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ON A MAGNETIC GINZBURG-LANDAU TYPE ENERGY WITH WEIGHT

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Abstract. In the presence of applied magnetic fields H in the order of H_{c_1} the first critical field, we determine the limiting vorticities of the minimal Ginzburg-Landau energy with weight. Our result is analogous to the work of E. Sandier and S. Serfaty [26].

1. Introduction

A Ginzburg-Landau type energy with weight of a superconducting sample is given by the functional

$$J(u,A)=\int_\Omega\left(|(\nabla-iA)u|^2+q(x)\,|\mathrm{curl}\,A-H|^2+\frac{p(x)}{2\varepsilon^2}(1-|u|^2)^2\right)\mathrm{d}x. \quad (1.1)$$

Ω an open, smooth and simply connected subset of $\mathbb{R}^2$. The superconductor is assumed to be an infinite vertical cylinder of section Ω. A is the vector potential, and the induced magnetic field in the material is $h=\mathrm{curlA}$. The complex-valued function u is called the "order parameter" and $\frac{1}{\varepsilon}$ is the Ginzburg-Landau parameter. In this work, let p and q be smooth maps from Ω into $\mathbb{R}$ such that $q(x)\geq\alpha$ and $\beta\leq p(x)\leq\gamma$ for all $x\in\Omega$ for some positive reals α, β and γ. Again we let the applied field H be such that

$$\lambda=\lim_{\varepsilon\longrightarrow 0}\frac{H}{|\ln\varepsilon|}\ \text{ exists and is finite.}$$

In general, weight terms can be studied to model pinning sites by impurities in the material, or variable thickness. More precisely, many authors studied the case of the usual Ginzburg-Landau energy ($p(x)=q(x)=1$) where the potential is like $(a(x)-|u|^2)^2$ for some situations of the function $a(x)$. They gave the limiting vorticities when $\varepsilon\longrightarrow 0$, in particular a pinning phenomenon appears. For more details see [1, 2, 3, 4, 8, 9, 10, 11, 12, 17, 19, 20, 21, 24]. For weight terms giving the case of thin films, we can refer in particular to [14, 15, 16]. There, we find a discussion on the vortex structure of the superconducting thin films placed in a

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magnetic field.
This paper is concerned with the Ginzburg-Landau functional with particulars weights, it is given by (1.1), i.e. we have a weight before the magnetic term $|\mathrm{curl}\mathrm{A}-H|^2$ and another before the potential $(1-|u|^2)^2$. The result here follows the work of Sandier-Serfaty [26] (see also [6, 7]) who studied the Gamma-convergence of the of the usual Ginzburg-Landau energy without weight, that is with $q(x)=1$ and $p(x)=1$. More precisely, they gave the vortex nucleation for minimizers of the energy J for applied magnetic fields comparable to H_{c_1} the first critical field. Our motivation in this paper is to address the same question, but in the presence of the weights $q(x)$ and $p(x)$. Although, we haven't the cases neither pinning model or variable thickness, so no particular physical sense appears, but it is of interest to find mathematically the vortex structure of minimizers and the influence of $q(x)$ and $p(x)$ in the limit of $\varepsilon \longrightarrow 0$. We will show that the same method as in [26] can be adapted to treat the case with weight, thus deducing the optimal distribution of vortices. Moreover, only the function q appears in the limiting vorticities.
Let $\mathcal{M}(\Omega)$ be the space of bounded Radon measures on Ω, i.e. the topological dual of $C_0^0(\Omega)$. Given $\lambda>0$, we introduce an energy E_λ defined on $\mathcal{M}(\Omega)\cap H^{-1}(\Omega)$ as follows. For $\mu\in\mathcal{M}(\Omega)\cap H^{-1}(\Omega)$, let $h_\mu\in H^1(\Omega)$ be the solution of

$$\begin{cases} -\Delta[q(x)(h_\mu-1)]+h_\mu=\mu \quad \text{in } \Omega\,, \\ h_\mu=1 \quad \text{on } \partial\Omega\,, \end{cases} \tag{1.2}$$

