Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 3 Issue 2(2011), Pages 127-133.

REFINEMENTS OF CHOI–DAVIS–JENSEN'S INEQUALITY

(COMMUNICATED BY JOZSEF SANDOR)

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Abstract. Let $\Phi_1, \dots, \Phi_n$ be strictly positive linear maps from a unital C^*-algebra $\mathscr{A}$ into a C^*-algebra $\mathscr{B}$ and let $\Phi = \sum_{i=1}^n \Phi_i$ be unital. If f is an operator convex function on an interval J, then for every self-adjoint operator $A \in \mathscr{A}$ with spectrum contained in J, the following refinement of the Choi–Davis–Jensen inequality holds:

$$f(\Phi(A)) \leq \sum_{i=1}^{n} \Phi_i(I)^{\frac{1}{2}} f\left(\Phi_i(I)^{-\frac{1}{2}} \Phi_i(A) \Phi_i(I)^{-\frac{1}{2}}\right) \Phi_i(I)^{\frac{1}{2}} \leq \Phi(f(A)).$$

1. Introduction and Preliminaries

Let $\mathbb{B}(\mathscr{H})$ stand for the algebra of all bounded linear operators on a complex Hilbert space $(\mathscr{H}, \langle \cdot, \cdot \rangle)$. For self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$ the order relation $A \leq B$ means that $\langle A\xi, \xi \rangle \leq \langle B\xi, \xi \rangle$ $(\xi \in \mathscr{H})$. In particular, if $0 \leq A$, then A is called *positive*. If a positive operator A is invertible, then we say that it is *strictly positive* and write $0 < A$. Every positive operator B has a unique positive square root $B^{1/2}$, in particular, the absolute value of $A \in \mathbb{B}(\mathscr{H})$ is defined to be $|A| = (A^*A)^{1/2}$. Throughout the paper any C^*-algebra $\mathscr{A}$ is regarded as a closed $*$-subalgebra of $\mathbb{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$. We also use the same notation I for denoting the identity of C^*-algebras in consideration.

A continuous real function f defined on an interval J is called *operator convex* if

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B)$$

for all $0 \leq \lambda \leq 1$ and all self-adjoint operators A, B with spectra in J. A function f is called *operator concave* if $-f$ is operator convex.

A linear map $\Phi : \mathscr{A} \to \mathscr{B}$ between C^*-algebras is said to be *positive* if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is *unital* if Φ preserves the identity. The linear map Φ is called *strictly positive* if $\Phi(A)$ is strictly positive whenever A is strictly positive. It can be easily seen that a positive linear map Φ is strictly positive if and only if $\Phi(I) > 0$. For a comprehensive account on positive linear maps see [3].

2000 *Mathematics Subject Classification.* Primary 47A63; Secondary 47A63.
Key words and phrases. Choi–Davis–Jensen's inequality; operator convex; operator inequality; Hilbert C^*-module; positive map.

Submitted February 2, 2011. Published May 2, 2011.

The classical Jensen inequality states that if f is a convex function on an interval J then for elements $x_1, \ldots, x_n \in J$, we have

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \leq \sum_{i=1}^{n} t_i f(x_i)\,,$$

where $t_1, \ldots, t_n$ are positive real numbers with $\sum_{i=1}^{n} t_i = 1$. Using the integral theory, Brown and Kosaki [4] proved that if f is continuous convex function on the interval $[0, +\infty)$ with $f(0) = 0$, then for each $x \in \mathscr{H}$ with $\|x\| \leq 1$, and positive operator A we have

$$f(\langle C^*ACx, x\rangle) \leq \langle C^* f(A)Cx, x\rangle,$$

where C is a contraction operator ($\|C\| \leq 1$). It can be proved that if $\mathscr{A}$ is a C^*-algebra and φ is a state on $\mathscr{A}$, then for every convex function f, the inequality

