\pi}\left|g^{(r)}(x)-\left(N_n^{(3)}(x)\right)^{(r)}\right|dx \\
&= \int_0^{\pi}\sum_{k=n}^{\infty}\left|\Delta^5 a_{k-3}-\Delta^5 a_{k-2}\right|\left|\left(\frac{\tilde{D}_k(x)}{2\sin^6\frac{x}{2}}\right)^{(r)}\right|dx \\
&= O_{r,\epsilon}\left(\sum_{k=n}^{\infty}k^{r+1}\left|\Delta^5 a_{k-3}-\Delta^5 a_{k-2}\right|\right) = o(1), n\to\infty,
\end{aligned}$$

which proves the theorem. □

Corollary 3.2. *Let (a_n) be a r-quasi convex null sequence, then the necessary and sufficient condition for L^1-convergence of the $r-th$ derivative of the series (2) is $n^{r+1}|a_n| = o(1)$, as $n\to\infty$.*

Proof. We have

$$\left\|g^{(r)}(x)-\left(N_n^{(3)}(x)\right)^{(r)}\right\| \leq \left\|g^{(r)}(x)-S_n^{(r)}(x)\right\| + \left\|S_n^{(r)}(x)-\left(N_n^{(3)}(x)\right)^{(r)}\right\|$$

$$= o(1)+2\left\|(\Delta^5 a_{n-2})\left(\frac{\tilde{D}_n(x)}{2\sin^6\frac{x}{2}}\right)^{(r)}\right\| + \left\|a_n\cdot\left(\frac{\cos(n+3)(x)}{2\sin^6\frac{x}{2}}\right)^{(r)} - a_{n+3}\cdot\left(\frac{\cos nx}{2\sin^6\frac{x}{2}}\right)^{(r)}\right\|$$

$$+\left\|a_{n-1}\cdot\left(\frac{\cos(n+2)(x)}{2\sin^6\frac{x}{2}}\right)^{(r)} - a_{n+2}\cdot\left(\frac{\cos(n-1)x}{2\sin^6\frac{x}{2}}\right)^{(r)}\right\| +$$

$$\left\|a_{n-2}\cdot\left(\frac{\cos(n+1)(x)}{2\sin^6\frac{x}{2}}\right)^{(r)} - a_{n+1}\cdot\left(\frac{\cos(n-2)x}{2\sin^6\frac{x}{2}}\right)^{(r)}\right\| +$$

$$\left\|6a_{n-1}\cdot\left(\frac{\cos(n+1)(x)}{2\sin^6\frac{x}{2}}\right)^{(r)} - 6a_{n+1}\cdot\left(\frac{\cos(n-1)x}{2\sin^6\frac{x}{2}}\right)^{(r)}\right\| +$$

$$\left\|6a_n\cdot\left(\frac{\cos(n+2)(x)}{2\sin^6\frac{x}{2}}\right)^{(r)} - 6a_{n+2}\cdot\left(\frac{\cos nx}{2\sin^6\frac{x}{2}}\right)^{(r)}\right\|$$

$$+\left\|15a_n\cdot\left(\frac{\cos(n+1)(x)}{2\sin^6\frac{x}{2}}\right)^{(r)} - 15a_{n+1}\cdot\left(\frac{\cos nx}{2\sin^6\frac{x}{2}}\right)^{(r)}\right\|$$

In similar way like as in proof of the Theorem 3.1 (relations (11), (12) and (13))we get that all the above summands are $o(1)$, respectively

$$\left\|g^{(r)}(x) - \left(N_n^{(3)}(x)\right)^{(r)}\right\| = o(1),$$

where $n \to \infty$. □

From Theorem 3.1 and Corollary 3.2 we deduce the following corollaries ($r = 0$):

Corollary 3.3. *If (a_n) is 0-quasi convex null sequence of scalars, then the necessary and sufficient condition for L_1-convergence of the cosine series (2) is* $\lim_{n\to\infty} n \cdot |a_n| = 0$.

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