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MAXIMUM TERM AND LOWER ORDER OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

(COMMUNICATED BY VIJAY GUPTA)

SUSHEEL KUMAR AND G. S. SRIVASTAVA

Abstract. In the present paper, we study the growth properties of entire functions of several complex variables. The characterizations of lower order of entire functions of several complex variables have been obtained in terms of their Taylor's series coefficients. Also we have obtained some inequalities between order, type, maximum term and central index of entire functions of several complex variables.

1. Introduction

We denote complex $N-$space by C^N. Thus, $z \in C^N$ means that $z = (z_1, z_2, ..., z_N)$, where $z_1, z_2, ..., z_N$ are complex numbers. A function $f(z)$, $z \in C^N$ is said to be analytic at a point $\xi \in C^N$ if it can be expanded in some neighborhood of ξ as an absolutely convergent power series. If we assume $\xi = (0, 0, ..., 0)$, then $f(z)$ has representation

$$f(z) = \sum_{|k|=0}^{\infty} a_{k_1,k_2,...,k_N} \, z_1^{k_1} \, z_2^{k_2} ... \, z_N^{k_N} = \sum_{n=0}^{\infty} a_k \, z^k \,, \tag{1.1}$$

where $k = (k_1, k_2, ..., k_N) \in \mathbb{N}_0^N$ and $n = |k| = k_1 + k_2 + ... + k_N$. For $r > 0$, the maximum modulus $M(r, f)$ of entire function $f(z)$ is given by (see [1], p.321)

$$M(r) = M(r, f) = \sup\{|f(z)| \ : \ |z_1|^2 + |z_2|^2 + ... + |z_N|^2 = r^2\} \,.$$

For $r > 0$, the maximum term $\mu(r)$ of entire function $f(z)$ is defined as (see [2] and [3])

$$\mu(r) = \mu(r, f) = \max_{n \geq 0}\{||a_k||r^n\}.$$

Also the index k with maximal length n for which maximum term is achieved is called the central index and is denoted by $\nu(r) = \nu(r, f) = k$.

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Following Valiron ([9], p. 31), for $0 < r_0 < r$ we have

$$\log \mu(r) = \log \mu(r_0) + \int_{r_0}^{r} \frac{|\nu(t)|}{t}\, dt\,. \tag{1.2}$$

Krishna ([3], Thm. 3.2) proved that if $f(z)$ is an entire function of finite order, then

$$\log \mu(r) \simeq \log M(r).$$

The order ρ of entire function $f(z)$ is defined as [10]

$$\rho = \lim_{r\to\infty} \sup \frac{\log\log M(r)}{\log r}. \tag{1.3}$$

Further, for $0 < \rho < \infty$, the type T of entire function $f(z)$ is defined as [10]

$$T = \lim_{r\to\infty} \sup \frac{\log M(r)}{r^{\rho}}. \tag{1.4}$$

For an entire function $f(z)$, we define the lower order λ of $f(z)$ as

$$\lambda = \lim_{r\to\infty} \inf \frac{\log\log M(r)}{\log r}. \tag{1.5}$$

Further, for $0 < \rho < \infty$, we define the lower type t of entire function $f(z)$ as

$$t = \lim_{r\to\infty} \inf \frac{\log M(r)}{r^{\rho}}. \tag{1.6}$$

We define the order $\rho\,(0 < \rho < \infty)$ and the lower order $\lambda\,(0 < \lambda < \infty)$ of entire function $f(z)$ in terms of central index as

$$\begin{matrix}\rho \\ \lambda\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log|\nu(r)|}{\log r}. \tag{1.7}$$

Further for $0 < \rho, \lambda < \infty$, we define

$$\begin{matrix}\gamma \\ \delta\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{|\nu(r)|}{r^{\rho}}. \tag{1.8}$$

2. Main Results

We now prove

Theorem 2.1. *Let $f(z)$ be an entire function whose Taylor's series representation is given by* (1.1)*. Then the lower order λ of this entire function $f(z)$ satisfies*

$$\lambda \geq \lim_{n\to\infty} \inf \frac{n\log n}{-\log||a_k||}. \tag{2.1}$$

Also if $||a_k||/||a_{k'}||$, where $|k'| = n+1$, is a non-decreasing function of n, then equality holds in (2.1)*.*

Proof. Write

$$\Phi = \lim_{n\to\infty} \inf \frac{n\log n}{-\log||a_k||}.$$

First we prove that $\lambda \geq \Phi$. The coefficients of an entire Taylor's series satisfy Cauchy's inequality, that is

$$||a_k|| \leq M(r)\, r^{-n}. \tag{2.2}$$

Also from (1.5), for arbitrary $\varepsilon > 0$ and a sequence $r = r_s \to \infty$ as $s \to \infty$, we have

$$M(r) \leq \exp\left(r^{\overline{\lambda}}\right) \quad , \quad \overline{\lambda} = \lambda + \varepsilon.$$