$$f(\varphi(a)) \leq \varphi(f(a)) \tag{1.1}$$

holds for each $a \in \mathscr{A}$. But it is not generally true when the state φ is replaced by an arbitrary positive linear map between C^*-algebras. However, Davis [7] showed that if Φ is a unital completely positive linear mapping from a C^*-algebra into $B(\mathscr{H})$ for some Hilbert space $\mathscr{H}$ and if f is an operator convex function on an interval J, then

$$f(\Phi(A)) \leq \Phi(f(A)) \tag{1.2}$$

holds for every self-adjoint operator A, whose spectrum is contained in J. Subsequently Choi [5] proved that it is sufficient to assume that Φ is a unital positive map. Ando [1] gave an alternative proof for this inequality by using the integral representation of operator convex functions. The equivalence of the Choi–Davis–Jensen inequality (1.2) and the operator convexity of f was first proved by Hansen and Pedersen [10, 11], see also [8]. In addition, Hansen et al. [12] presented a general Jensen operator inequality for a unital filed of positive linear mappings extending a previous result of Mond and Pečarić. [14]. A number of mathematicians investigated some different types of inequality (1.1), when f is not necessarily operator convex. In [2], Antezana, Massey and Stojanoff proved that if $\Phi : \mathscr{A} \to \mathscr{B}$ is a positive unital map between unital C^*-algebras $\mathscr{A}, \mathscr{B}$ and f is a convex function, and $a \in \mathscr{A}$ is such that $\Phi(f(a))$ and $\Phi(a)$ commute, then $f(\Phi(a)) \leq \Phi(f(a))$.

The notion of Hilbert C^*-module is an extension of that of Hilbert space, where the inner product takes its values in a C^*-algebra. When $(\mathscr{X}, \langle\cdot,\cdot\rangle)$ is a Hilbert C^*-module over a C^*-algebra, $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ defines a norm on $\mathscr{X}$, where the latter norm denotes that in the C^*-algebra. Any Hilbert space can be regarded as a Hilbert $\mathbb{C}$-module and any C^*-algebra $\mathscr{A}$ is a Hilbert C^*-module over itself via $\langle a, b\rangle = a^*b$ $(a, b \in \mathscr{A})$.
The set of all maps T on a Hilbert C^*-module $\mathscr{X}$ such that there is a map T^* on $\mathscr{X}$ with the property $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ $(x, y \in \mathscr{X})$ is denoted by $\mathcal{L}(\mathscr{X})$. This space is in fact a unital C^*-algebra in a natural fashion. For every $x \in \mathscr{X}$ we define the absolute value of x as the unique positive square root of $\langle x, x\rangle$, that is, $|x| = \langle x, x\rangle^{\frac{1}{2}}$. For any $x, y \in \mathscr{X}$ the operator $x \otimes y$ on $\mathscr{X}$ is defined by $(x \otimes y)(z) = x\langle y, z\rangle$ $(z \in \mathscr{X})$.
We refer the reader to [15] for undefined notions on C^*-algebra theory, to [13] for Hilbert C^*-modules and to [9] for more information on operator inequalities.

In this paper we obtain some refinements of the Choi–Davis–Jensen inequality $f(\Phi(A)) \leq \Phi(f(A))$ for strictly positive linear maps in the framework of Hilbert C^*-modules.

2. Main results

We first slightly improve the condition $\Phi(I) = I$ in (1.2) under some reasonable conditions on f.

Proposition 2.1. *If f is an operator convex function on an interval J containing 0 and $f(0) = 0$, then*

$$f(\Phi(A)) \leq \Phi(f(A))$$

for every self-adjoint operator A in a unital C^*-algebra $\mathscr{A}$ with spectrum in J and every positive linear map $\Phi : \mathscr{A} \to \mathbb{B}(\mathscr{H})$ with $0 < \Phi(I) \leq I$.