Now $E_\lambda(\mu)$ is by definition

$$E_\lambda(\mu)=\frac{1}{\lambda}\int_\Omega |\mu|\,\mathrm{d}x+\int_\Omega \left(|\nabla[q(x)(h_\mu-1)]|^2+|h_\mu-1|^2\right)\,\mathrm{d}x\,. \tag{1.3}$$

A standard choice of gauge permits one to assume that the magnetic potential satisfies

$$\operatorname{div} A=0 \quad \text{in } \Omega, \quad \nu\cdot A=0 \quad \text{on } \partial\Omega, \tag{1.4}$$

where ν is the outward unit normal vector of $\partial\Omega$. Notice that the existence of minimizers of J is standard starting from a minimizing sequence (cf. e.g. [18]). Our result is the following

Theorem 1.1. *Given $\lambda>0$, assume that*

$$\lim_{\varepsilon\to 0}\frac{H}{|\ln\varepsilon|}=\lambda\,,$$

then

$$\frac{J}{H^2}\to E_\lambda$$

in the sense of Γ-convergence.

The convergence in Theorem 1.1 is precisely described in Propositions 2.1 and 3.2 below.
Minimizers of (1.3) can be characterized by means of minimizers of the following problem,

$$\min_{\substack{h\in H_0^1(\Omega)\\ -\Delta(q(x)h)+h\in\mathcal{M}(\Omega)}} \int_\Omega \left(\frac{1}{\lambda}\,|-\Delta[q(x)h]+h+1|+|\nabla[q(x)h]|^2+|h|^2\right)\,\mathrm{d}x\,. \tag{1.5}$$

The above functional being strictly convex and lower semi-continuous, it admits a unique minimizer, and so the functional E_λ. Therefore, as a corollary of Theorem 1.1, we may describe the limiting vorticity measure in terms of the minimizer of the limiting energy E_λ.

Theorem 1.2. *Under the hypothesis of Theorem 1.1, if $(u_\varepsilon, A_\varepsilon)$ is a minimizer of (1.1), then, denoting by*

$$h_\varepsilon = \operatorname{curl} A_\varepsilon\,, \quad \mu(u_\varepsilon, A_\varepsilon) = h_\varepsilon + \operatorname{curl}(iu_\varepsilon\,,\,(\nabla - iA_\varepsilon)u_\varepsilon)\,, \tag{1.6}$$

the 'induced magnetic field' and 'vorticity measure' respectively, the following convergences hold,

$$\mu_\varepsilon =: \frac{\mu(u_\varepsilon, A_\varepsilon)}{H} \to \mu_* \quad \text{in } \mathcal{M}(\Omega)\,, \tag{1.7}$$

$$\frac{h_\varepsilon}{H} \to h_{\mu_*} \quad \text{weakly in } H^1(\Omega) \text{ and strongly in } W^{1,p}(\Omega)\,,\ \forall\, p < 2\,. \tag{1.8}$$

Here $\mu_* = -\Delta[q(h_{\mu_*} - 1)] + h_{\mu_*}$ is the unique minimizer of E_λ.

In [26], Sandier-Serfaty obtained for the classic Ginzburg-Landau energy (the case of $q(x) = p(x) = 1$)

$$\lim_{\varepsilon \longrightarrow 0} \frac{J(u_\varepsilon, A_\varepsilon)}{H^2} = \frac{1}{\lambda}\int_\Omega |\mu_*|\,\mathrm{d}x + \int_\Omega \left(|\nabla h_{\mu_*}|^2 + |h_{\mu_*} - 1|^2\right)\,\mathrm{d}x, \tag{1.9}$$

where $\mu_* = -\Delta h_{\mu_*} + h_{\mu_*}$. This shows that our results given in Theorems 1.1-1.2 are an extension of (1.9).