So from (2.2), we get

$$||a_k|| \leq r^{-n} \exp\left(r^{\overline{\lambda}}\right).$$

Putting $r = \left(n/\overline{\lambda}\right)^{1/\overline{\lambda}}$ in the above inequality we get

$$||a_k|| \leq \left(n/\overline{\lambda}\right)^{-n/\overline{\lambda}} \exp\left(n/\overline{\lambda}\right)$$

or

$$\log ||a_k||^{-1} \geq \frac{n \log n}{\overline{\lambda}} \left[1 - \frac{\log \overline{\lambda}}{\log n} - \frac{1}{\log n}\right]$$

or

$$\lim_{n\to\infty} \inf \frac{n \log n}{-\log ||a_k||} \leq \overline{\lambda}$$

or

$$\Phi \leq \overline{\lambda}.$$

Since $\varepsilon > 0$ is arbitrarily small so finally we get

$$\Phi \leq \lambda .$$

Now we prove that $\lambda \leq \Phi$. Let

$$\psi(n) = ||a_k|| / ||a_{k'}||,$$

then

$$\psi(n) \to \infty \text{ as } n \to \infty.$$

Also

$$\psi(|k^{'}|) > \psi(n).$$

Now suppose that $||a_{k^1}|| r^{|k^1|}$ and $||a_{k^2}|| r^{|k^2|}$ are two consecutive maximum terms. Then

$$|k^1| \leq |k^2| - 1.$$

Let

$$|k^1| \leq n \leq |k^2|,$$

then

$$|\nu(r)| = |k^1|$$

for

$$\psi(|k^{1^*}|) \leq r < \psi(|k^1|)\,,$$

where

$$|k^{1^*}| = |k^1| - 1.$$

Hence from (1.7), for arbitrary $\varepsilon > 0$ and all $r > r_0(\varepsilon)$, we have

$$|k^1| = |\nu(r)| > r^{\lambda'} , \ \lambda' = \lambda - \varepsilon$$

or

$$|k^1| = |\nu(r)| \geq \left\{\psi(|k^1|) - q\right\}^{\lambda'} ,$$

where q is a constant such that $0 < q < \min\left\{1\,,\ [\psi(|k^1|) - \psi(|k^{1^*}|)]/2\right\}$

or

$$\log\psi(|k^1|) \leq O(1) + \frac{\log|k^1|}{\lambda'}.$$

Further we have

$$\psi(|k^1|) = \psi(|k^1|+1) = ... = \psi(n-1).$$

Now we can write

$$\psi(|k^0|)\ ...\ \psi(|k^*|) = \frac{||a_{k^0}||}{||a_k||} \leq [\psi(|k^*|]^{n-|k^0|} ,$$

where $|k^*| = n-1$ and $n \gg |k^0|$

or

$$\log||a_k||^{-1} \leq n\log\psi(|k^1|) + O(1)$$

or

$$\log||a_k||^{-1} \leq n\frac{\log|k^1|}{\lambda'}\ [1+o(1)]$$

or

$$\frac{1}{n}\log||a_k||^{-1} \leq \frac{\log|k^1|}{\lambda'}\ [1+o(1)]$$

or

$$\frac{1}{n}\log||a_k||^{-1} \leq \frac{\log n}{\lambda'}\ [1+o(1)]$$

or

$$\lambda' \leq \frac{n\log n}{-\log||a_k||}\ [1+o(1)].$$

Now taking limits as $n \to \infty$, we get $\lambda \leq \Phi$. Hence the Theorem 2.1 is proved. $\square$

Next we prove

Theorem 2.2. *Let $f(z)$ be an entire function whose Taylor's series representation is given by* (1.1). *Then for $0 < \rho\,, \lambda < \infty$, following inequalities hold*

$$\lim_{r\to\infty} \inf \frac{\log \mu(r)}{|\nu(r)|} \;\le\; \frac{1}{\rho} \;\le \frac{1}{\lambda} \;\le \lim_{r\to\infty} \sup \frac{\log \mu(r)}{|\nu(r)|} \;.$$