Proof. The mapping $\Psi(A) = \Phi(I)^{-1/2}\Phi(A)\Phi(I)^{-1/2}$ is a unital positive map. Hence, by (1.2) and (ii) of Theorem 5.2,

$$f\left(\Phi(I)^{1/2}\Psi(A)\Phi(I)^{1/2}\right) \leq \Phi(I)^{1/2}f(\Psi(A))\Phi(I)^{1/2} \leq \Phi(I)^{1/2}\Psi(f(A))\Phi(I)^{1/2}\,,$$

whence $f(\Phi(A)) \leq \Phi(f(A))$. □

We are ready to present our main result as a refinement of the Choi–Davis–Jensen inequality.

Theorem 2.2. *Let $\Phi_1, \ldots, \Phi_n$ be strictly positive linear maps from a unital C^*-algebra $\mathscr{A}$ into a unital C^*-algebra $\mathscr{B}$ and let $\Phi = \sum_{i=1}^n \Phi_i$ be unital. If f is an operator convex function on an interval J, then*

$$f(\Phi(A)) \leq \sum_{i=1}^n \Phi_i(I)^{\frac{1}{2}} f\left(\Phi_i(I)^{-\frac{1}{2}}\Phi_i(A)\Phi_i(I)^{-\frac{1}{2}}\right)\Phi_i(I)^{\frac{1}{2}} \leq \Phi(f(A)) \tag{2.1}$$

for every self-adjoint operator $A \in \mathscr{A}$ with spectrum contained in J. For the concave operator functions, the inequalities will be reversed.

Proof. We can simply write

$$f(\Phi(A)) = f\left(\sum_{i=1}^n \Phi_i(A)\right) = f\left(\sum_{i=1}^n \Phi_i(I)^{\frac{1}{2}}(\Phi_i(I)^{-\frac{1}{2}}\Phi_i(A)\Phi_i(I)^{-\frac{1}{2}})\Phi_i(I)^{\frac{1}{2}}\right).$$

Since $\sum_{i=1}^n \Phi_i(I) = I$, from (1.2), the first inequality follows.

For the second inequality, let $\Psi_i(A) = \Phi_i(I)^{-\frac{1}{2}}\Phi_i(A)\Phi_i(I)^{-\frac{1}{2}}$. Then Ψ_i is a unital positive linear map. Again by applying the Choi–Davis–Jensen inequality, we have $f(\Psi_i(A)) \leq \Psi_i(f(A))$, so

$$f\left(\Phi_i(I)^{-\frac{1}{2}}\Phi_i(A)\Phi_i(I)^{-\frac{1}{2}}\right) \leq \Phi_i(I)^{-\frac{1}{2}}\Phi_i(f(A))\Phi_i(I)^{-\frac{1}{2}}.$$

Summing these inequalities over i from 1 to n, the second inequality will be obtained. □

Remark. *Let Φ be a unital positive linear map from a unital C^*-algebra $\mathscr{A}$ into a unital C^*-algebra $\mathscr{B}$. If f is a non-negative operator concave function on $[0, \infty)$*

and σ is its corresponding operator mean via the Kubo–Ando theory (see [9]*), then for every positive operator $A \in \mathscr{A}$, inequality* (2.1) *can be restated as*

$$\Phi(f(A)) \le \sum_{i=1}^{n} \big(\Phi_i(I)\sigma\Phi_i(A)\big) \le f(\Phi(A)).$$

The next corollaries are of special interest.

Corollary 2.3. *Let $\mathscr{X}$ be a Hilbert C^*-module and $T_1, \ldots, T_n \in \mathcal{L}(\mathscr{X})$ be self-adjoint operators with spectra contained in J. Then*

$$f\left(\sum_{i=1}^{n}\langle x_i, T_i x_i\rangle\right) \le \sum_{i=1}^{n} |x_i| f\left(|x_i|^{-1}\langle x_i, T_i x_i\rangle |x_i|^{-1}\right)|x_i| \le \sum_{i=1}^{n}\langle x_i, f(T_i)x_i\rangle,$$

for every elements $x_1, \ldots, x_n \in \mathscr{X}$ with $|x_i| > 0$ and $\sum_{i=1}^{n}|x_i|^2 = I$.