Sketch of the proof.
The proof of Theorems 1.1-1.2 is obtained by getting first a lower bound, Proposition 2.1, proved in Section 2, and then an upper bound on the minimal energy of J, Proposition 3.2, proved in Section 3. The upper bound will be done by construction of a test configuration which goes with the same idea of [26].
Remarks on the notation.

- The letters $C, \widetilde{C}, M$, etc. will denote positive constants independent of ε.
- For $n \in \mathbb{N}$ and $X \subset \mathbb{R}^n$, $|X|$ denotes the Lebesgue measure of X. $B(x,r)$ denotes the open ball in $\mathbb{R}^n$ of radius r and center x.
- $J(u, A, U)$ means that the energy density of (u, A) is integrated only on $U \subset \Omega$.
- For two positive functions $a(\varepsilon)$ and $b(\varepsilon)$, we write $a(\varepsilon) \ll b(\varepsilon)$ as $\varepsilon \to 0$ to mean that $\lim_{\varepsilon\to 0} \frac{a(\varepsilon)}{b(\varepsilon)} = 0$.

2. Lower bound of the energy

First we take $\lambda > 0$, i.e. H is of the order of $|\ln \varepsilon|$. The objective of this section is to prove the lower bound stated in Proposition 2.1 below. Given a family of configurations $\{(u_\varepsilon, A_\varepsilon)\}$, we denote by

$$j_\varepsilon = (iu_\varepsilon, (\nabla - iA_\varepsilon)u_\varepsilon)\ , \quad h_\varepsilon = \operatorname{curl} A_\varepsilon\,. \tag{2.1}$$

Proposition 2.1. *Assume that $\lim_{\varepsilon\to 0} \dfrac{H}{|\ln \varepsilon|} = \lambda$ with $\lambda > 0$. Let $\{(u_\varepsilon, A_\varepsilon)\}_n$ be a family of configurations satisfying $J(u_\varepsilon, A_\varepsilon) \le CH^2$ and $\|u_\varepsilon\|_{L^\infty(\Omega)} \le 1$ for a given*

constant $C > 0$.
Then, up to the extraction of a subsequence ε_n converging to 0, one has

$$\mu_{\varepsilon_n} =: \frac{\mu(u_{\varepsilon_n}, A_{\varepsilon_n})}{H} \longrightarrow \mu_0 \quad \text{in } \mathcal{M}(\Omega), \tag{2.2}$$

$$\frac{j_{\varepsilon_n}}{H} \rightharpoonup j_0, \quad \frac{h_{\varepsilon_n}}{H} \rightharpoonup h_0 \quad \text{weakly in } L^2(\Omega)\,. \tag{2.3}$$

where $\mu(u_{\varepsilon_n}, A_{\varepsilon_n})$ is given in (1.6). Moreover, $\mu_0 = \mathrm{curl} j_0 + h_0$ and

$$\liminf_{\varepsilon_n \longrightarrow 0} \frac{J(u_\varepsilon, A_\varepsilon)}{H^2} \geq E_\lambda(\mu_0) + \int_\Omega |j_0 + \nabla^\perp[q(h_{\mu_0} - 1)]|^2 + q|h_0 - h_{\mu_0}|^2. \tag{2.4}$$

Here, h_{μ_0} and the energy E_λ are introduced in (1.2) and (1.3) respectively.

In order to achieve the above lower bound on the minimal energy $J(u_\varepsilon, A_\varepsilon)$ we adapt results from [23] and [25] regarding energy concentration on balls. First, if $(u_\varepsilon, A_\varepsilon)$ is a minimizer of J, then it is a critical point. It verifies in Ω

$$-\nabla^2_{A_\varepsilon} u_\varepsilon = \frac{p(x)}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2).$$

Under the Coulomb gauge (1.4), we rewrite this as

$$-\Delta u_\varepsilon = \frac{p(x)}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) - 2i(A.\nabla) u_\varepsilon - |A_\varepsilon|^2 u_\varepsilon.$$