Proof. Let

$$\lim_{r\to\infty} \sup \frac{\log \mu(r)}{|\nu(r)|} \;=\; A.$$

Then for $\varepsilon > 0$ and $r > r_0(\varepsilon)$, we have

$$\log \mu(r) \;<\; (A+\varepsilon)\,|\nu(r)|. \tag{2.3}$$

From (1.2), we have

$$\frac{\mu^{'}(r)}{\mu(r)} = \frac{|\nu(r)|}{r}.$$

So from (2.3), we get

$$\log \mu(r) \;<\; (A+\varepsilon)\,\frac{\mu^{'}(r)}{\mu(r)}\; r$$

or

$$\frac{\mu^{'}(r)}{\mu(r)\log \mu(r)} \;>\; \frac{1}{(A+\varepsilon)\,r}$$

or

$$\log\log \mu(r) \;>\; \frac{1}{(A+\varepsilon)} \;\; \log r \;+\; O(1)$$

or

$$\frac{\log\log \mu(r)}{\log r} \;>\; \frac{1}{(A+\varepsilon)} \;\;+\; o(1).$$

Proceeding to limits as $r \to \infty$ and taking inf on both sides we get

$$\lambda \;\ge\; \frac{1}{A}. \tag{2.4}$$

Now let us assume that

$$\lim_{r\to\infty} \inf \frac{\log \mu(r)}{|\nu(r)|} \;=\; B.$$

Proceeding as above and using definations of limit inf and limit sup, we obtain

$$\rho \;\ge\; \frac{1}{B}. \tag{2.5}$$

Combining (2.4) and (2.5), we get

$$\lim_{r\to\infty} \inf \frac{\log \mu(r)}{|\nu(r)|} \;\le\; \frac{1}{\rho} \;\le \frac{1}{\lambda} \;\le \lim_{r\to\infty} \sup \frac{\log \mu(r)}{|\nu(r)|} \;.$$

Hence the Theorem 2.2 is proved. □

Next we prove

Theorem 2.3. *Let $f(z)$ be an entire function whose Taylor's series representation is given by* (1.1)*. Then for $0<\rho<\infty$, following inequalities hold*

$$\delta \ \leq\ \frac{\gamma}{e}e^{\delta/\gamma} \ \leq\ \delta T \ \leq\ \gamma\ ,$$
$$\delta \ \leq\ \rho\, t \ \leq\ \delta(1+\log\frac{\gamma}{\delta}) \ \leq\ \gamma\ ,$$

and

$$\gamma+\delta \ \leq\ e\delta T.$$

Proof. From (1.2), for $r \geq r_0$ and $k \geq 1$ we have

$$\log\mu(kr) \ =\ O(1) \ +\ \int_{r_0}^{r}\frac{|\nu(t)|}{t}dt \ +\ \int_{r}^{kr}\frac{|\nu(t)|}{t}dt \tag{2.6}$$

or

$$\log\mu(kr) \ >\ O(1) \ +\ \frac{(\delta-\varepsilon)\,r^{\rho}}{\rho} \ +\ |\nu(r)|\log k\,.$$

Dividing both sides by $(kr)^{\rho}$, we get

$$\frac{\log\mu(kr)}{(kr)^{\rho}} \ >\ o(1) \ +\ \frac{(\delta-\varepsilon)}{\rho\,k^{\rho}} \ +\ \frac{|\nu(r)|}{r^{\rho}}\ \frac{\log k}{k^{\rho}}\,. \tag{2.7}$$

Proceeding to limits as $r\to\infty$ and taking *sup* on both sides of (2.7), we get

$$T \ \geq\ \frac{\delta+\rho\,\gamma\log k}{\rho\,k^{\rho}}\,. \tag{2.8}$$

Also proceeding to limits as $r\to\infty$ and taking inf on both sides of (2.7), we get

$$t \ \geq\ \frac{\delta(1+\rho\log k)}{\rho\,k^{\rho}}\,. \tag{2.9}$$

Taking $k=\exp[(\gamma-\delta)/(\gamma\rho)]$ in (2.8), we get

$$e\rho T \ \geq\ \gamma\, e^{\delta/\gamma}\,.$$

Since $\exp(t) \ \geq\ e\,t$ for all $t\ \geq\ 0$. Therefore finally, we get

$$e\rho T \ \geq\ \gamma\, e^{\delta/\gamma} \ \geq\ e\delta\,. \tag{2.10}$$

Also taking $k=1$ in (2.9), we get

$$t \ \geq\ \frac{\delta}{\rho}\,. \tag{2.11}$$

Again from (2.6), we have

$$\log\mu(kr) \ <\ O(1) \ +\ \frac{(\gamma+\varepsilon)\,r^{\rho}}{\rho} \ +\ |\nu(kr)|\log k\,.$$

Dividing both sides by $(kr)^{\rho}$, we get

$$\frac{\log\mu(kr)}{(kr)^{\rho}} \ <\ o(1) \ +\ \frac{(\gamma+\varepsilon)}{\rho\,k^{\rho}} \ +\ \frac{|\nu(kr)|}{(kr)^{\rho}}\ \log k. \tag{2.12}$$