Proof. For $1 \le i \le n$, define $\Phi_i : \oplus_{k=1}^{n}\mathcal{L}(\mathscr{X}) \to \mathscr{A}$ by $\Phi_i(\{T_k\}_{k=1}^{n}) = \langle x_i, T_i x_i\rangle$. Then Φ_i is a positive map and $\Phi_i(\{I\}_{k=1}^{n}) = |x_i|^2 > 0$. Also it follows from the hypothesis that $\sum \Phi_i$ is unital. So by using inequality (2.1), the desired result follows. □

Corollary 2.4. *If $A_1, \ldots, A_n$ are self-adjoint elements in a unital C^*-algebra $\mathscr{A}$ and $U_1, \ldots, U_n \in \mathscr{A}$ such that $\sum_{i=1}^{n} U_i^* U_i = I$ and $U_i^* U_i > 0$, then*

$$f\left(\sum_{i=1}^{n} U_i^* A_i U_i\right) \le \sum_{i=1}^{n} |U_i| f\left(|U_i|^{-1} U_i^* A_i U_i |U_i|^{-1}\right)|U_i| \le \sum_{i=1}^{n} U_i^* f(A_i) U_i.$$

Proof. As in Corollary 2.3, for $\{A_i\}_{i=1}^{n} \in \oplus_{i=1}^{n}\mathscr{A}$, set $\Phi_i(A) = U_i^* A_i U_i$ in inequality (2.1). □

Since the functions $f(t) = t^{-1}$ and $f(t) = t^p$ for $p \in [1,2]$ are operator convex and $f(t) = t^p$ for $p \in [0,1]$ is an operator concave function, the following inequalities holds.

Corollary 2.5. *If $\mathscr{X}$ is a Hilbert C^*-module and $T_1, \ldots, T_n$ are positive operators in $\mathcal{L}(\mathscr{X})$ and $x_1, \ldots, x_n \in \mathscr{X}$ with $|x_i| > 0$ and $\sum_{i=1}^{n}|x_i|^2 = I$, then*

(1) $(\sum_{i=1}^{n}\langle x_i, T_i x_i\rangle)^{-1} \le \sum_{i=1}^{n}|x_i|^2(\langle x_i, T_i x_i\rangle)^{-1}|x_i|^2 \le \sum_{i=1}^{n}\langle x_i, T_i^{-1}x_i\rangle \qquad (T_i > 0)$;

(2) $(\sum_{i=1}^{n}\langle x_i, T_i x_i\rangle)^p \le \sum_{i=1}^{n}|x_i|(|x_i|^{-1}\langle x_i, T_i x_i\rangle|x_i|^{-1})^p|x_i| \le \sum_{i=1}^{n}\langle x_i, T_i^p x_i\rangle \quad p \in [1,2]$,

(3) $(\sum_{i=1}^{n}\langle x_i, T_i x_i\rangle)^p \ge \sum_{i=1}^{n}|x_i|(|x_i|^{-1}\langle x_i, T_i x_i\rangle|x_i|^{-1})^p|x_i| \ge \sum_{i=1}^{n}\langle x_i, T_i^p x_i\rangle \quad p \in [0,1]$.

Corollary 2.6. *For positive elements $A_1, \ldots, A_n$ in a unital C^*-algebra $\mathscr{A}$ and each $U_1, \ldots, U_n \in \mathscr{A}$ such that $\sum_{i=1}^{n} U_i^* U_i = I$ and $U_i^* U_i > 0$ $(1 \le i \le n)$,*

(1') $(\sum_{i=1}^{n} U_i^* A_i U_i)^{-1} \le \sum_{i=1}^{n}|U_i|^2(U_i^* A_i U_i)^{-1}|U_i|^2 \le \sum_{i=1}^{n} U_i^* A_i^{-1} U_i \qquad (A_i > 0)$;

(2') $(\sum_{i=1}^{n} U_i^* A_i U_i)^p \le \sum_{i=1}^{n}|U_i|(|U_i|^{-1}U_i^* A_i U_i|U_i|^{-1})^p|U_i| \le \sum_{i=1}^{n} U_i^* A_i^p U_i \quad p \in [1,2]$,

(3') $(\sum_{i=1}^{n} U_i^* A_i U_i)^p \ge \sum_{i=1}^{n}|U_i|(|U_i|^{-1}U_i^* A_i U_i|U_i|^{-1})^p|U_i| \ge \sum_{i=1}^{n} U_i^* A_i^p U_i \quad p \in [0,1]$.