Adjusting now as in [27, proposition 3.9], one gets easily

$$|u_\varepsilon| \leq 1.$$

Again, notice that by using $(1, 0)$ as a test configuration for the function (1.1), we deduce an upper bound of the form :

$$J(u_\varepsilon, A_\varepsilon) \leq CH^2\,, \tag{2.5}$$

The hypotheses of proposition 2.1 are satisfied for a minimizer $(u_\varepsilon, A_\varepsilon)$. We recall the hypothesis on H that there exists a positive constant $C > 0$ such that the applied magnetic field H satisfies

$$H \leq C|\ln \varepsilon|\,. \tag{2.6}$$

The upper bound (2.5) provides us, as in [27], with the construction of suitable 'vortex-balls'. We find it in [5, Proposition 4.1]. Note that in what coming (2.7) stays true even in the presence of the weight $p(x)$, because of its uniform upper and lower bound.

Proposition 2.2. *Assume the hypothesis (2.6) holds. Given a number $p \in]1, 2[$, there exists a constant $C > 0$ and a finite family of disjoint balls $\{B_i((a_i, r_i)\}_{i \in I}$ such that, (u, A) being a configuration satisfying the bound (2.5), the following properties hold:*

(1) $\overline{B_i(a_i, r_i)} \subset \Omega$ *for all i ;*
(2) $w = \{x \in \Omega \ : \ |u(x)| \leq 1 - |\ln \varepsilon|^{-4}\} \subset \bigcup_{i \in I} B(a_i, r_i).$
(3) $\sum_{i \in I} r_i \leq C\,|\ln \varepsilon|^{-10}.$

(4) *Letting d_i be the degree of the function $u/|u|$ restricted to $\partial B(a_i, r_i)$ if $B_i(a_i, r_i) \subset \Omega$ and $d_i = 0$ otherwise, then we have:*

$$\int_{B_i(a_i,r_i)} |(\nabla - iA)u_\varepsilon|^2 \,\mathrm{d}x + \frac{p(x)}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 \geq 2\pi|d_i|\left(|\ln\varepsilon| - C\ln|\ln\varepsilon|\right). \quad (2.7)$$

(5) $\left|2\pi\sum_{i\in I} d_i\delta_{a_i} - \mathrm{curl}\big(A + (iu, \nabla_A u)\big)\right|_{W^{-1,p}(\Omega)} \leq C|\ln\varepsilon|^{-4}.$

2.1. **Proof of Proposition 2.1.** We assume that $J(u_\varepsilon, A_\varepsilon) \leq CH^2$ and $\|u_\varepsilon\| \leq 1$. Then, knowing that

$$|\nabla_{A_\varepsilon} u_\varepsilon|^2 \geq |u_\varepsilon|^2 |\nabla_{A_\varepsilon} u_\varepsilon|^2 \geq |j_\varepsilon|^2,$$

with $j_\varepsilon = (iu_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon)$, we obtain

$$\int_\Omega |j_\varepsilon|^2 + q|h_\varepsilon - H|^2 \leq J(u_\varepsilon, A_\varepsilon) \leq CH^2. \quad (2.8)$$

We use the fact that $q(x) \geq \alpha$ to write

$$\int_\Omega |j_\varepsilon|^2 + \alpha\int_\Omega |h_\varepsilon - H|^2 \leq CH^2. \quad (2.9)$$

Then, up to the extraction of a subsequence, $\frac{j_\varepsilon}{H}$ and $\frac{h_\varepsilon}{H}$ respectively converge weakly to j_0 and h_0 in $L^2(\Omega)$. Moreover, since $\mu(u_\varepsilon, A_\varepsilon) = \mathrm{curl} j_\varepsilon + h_\varepsilon$, we have

$$\mu_\varepsilon =: \frac{\mu(u_\varepsilon, A_\varepsilon)}{H} \longrightarrow \mu_0 =: \mathrm{curl} j_0 + h_0, \quad (2.10)$$

weakly in H^{-1}. Using now the balls concentration given in Proposition 2.2 and referring to (2.7)

