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# UNIQUE SOLUTIONS FOR SYSTEMS OF FRACTIONAL ORDER DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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ZHENYU GUO, MIN LIU

Abstract. Two systems of fractional order functional and neutral functional differential equations with infinite delay are studied in this article, and some sufficient conditions for existence of unique solutions for these two systems are established by Banach fixed point theorem.

## 1. Introduction and preliminaries

This article is concerned with the existence of unique solutions of initial value problems (IVP for short) of a system of fractional order differential equations with infinite delay. Consider the IVP of the form

$$\begin{aligned} D^\alpha y_1(t) &= f_1[t, y_{1t}, y_{2t}], \quad t \in [0, b], \\ y_1(t) &= \phi_1(t), \quad t \in (-\infty, 0], \\ D^\alpha y_2(t) &= f_2[t, y_{1t}, y_{2t}], \quad t \in [0, b], \\ y_2(t) &= \phi_2(t), \quad t \in (-\infty, 0], \end{aligned}$$

which may be rewritten as

$$\begin{aligned} D^\alpha y_i(t) &= f_i[t, y_{1t}, y_{2t}], \quad t \in [0, b], \\ y_i(t) &= \phi_i(t), \quad t \in (-\infty, 0], \end{aligned} \tag{1}$$

where $0 < \alpha < 1, D^\alpha$ is the standard Riemman-Liouville fractional derivative, $f_i : [0, b] \times B^2 \to \mathbb{R}, \phi_i \in B, \phi_i(0) = 0$, $i \in \{1, 2\}$ and $B$ is a phase space. In the theory of functional differential equations, a usual choice of the phase space $B$ is a seminormed space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$ satisfying the following fundamental axioms which were introduced by Hale and Kato [3].

(A) If $y : (-\infty, b] \to \mathbb{R}$, and $y_0 \in B$, then for all $t \in [0, b]$ the following conditions hold:

(i) $y_t$ is in $B$,

(ii) $\|y_t\|_B \le K(t) \sup\{|y(s)| : 0 \le s \le t\} + M(t)\|y_0\|_B$,

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(iii) $|y(t)| \leq H\|y_t\|_B$,

where $H \geq 0$ is a constant, $K : [0, b] \to [0, +\infty)$ is continuous, $M : [0, +\infty) \to [0, +\infty)$ is locally bounded

and $H, K, M$ are independent of $y(\cdot)$.

(A-1) For the function $y(\cdot)$ in (A), $y_t$ is a $B$-valued continuous function on [0,b].

(A-2) The space $B$ is complete.

For any function $y$ defined on $(-\infty, b]$ and any $t \in [0, b]$, $y_t$ is the element of $B$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in (-\infty, 0].$$

For details, see the book by Hino et al.[4].

At last, we will consider the IVP of the following system of neutral fractional functional differential equations

$$\begin{aligned} D^\alpha[y_1(t) - g_1(t, y_{1t})] &= f_1[t, y_{1t}, y_{2t}], \quad t \in [0, b], \\ y_1(t) &= \phi_1(t), \quad t \in (-\infty, 0], \\ D^\alpha[y_2(t) - g_2(t, y_{2t})] &= f_2[t, y_{1t}, y_{2t}], \quad t \in [0, b], \\ y_2(t) &= \phi_2(t), \quad t \in (-\infty, 0], \end{aligned}$$

which may be rewritten as

$$\begin{aligned} D^\alpha[y_i(t) - g_i(t, y_{it})] &= f_i[t, y_{1t}, y_{2t}], \quad t \in [0, b], \\ y_i(t) &= \phi_i(t), \quad t \in (-\infty, 0], \end{aligned} \tag{2}$$

where $\alpha, f_i, \phi_i$ are as in IVP (1), $g_i : [0, b] \times B \to \mathbb{R}$ is a given function such that $g_i(0, \phi_i) = 0$, and $i \in \{1, 2\}$.

Recently, fractional differential equations have been of great interest. For detailed discussion on this topic, refer to the monographs of Miller and Ross [5], Podlubny [6], and the papers by Benchohra et al.[1], Delboso and Rodino [2], Yu and Gao [7] and the references therein.

Applying Banach fixed point theorem, we obtain a few sufficient conditions for the existence of solutions of the systems (1) and (2).

The following notations, definitions, and preliminary facts will be used throughout this paper.

Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$, $C^0(\mathbb{R}^+)$ be the space of all continuous functions on $\mathbb{R}^+$, $C^0(\mathbb{R}_0^+)$ be the space of all continuous real functions on $\mathbb{R}_0^+ = \{x \in \mathbb{R} : x \geq 0\}$ with the class of all $f \in C^0(\mathbb{R}^+)$ such that $\lim_{t \to 0^+} f(t) = f(0^+) \in \mathbb{R}$, and $C([0, b], \mathbb{R})$ be the Banach space of all continuous functions from $[0, b]$ into $\mathbb{R}$ with the norm $\|y\|_\infty := \sup\{|y(t)| : t \in [0, b]\}$.

**Definition 1.1.** [1] *The fractional primitive of order $\alpha > 0$ of a function $h : \mathbb{R}^+ \to \mathbb{R}$ of order $\alpha \in \mathbb{R}^+$ is defined by*

$$I_0^\alpha h(t) = \int_0^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} h(s) ds,$$

*provided the right side exists pointwise on $\mathbb{R}^+$. $\Gamma$ is the gamma function. For instance, $I^\alpha h$ exists for all $\alpha > 0$, when $h \in C^0(\mathbb{R}^+) \bigcap L^1_{loc}(\mathbb{R}^+)$. If $h \in C^0(\mathbb{R}_0^+)$, then $I^\alpha h \in C^0(\mathbb{R}_0^+)$, and moreover $I^\alpha h(0) = 0$.*

**Definition 1.2.** [1] *The fractional derivative of order $\alpha > 0$ of a continuous function $h : \mathbb{R}^+ \to \mathbb{R}$ is given by*

$$\frac{d^\alpha h(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} h(s) ds = \frac{d}{dt} I_a^{1-\alpha} h(t).$$

**Lemma 1.3.** [2] *If $0 < \alpha < 1$, $h : (0, b] \to \mathbb{R}$ is continuous and $\lim_{t \to 0^+} h(t) = h(0^+) \in \mathbb{R}$, then $y$ is a solution of the fractional integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds,$$

*if and only if $y$ is a solution of the IVP for the fractional differential equation*

$$\begin{aligned} D^\alpha y(t) &= h(t), \quad t \in (0, b], \\ y(0) &= 0. \end{aligned}$$

Define the space

$$\begin{aligned} \Omega = \big\{ y = (y_1, y_2) : (-\infty, b] \to \mathbb{R}^2 \big| y_i|_{(-\infty,0]} \in B, \\ y_i|_{[0,b]} \text{ is continuous, and } i \in \{1,2\} \big\}. \end{aligned}$$

**Definition 1.4.** *A vector function $y \in \Omega$ is said to be a solution of (1) if $y_1$ and $y_2$ satisfy the equations $D^\alpha y_1(t) = f_1[t, y_{1t}, y_{2t}]$ and $D^\alpha y_2(t) = f_2[t, y_{1t}, y_{2t}]$ on $[0, b]$ and the conditions $y_1(t) = \phi_1(t)$ and $y_2(t) = \phi_2(t)$ on $(-\infty, 0]$.*

**Definition 1.5.** *A vector function $y \in \Omega$ is said to be a solution of (2) if $y_1$ and $y_2$ satisfy the equations $D^\alpha[y_1(t) - g_1(t, y_{1t})] = f_1[t, y_{1t}, y_{2t}]$ and $D^\alpha[y_2(t) - g_2(t, y_{2t})] = f_2[t, y_{1t}, y_{2t}]$ on $[0, b]$ and the conditions $y_1(t) = \phi_1(t)$ and $y_2(t) = \phi_2(t)$ on $(-\infty, 0]$.*

## 2. Existence of a solution

In this section, a few sufficient conditions of existence of unique solutions for system (1) and (2) will be given.