By the following example we shall show that inequality (2.1) is a refinement of the Choi–Davis–Jensen inequality. More precisely, there are some examples for which both inequalities in (2.1) are strict.

Example 2.7. *Let $x_1,\dots,x_n$ be elements of a Hilbert C^*-module $\mathscr{X}$ such that $|x_i|>0$ and $\sum_{i=1}^n |x_i|^2 = I$. For $1\le i\le n$, set $T_i = y_i\otimes y_i$, where $y_1,\dots,y_n\in\mathscr{X}$. From Corollary 2.5 (2) with $p=2$, we have*

$$\Big(\sum_{i=1}^n |\langle y_i,x_i\rangle|^2\Big)^2 \le \sum_{i=1}^n |\langle y_i,x_i\rangle|^2|x_i|^{-2}|\langle y_i,x_i\rangle|^2 \le \sum_{i=1}^n |y_i\langle y_i,x_i\rangle|^2. \tag{2.2}$$

Let $\mathscr{H}$ be a Hilbert space of dimension greater than 3, e_1,e_2,e_3 be orthonormal vectors in $\mathscr{H}$ and $x_i=\frac{\sqrt{2}}{2}e_i$ and $y_i = ie_i+e_3$, for $i=1,2$. A straightforward computation shows that $\big(\sum_{i=1}^n |\langle y_i,x_i\rangle|^2\big)^2 = \frac{25}{4}, \sum_{i=1}^n |\langle y_i,x_i\rangle|^2|x_i|^{-2}|\langle y_i,x_i\rangle|^2 = \frac{17}{2}$ and $\sum_{i=1}^n |y_i\langle y_i,x_i\rangle|^2 = 11$. Thus

$$\Big(\sum_{i=1}^n |\langle y_i,x_i\rangle|^2\Big)^2 < \sum_{i=1}^n |\langle y_i,x_i\rangle|^2|x_i|^{-2}|\langle y_i,x_i\rangle|^2 < \sum_{i=1}^n |y_i\langle y_i,x_i\rangle|^2.$$

As another application of inequality (2.1), we have the following refinement of Choi's inequality [6, Proposition 4.3].

Theorem 2.8. *Suppose that $\Phi_1,\dots,\Phi_n$ are strictly positive linear maps from a unital C^*-algebra $\mathscr{A}$ into a unital C^*-algebra $\mathscr{B}$ and $\Phi=\sum_{i=1}^n\Phi_i$. Then*

$$\Phi(S)\Phi(T)^{-1}\Phi(S) \le \sum_{i=1}^n \Phi_i(S)\Phi_i(T)^{-1}\Phi_i(S) \le \Phi(ST^{-1}S).$$

for every self-adjoint element S and every $T>0$ in $\mathscr{A}$.

Proof. Set $\Psi_i(X) = \Phi(T)^{-1/2}\Phi_i(T^{1/2}XT^{1/2})\Phi(T)^{-1/2}$ and $\Psi=\sum_{i=1}^n\Psi_i$. Then Ψ_i's are strictly positive linear maps and Ψ is unital. It follows from the operator convexity of $f(t)=t^2$ and Theorem 2.2 that

$$\Psi(X)^2 \le \sum_{i=1}^n \Psi_i(X)\Psi_i(I)^{-1}\Psi_i(X) \le \Psi(X^2),$$

for every positive element X. Now if $X = T^{-1/2}ST^{-1/2}$ we get

$$\begin{aligned}\Phi(T)^{-1/2}\Phi(S)\Phi(T)^{-1}\Phi(S)\Phi(T)^{-1/2} &\le \textstyle\sum_{i=1}^n \Phi(T)^{-1/2}\Phi_i(S)\Phi_i(T)^{-1}\Phi_i(S)\Phi(T)^{-1/2}\\ &\le \Phi(T)^{-1/2}\Phi(ST^{-1}S)\Phi(T)^{-1/2},\end{aligned}$$