$$2\pi\sum_i |d_i|(|\ln\varepsilon| - C\ln|\ln\varepsilon|) \leq J(u_\varepsilon, A_\varepsilon, \cup_i B_i) \leq J(u_\varepsilon, A_\varepsilon, \Omega) \leq CH^2,$$

which yields that $\frac{1}{H}\sum_i |d_i|$ remains bounded. Hence, the measure $2\pi\frac{\sum_{i\in I} d_i\delta_{a_i}}{H}$ is weakly compact in the sense of measures, and thus, up to extraction of subsequences, it converges to a measure ν in $\mathcal{M}(\Omega)$. Again, the last property of Proposition 2.2 reads

$$\left|2\pi\sum_{i\in I} d_i\delta_{a_i} - \mu(u_\varepsilon, A_\varepsilon)\big)\right|_{W^{-1,p}_{p<2}(\Omega)} \longrightarrow 0.$$

Thanks to this, one can assert that ν is also the limit of μ_ε. From (2.10), we have $\nu = \mu_0 = \mathrm{curl} j_0 + h_0$. Hence, μ_ε converges to μ_0 in $\mathcal{M}(\Omega)$.
(B_i) being the family of disjoint balls constructed in Proposition 2.2, then thanks to (2.7) and the fact that $|j_\varepsilon| \leq |\nabla u_\varepsilon - iA_\varepsilon u_\varepsilon|$

$$\begin{aligned} J(u_\varepsilon, A_\varepsilon, \Omega) =& J(u_\varepsilon, A_\varepsilon, \cup_i B_i) + J(u_\varepsilon, A_\varepsilon, \Omega\setminus\cup_i B_i) \\ \geq& 2\pi\sum_i |d_i|(|\ln\varepsilon| - C\ln|\ln\varepsilon|) + \int_{\Omega\setminus\cup_i B_i} |j_\varepsilon|^2 + q|h_\varepsilon - H|^2. \end{aligned} \quad (2.11)$$

Adjusting as in [26], then passing to the lim inf, one can find

$$\liminf_{\varepsilon\longrightarrow 0} \frac{J(u_\varepsilon, A_\varepsilon, \Omega)}{H^2} \geq \frac{1}{\lambda}\int_\Omega |\mu_0| + \int_\Omega |j_0|^2 + \int_\Omega q|h_0 - 1|^2. \quad (2.12)$$

Denoting by h_{μ_0} the solution of (1.2), writing

$$j_0 = -\nabla^{\perp}[q(h_{\mu_0} - 1)] + \big(j_0 + \nabla^{\perp}[q(h_{\mu_0} - 1)]\big)\ , \quad h_0 = h_{\mu_0} + (h_0 - h_{\mu_0})$$

and observing that

$$\mathrm{curl}\left(j_0 + \nabla^{\perp}[q(h_{\mu_0} - 1)]\right) + h_0 - h_{\mu_0} = 0, \tag{2.13}$$

we have

$$\begin{aligned}
\int_\Omega |j_0|^2 + \int_\Omega q|h_0 - 1|^2 &= \int_\Omega |\nabla^{\perp}[q(h_{\mu_0} - 1)]|^2 + q|h_{\mu_0} - 1|^2 + |j_0 + \nabla^{\perp}[q(h_{\mu_0} - 1)]|^2 + q|h_0 - h_{\mu_0}|^2 \\
&\quad +2\int_\Omega (-\nabla^{\perp}[q(h_{\mu_0} - 1)]).(j_0 + \nabla^{\perp}[q(h_{\mu_0} - 1)]) + q(h_{\mu_0} - 1)(h_0 - h_{\mu_0}) \\
&= \int_\Omega |\nabla^{\perp}[q(h_{\mu_0} - 1)]|^2 + q|h_{\mu_0} - 1|^2 + |j_0 + \nabla^{\perp}[q(h_{\mu_0} - 1)]|^2 + q|h_0 - h_{\mu_0}|^2 \\
&\quad +2\int_\Omega q(h_{\mu_0} - 1)\mathrm{curl}\left(j_0 + \nabla^{\perp}[q(h_{\mu_0} - 1)]\right) + q(h_{\mu_0} - 1)(h_0 - h_{\mu_0}) \\
&= \int_\Omega |\nabla^{\perp}[q(h_{\mu_0} - 1)]|^2 + q|h_{\mu_0} - 1|^2 + |j_0 + \nabla^{\perp}[q(h_{\mu_0} - 1)]|^2 + q|h_0 - h_{\mu_0}|^2.
\end{aligned}$$