**Theorem 2.1.** *Assume*
*$(C)$ there exist $l_1, l_2 > 0$ such that*

$$|f_i(t, u_1, v_1) - f_i(t, u_2, v_2)| \le l_i \max\{\|u_1 - u_2\|_B, \|v_1 - v_2\|_B\}, \quad \forall t \in [0, b], u_i, v_i \in B, i \in \{1, 2\}.$$

*If $b^\alpha l \max\{K_{1b}, K_{2b}\} < \Gamma(\alpha + 1)$, where*

$$l = \max\{l_1, l_2\}, \quad K_{ib} = \sup\{|K_i(t)| : t \in [0, b]\},$$

*then there exists a unique solution for the IVP (1) on $(-\infty, b]$.*

*Proof.* In this proof, $i \in \{1, 2\}$. Define a mapping $N = (N_1, N_2) : \Omega \to \Omega$ by

$$(N_i y_i)(t) = \begin{cases} \phi_i(t), & t \in (-\infty, 0], \\ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_i(s, y_{1s}, y_{2s}) ds, & t \in [0, b]. \end{cases}$$

Let $x(\cdot) = \big(x_1(\cdot), x_2(\cdot)\big) : (-\infty, b] \to \mathbb{R}^2$ be the vector function defined by

$$(x_i)(t) = \begin{cases} \phi_i(t), & t \in (-\infty, 0], \\ 0, & t \in [0, b]. \end{cases}$$

Then, $x_{i0}(\theta) = x_i(0+\theta) = x_i(\theta) = \phi_i(\theta)$, $\forall \theta \in (-\infty, 0]$, which is $x_{i0} = \phi_i$. Let $C([t_0, +\infty), \mathbb{R}^2)$ be the set of all continuous vector functions $z(\cdot) = \big(z_1(\cdot), z_2(\cdot)\big)$ with $z(0) = (0,0)$. Define a vector function $\overline{z} = (\overline{z}_1, \overline{z}_2)$ by

$$(\overline{z}_i)(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ z_i(t), & t \in [0, b]. \end{cases}$$

Suppose $y_i(\cdot)$ satisfy the integral equations

$$y_i(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_i(s, y_{1s}, y_{2s}) ds.$$

$y_i(\cdot)$ can be decomposed as $y_i(t) = \overline{z}_i(t) + x_i(t)$, $t \in [0, b]$, which imply $y_{it} = \overline{z}_{it} + x_{it}$, $t \in [0, b]$. Then the functions $z_i(\cdot)$ satisfy

$$z_i(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_i(s, \overline{z}_{1s} + x_{1s}, \overline{z}_{2s} + x_{2s}) ds.$$

Set

$$C_0 = \big\{ z \in C([t_0, +\infty), \mathbb{R}^2) : z_0 = (z_{10}, z_{20}) = (0,0) \big\}$$

with the seminorm $\|\cdot\|_b$ defined by

$$\begin{aligned} \|z\|_b &= \|z_0\|_B + \sup\{|z_1(t)|, |z_2(t)| : t \in [0, b]\} \\ &= \sup\{|z_1(t)|, |z_2(t)| : t \in [0, b]\}, \quad z \in C_0. \end{aligned}$$

Obviously, $C_0$ is a Banach space. Now, define a mapping $P = (P_1, P_2) : C_0 \to C_0$ by

$$(P_i z_i)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_i(s, \overline{z}_{1s} + x_{1s}, \overline{z}_{2s} + x_{2s}) ds, \quad t \in [0, b].$$

That the mapping $N$ has a fixed point is equivalent to that $P$ has a fixed point. So we transform the IVP (1) to that $P$ has a fixed point. Now, prove that $P : C_0 \to C_0$ is a contraction mapping. Actually, for $z, z^* \in C_0$, $t \in [0, b]$, we have

$$\begin{aligned} \big|P_i(z_i)(t) - P_i(z_i^*)(t)\big| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big| f_i(s, \overline{z}_{1s} + x_{1s}, \overline{z}_{2s} + x_{2s}) \\ &\qquad - f_i(s, \overline{z}^*_{1s} + x_{1s}, \overline{z}^*_{2s} + x_{2s}) \big| ds \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} l_i \max\{\|\overline{z}_{1s} - \overline{z}^*_{1s}\|_B, \|\overline{z}_{2s} - \overline{z}^*_{2s}\|_B\} ds \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} l \max\{K_1(t) \sup_{s \in [0,t]} |z_1(s) - z_1^*(s)|, \\ &\qquad K_2(t) \sup_{s \in [0,t]} |z_2(s) - z_2^*(s)|\} ds \\ &\le \frac{l}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \max\{K_{1b}, K_{2b}\} ds \|z - z^*\|_b \\ &\le \frac{l \max\{K_{1b}, K_{2b}\}}{\Gamma(\alpha)} \cdot \frac{t^\alpha}{\alpha} \|z - z^*\|_b, \end{aligned}$$

which imply

$$\|P(z) - P(z^*)\|_b \le \frac{b^\alpha l \max\{K_{1t}, K_{2t}\}}{\Gamma(\alpha+1)} \|z - z^*\|_b < \|z - z^*\|_b.$$