or

$$\Phi(S)\Phi(T)^{-1}\Phi(S) \le \sum_{i=1}^n \Phi_i(S)\Phi_i(T)^{-1}\Phi_i(S) \le \Phi(ST^{-1}S)\,,$$

as desired. □

Remark. *The second inequality is a simple result of Choi–Davis–Jensen's inequality. It is a well-known theorem that for operators R,S,T on a Hilbert space $\mathscr{H}$, if T is invertible then*

$$\begin{bmatrix} T & S \\ S^* & R \end{bmatrix} \ge 0 \Leftrightarrow R \ge S^*T^{-1}S.$$

Thus for each $1 \leq i \leq n$,

$$\begin{bmatrix} \Phi_i(T) & \Phi_i(S) \\ \Phi_i(S) & \Phi_i(S)\Phi_i(T)^{-1}\Phi_i(S) \end{bmatrix} \geq 0\,.$$

If we sum these matrices over i, *we obtain*

$$\begin{bmatrix} \Phi(T) & \Phi(S) \\ \Phi(S) & \sum_{i=1}^n \Phi_i(S)\Phi_i(T)^{-1}\Phi_i(S) \end{bmatrix} \geq 0,$$

or equivalently

$$\sum_{i=1}^n \Phi_i(S)\Phi_i(T)^{-1}\Phi_i(S) \geq \Phi(S)\Phi(T)^{-1}\Phi(S)\,.$$

Acknowledgement. The fourth author was supported by a grant from Ferdowsi University of Mashhad (No. MP89142MOS).

References

[1] T. Ando, *Topics on operator inequalities*, Hokkaido University, Sapporo, 1978.
[2] J. Antezana, P. Massey and D. Stojanoff, *Jensen's inequality for spectral order and submajorization*, J. Math. Anal. Appl. **331** (2007), no. 1, 297–307.
[3] R. Bhatia, *Positive Definite Matrices*, Princeton University Press, 2007.
[4] L.G. Brown and H. Kosaki, *Jensen's inequality in semi-finite von Neumann algebras*, J. Operator Theory **23** (1990), no. 1, 3–19.
[5] M.D. Choi, *A Schwarz inequality for positive linear maps on C^*-algebras*, Illinois J. Math. **18** (1974), 565–574.
[6] M.D. Choi, *Some assorted inequalities for positive linear maps on C^*-algebras*, J. Operator Theory, **4** (1980), 271–285.
[7] C. Davis, *A Schwarz inequality for convex operator functions*, Proc. Amer. Math. Soc., **8** (1957), 42–44.
[8] J.I. Fujii and M. Fujii, *Jensen's inequalities on any interval for operators*, Nonlinear analysis and convex analysis, 29–39, Yokohama Publ., Yokohama, 2004.
[9] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond–Pečarić Method in Operator Inequalities*, Element, Zagreb, 2005.
[10] F. Hansen and G.K. Pedersen, *Jensen's inequality for operators and Lowner's theorem*, Math. Ann. **258** (1981/82), no. 3, 229–241.
[11] F. Hansen and G.K. Pedersen, *Jensen's operator inequality*, Bull. London Math. Soc., **35** (2003), 553–564.
[12] F. Hansen, Pečarić and I. Perić, *Jensen's operator inequality and it's converses*, Math. Scand. **100** (2007) 61-?73.
[13] E.C. Lance: *Hilbert C^*-Modules*, London Math. Soc. Lecture Note Series 210, Cambridge University Press, Cambridge, 1995.
[14] B. Mond and J.E. Pečarić, *Convex inequalities in Hilbert space*, Houston J. Math. **19** (1993), 405–420.
[15] J.G. Murphy, *C^*-Algebras and Operator Theory* , Academic Press, San Diego, 1990.

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