Inserting this in (2.12), we deduce that (2.4) holds. This completes the proof of proposition 2.1.

3. Upper bound of the energy

In this section $0 \leq \lambda < +\infty$. We define the expression

$$F_\lambda(f) = \frac{1}{\lambda}\int_\Omega |-\Delta[q(f-1)] + f| + \int_\Omega |\nabla[q(f-1)]|^2 + q|f-1|^2$$

defined over

$$V = \{f \in H^1_1(\Omega), \quad \mu = -\Delta[q(f-1)] + f \quad \text{is a Radon measure}\},$$

where $H^1_1(\Omega)$ is the space of Sobolev functions f such that $f - 1 \in H^1_0(\Omega)$. For $f \in V$ with $\mu = -\Delta[q(f-1)] + f$, we can write

$$F_\lambda(f) = E_\lambda(\mu).$$

In the next section, it will be proved that the minimum of F_λ is achieved uniquely over V by the function h_* for which $\mu_* = -\Delta[q(h_*-1)] + h_*$ is the unique minimum of E_λ over $H^{-1} \cap \mathcal{M}(\Omega)$. For any $f \in V$, we have

$$(f-1)(x) = \int_\Omega G(x,y)d(\mu - 1)(y), \tag{3.1}$$

where $G(x,y)$ is the Green solution of

$$-\Delta_x[q(x)G(x,y)] + G(x,y) = \delta_y(x) \ \text{ in } \ \Omega \ \text{ and } \ G(x,y) = 0 \ \text{ for } \ x \in \partial\Omega. \tag{3.2}$$

It is clear that for any $f \in V$

$$F_\lambda(f) = I(\mu) = \frac{1}{\lambda}\int_\Omega |\mu| + \int_{\Omega\times\Omega} q(x)G(x,y)d(\mu-1)(x)d(\mu-1)(y). \tag{3.3}$$

Lemma 3.1. *The function G solution of (3.2) verifies*
i) $G(x,y)$ is symmetric and positive.
ii) $q(x)G(x,y)+\frac{1}{2\pi}\ln|x-y|$ is continuous on $\Omega\times\Omega$.
iii) There exists $C>0$ such that for all $x,y\in\Omega\times\Omega\backslash\Delta$

$$\frac{1}{2\pi}\ln|x-y|-C\leq q(x)G(x,y)\leq C(\frac{1}{2\pi}\ln|x-y|+1),$$

where Δ is the diagonal of $\mathbb{R}^2\times\mathbb{R}^2$.

3.1. **Main result.** The objective of this section is to establish the following upper bound, which corresponds to [26, Proposition 2.1].

Proposition 3.2. *Let H be such that $\lim_{\varepsilon\longrightarrow 0}\frac{H}{|\ln\varepsilon|}=\lambda$ with the additional condition, if $\lambda=0$, that $H\ll\frac{1}{\varepsilon^2}$, and μ be a positive Radon measure absolutely continuous with respect to the Lebesgue measure. Then, letting $(u_\varepsilon,A_\varepsilon)$ be a minimizer of J over $H^1\times H^1$,*

$$\limsup_{\varepsilon\longrightarrow 0}\frac{J(u_\varepsilon,A_\varepsilon)}{H^2}\leq I(\mu). \tag{3.4}$$

Again we can adjust the [26, Proposition 2.2].