So in this case we get

$$T \leq \frac{\gamma(1+\rho k^{\rho}\log k)}{\rho k^{\rho}} \tag{2.13}$$

and

$$t \leq \frac{\gamma+\rho\delta k^{\rho}\log k}{\rho k^{\rho}}. \tag{2.14}$$

Taking $k=1$ in (2.13), we get

$$T \leq \frac{\gamma}{\rho}. \tag{2.15}$$

Also taking $k=(\gamma/\delta)^{1/\rho}$ in (2.14), we get

$$\rho t \leq \delta(1+\log\frac{\gamma}{\delta}).$$

Since $\log(1+t)\leq t$ for all $t\geq 0$. Therefore finally we get

$$\rho t \leq \delta(1+\log\frac{\gamma}{\delta}) \leq \gamma. \tag{2.16}$$

Now from (2.10), (2.11), (2.15) and (2.16), we get

$$\delta \leq \frac{\gamma}{e}e^{\delta/\gamma} \leq \delta T \leq \gamma \tag{2.17}$$

and

$$\delta \leq \rho t \leq \delta(1+\log\frac{\gamma}{\delta}) \leq \gamma.$$

From (2.17), we have

$$\frac{\gamma}{e}e^{\delta/\gamma} \leq \delta T$$

or

$$\gamma\left[1+\frac{\delta}{\lambda}+....\right] \leq e\delta T$$

or

$$\gamma\left[1+\frac{\delta}{\lambda}\right] \leq e\delta T$$

or

$$\gamma+\delta \leq e\delta T.$$

Hence the Theorem 2.3 is proved. □

Next we prove

Theorem 2.4. *Let $f(z)$ be an entire function whose Taylor's series representation is given by* (1.1). *Then for $0<\rho<\infty$, following inequalities holds*

$$\gamma+\rho t \leq e\rho T$$

and

$$e\rho\, t \ \le \rho\, T + e\delta.$$

Proof. From (1.2) for $r \ge r_0$ and $k \ge 1$, we have

$$\log \mu(kr) \ = \ \log \mu(r) \ + \ \int_r^{kr} \frac{|\nu(t)|}{t} dt \tag{2.18}$$

or

$$\log \mu(kr) \ > \ (t - \varepsilon) r^\rho \ + \ |\nu(r)| \log k\,.$$

Dividing both sides by $(kr)^\rho$, we get

$$\frac{\log \mu(kr)}{(kr)^\rho} \ > \ \frac{(t-\varepsilon)}{k^\rho} \ + \ \frac{|\nu(r)|}{r^\rho}\ \frac{\log k}{k^\rho}.$$

Proceeding to limits as $r \to \infty$ and taking *sup* on both sides, we get

$$T \ \ge \ \frac{t}{k^\rho} + \frac{\gamma \log k}{k^\rho}\,.$$

Taking $k = e^{1/\rho}$ in above inequality, we get

$$T \ \ge \ \frac{t}{e} + \frac{\gamma}{\rho e}\,. \tag{2.19}$$

Again from (2.18), we have

$$\log \mu(kr) \ < \ (T + \varepsilon) r^\rho \ + \ |\nu(kr)| \log k\,.$$

Dividing both sides by $(kr)^\rho$, we get

$$\frac{\log \mu(kr)}{(kr)^\rho} \ < \ \frac{(T+\varepsilon)}{k^\rho} \ + \ \frac{|\nu(kr)|}{(kr)^\rho}\ \log k\,.$$

Proceeding to limits as $r \to \infty$ and taking inf on both sides, we get

$$t \ \le \ \frac{T}{k^\rho} + \delta \log k\,.$$

Taking $k = e^{1/\rho}$ in above inequality, we get

$$t \ \le \ \frac{T}{e} + \frac{\delta}{\rho}\,. \tag{2.20}$$

Now from (2.19) and (2.20), we get

$$\gamma + \rho\, t \ \ \le \ \ e\rho\, T$$

and

$$e\rho\, t \ \le \rho\, T + e\delta.$$

Hence the Theorem 2.4 is proved. □

Note: Similar results were obtained for entire functions of one variable by Shah ([5] , [6] , [7] and [8]).

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Dr Susheel Kumar
Department of Mathematics
Central University of Himachal Pradesh
Dharamshala-176215, INDIA.
E-mail address: sus83dma@gmail.com

Dr G. S. Srivastava
Department of Mathematics
Indian Institute of Technology Roorkee
Roorkee-247667, INDIA.
E-mail address: girssfma@iitr.ernet.in