Therefore, $P$ has a unique fixed point by Banach contraction principle. This completes the proof. □

**Theorem 2.2.** *Assume* $(C)$ *holds and there exist* $c_1, c_2 \geq 0$ *such that*

$$|g_i(t,u) - g_i(t,v)| \leq c_i \|u - v\|_B, \quad \forall t \in [0,b], u, v \in B, i \in \{1,2\}.$$

*If* $[c\Gamma(\alpha+1) + b^\alpha l]\max\{K_{1b}, K_{2b}\} < \Gamma(\alpha+1)$, *where*

$$c = \max\{c_1, c_2\}, \quad l = \max\{l_1, l_2\}, \quad K_{ib} = \sup\{|K_i(t)| : t \in [0,b]\},$$

*then there exists a unique solution for the IVP* $(2)$ *on* $(-\infty, b]$.

*Proof.* In this proof, $i \in \{1,2\}$. Define a mapping $\widetilde{N} = (\widetilde{N}_1, \widetilde{N}_2) : \Omega \to \Omega$ by

$$(\widetilde{N}_i y_i)(t) = \begin{cases} \phi_i(t), & t \in (-\infty, 0], \\ g_i(t, y_{it}) \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_i(s, y_{1s}, y_{2s}) ds, & t \in [0,b]. \end{cases}$$

Similar to Theorem 2.1, consider the mapping $\widetilde{P} = (\widetilde{P}_1, \widetilde{P}_2) : C_0 \to C_0$ defined by

$$(\widetilde{P}_i z_i)(t) = g_i(t, \overline{z}_{it} + x_{it}) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_i(s, \overline{z}_{1s} + x_{1s}, \overline{z}_{2s} + x_{2s}) ds, \quad t \in [0,b].$$

We shall prove that $\widetilde{P}$ is a contraction mapping. Indeed, for $z, z^* \in C_0$, $t \in [0,b]$, we have

$$\begin{aligned} \left|\widetilde{P}_i(z_i)(t) - \widetilde{P}_i(z_i^*)(t)\right| &\leq \left|g_i(t, \overline{z}_{it} + x_{it}) - g_i(t, \overline{z}_{it}^* + x_{it})\right| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big| f_i(s, \overline{z}_{1s} + x_{1s}, \overline{z}_{2s} + x_{2s}) \\ &\qquad - f_i(s, \overline{z}_{1s}^* + x_{1s}, \overline{z}_{2s}^* + x_{2s}) \big| ds \\ &\leq c_i \|\overline{z}_{it} - \overline{z}_{it}^*\|_B + \frac{b^\alpha l \max\{K_{1t}, K_{2t}\}}{\Gamma(\alpha+1)} \|z - z^*\|_b \\ &\leq c K_i(t) \sup_{s \in [0,t]} |z_i(s) - z_i^*(s)| + \frac{b^\alpha l \max\{K_{1t}, K_{2t}\}}{\Gamma(\alpha+1)} \|z - z^*\|_b \\ &\leq c \max\{K_{1t}, K_{2t}\} \|z - z^*\|_b + \frac{b^\alpha l \max\{K_{1t}, K_{2t}\}}{\Gamma(\alpha+1)} \|z - z^*\|_b, \end{aligned}$$

which imply

$$\|\widetilde{P}(z) - \widetilde{P}(z^*)\|_b \leq \max\{K_{1t}, K_{2t}\} \Big( c + \frac{b^\alpha l}{\Gamma(\alpha+1)} \Big) \|z - z^*\|_b < \|z - z^*\|_b.$$

Therefore, $\widetilde{P}$ has a unique fixed point by Banach contraction principle. This completes the proof. □

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Zhenyu Guo
School of Sciences, Liaoning Shihua University, Postal 113001, Fushun, Liaoning, China
*E-mail address*: guozy@163.com

Min Liu
School of Sciences, Liaoning Shihua University, Postal 113001, Fushun, Liaoning, China
*E-mail address*: min_liu@yeah.net