Proposition 3.3. *Let μ, H, λ be as in proposition 3.2. Then for $\varepsilon>0$ small enough there exist points a_i^ε, $1\leq i\leq n(\varepsilon)$, such that*

$$n(\varepsilon)\simeq\frac{H\mu(\Omega)}{2\pi},\quad |a_i^\varepsilon-a_j^\varepsilon|>4\varepsilon,$$

and letting μ_ε^i be the uniform measure on $\partial B(a_i^\varepsilon,\varepsilon)$ of mass 2π,

$$\mu_\varepsilon=\frac{1}{H}\sum_i\mu_\varepsilon^i\longrightarrow\mu$$

in the sense of measures as $\varepsilon\longrightarrow 0$. Finally,

$$\limsup_{\varepsilon\longrightarrow 0}\int_{\Omega\times\Omega}q(x)G(x,y)d\mu_\varepsilon(x)d\mu_\varepsilon(y)\leq\frac{1}{\lambda}\mu(\Omega)+\int_{\Omega\times\Omega}q(x)G(x,y)d\mu(x)d\mu(y). \tag{3.5}$$

The proof of the above proposition needs a construction of a test configuration for J.

3.2. **Proof of Proposition 3.2.** One may also follow step by step the proof given in [26]. The unique difference in the construction of the test configuration $(u_\varepsilon,A_\varepsilon)$ is in the definition of h_ε. Indeed, let h_ε be the solution to

$$\begin{cases}-\Delta[q(h_\varepsilon-H)]+h_\varepsilon=H\mu_\varepsilon & \text{in}\ \ \Omega\\ h_\varepsilon=H \quad \text{on}\ \ \partial\Omega.\end{cases}$$

Here $(h_\varepsilon-H)(x)=H\int_\Omega G(x,y)d(\mu_\varepsilon-1)(y)$. Let A_ε such that $curl A_\varepsilon=h_\varepsilon$. Therefore,

$$\int_\Omega|\nabla[q(x)(h_\varepsilon-H)]|^2+q(x)|h_\varepsilon-H|^2=H^2\int_{\Omega\times\Omega}q(x)G(x,y)d(\mu_\varepsilon-1)(y)d(\mu_\varepsilon-1)(x).$$

Again choosing $x_0\in\Omega_\varepsilon=\Omega\backslash\cup_i B(a_i^\varepsilon,\varepsilon)$, we let for any $x\in\Omega_\varepsilon$

$$\phi_\varepsilon(x)=\int_{(x_0,x)}A_\varepsilon.\tau-\nabla[q(h_\varepsilon-H)].\nu$$

By construction, one can obtain

$$\nabla\phi_\varepsilon - A_\varepsilon = -\nabla^\perp[q(h_\varepsilon - H)].$$

In other words, we let $\rho_\varepsilon \leq 1$ in order to find

$$\int_\Omega |\nabla\rho_\varepsilon|^2 + \frac{p(x)}{2\varepsilon^2}(1-\rho_\varepsilon^2)^2 \leq CH.$$

We take $u_\varepsilon = \rho_\varepsilon e^{i\phi_\varepsilon}$. In particular

$$\begin{aligned} J(u_\varepsilon, A_\varepsilon) &= \int_\Omega |\nabla\rho_\varepsilon|^2 + \frac{p(x)}{2\varepsilon^2}(1-\rho_\varepsilon^2)^2 + \rho_\varepsilon^2|\nabla\phi_\varepsilon - A_\varepsilon|^2 + q|h_\varepsilon - H|^2 \\ &= \int_\Omega |\nabla\rho_\varepsilon|^2 + \frac{p(x)}{2\varepsilon^2}(1-\rho_\varepsilon^2)^2 + \rho_\varepsilon^2|\nabla[q(h_\varepsilon - H)]|^2 + q|h_\varepsilon - H|^2 \\ &\leq CH + \int_\Omega |\nabla[q(h_\varepsilon - H)]|^2 + q|h_\varepsilon - H|^2 = CH + H^2\int_{\Omega\times\Omega} q(x)G(x,y)d(\mu_\varepsilon - 1)(x)d(\mu_\varepsilon - 1)(y). \end{aligned}$$

Using (3.3)-(3.5) in the above inequality completes the proof of Proposition 3.2. Now, thanks to a standard approximation argument, Proposition 3.2 remains true for the general case where $\mu \in H^{-1} \cap \mathcal{M}(\Omega)$. For more details, see [27, p. 149].

4. Minimization of the limiting energy

As we explained in the introduction, by convexity and lower semi-continuity, the limiting energy (1.3) admits a unique minimizer μ_* which is expressed by means of the unique minimizer $h_* =: h_{\mu_*}$ of (1.5) as follows,

$$\mu_* = -\Delta[q(h_{\mu_*} - 1)] + h_{\mu_*}\,. \tag{4.1}$$

Proceeding as in [26, 27], we may get an equivalent characterization of h_*.

Proposition 4.1. *The minimizer u_* of*

$$\min_{\substack{u\in H_0^1(\Omega)\\ -\Delta[qu]+u\in\mathcal{M}(\Omega)}} \int_\Omega \frac{1}{\lambda}|-\Delta(qu)+u+1| + |\nabla(qu)|^2 + q|u|^2)\,\mathrm{d}x\,,$$

is also the unique minimizer of the dual problem

$$\min_{\substack{v\in H_0^1(\Omega)\\ |v|\leq\frac{1}{2q\lambda}}} \int_\Omega \left(|\nabla(qv)|^2 + q|v|^2 + 2qv\right)\,\mathrm{d}x\,.$$

For instance, $h_* = u_* + 1$ minimizes the energy,

$$\min_{\substack{f\in H_1^1(\Omega)\\ (f-1)\geq-\frac{1}{2q\lambda}}} \left(\int_\Omega |\nabla(q(f-1))|^2 + q|f-1|^2\right)\,,$$

and satisfies $-\Delta(q(h_*-1)) + h_* \geq 0$.

Proof. Let us define the lower semi-continuous and convex functional

$$\Phi(u) = \int_\Omega \frac{1}{2\lambda}\,|-\Delta(qu)+u+1|$$

in the Hilbert space $H_0^1 = H_0^1(\Omega)$ endowed with the scalar product $\langle f,g\rangle_{H_0^1} = \int_\Omega \nabla(qf)\cdot\nabla(qg) + qfg$. Let us compute its conjugate Φ^*, i.e.

$$\Phi^*(f) = \sup_{\{g\,:\,\Phi(g)<\infty\}} \langle f,g\rangle - \Phi(g)\,.$$

Indeed, we have,

$$\Phi^*(f) \geq \sup_{\eta \in L^2} \int_\Omega qf\eta \,\mathrm{d}x - \frac{1}{2\lambda}\int_\Omega |\eta| \,\mathrm{d}x - \int_\Omega qf \,\mathrm{d}x\,,$$

from which we deduce that

$$\Phi^*(f) = \begin{cases} -\displaystyle\int_\Omega qf \,\mathrm{d}x & \text{if } |qf| \leq \frac{1}{2\lambda}\,, \\ +\infty & \text{otherwise.} \end{cases}$$

By convex duality (see [27, Lemma 7.2]),

$$\min_{u \in H} \left(\|u\|_H^2 + 2\Phi(u)\right) = -\min_{f \in H} \left(\|f\|_H^2 + 2\Phi^*(-f)\right)\,,$$

and minimizers coincide. Note that the measure $\mu_* = -\Delta(q(h_{\mu_*} - 1)) + h_{\mu_*}$ is positive and absolutely continuous measure, which is actually a consequence of the weak maximum principle, see [22, p. 131]. One may also follow step by step the proof given in [26]. □

Remark. *Combining the upper and lower bound of Propositions 3.2 and Proposition 2.1, then by uniqueness of the minimizer μ_* of E_λ, it is evident that $\mu_0 = \mu_*$ and $h_0 = h_{\mu_*}$. Here μ_0 and h_0 are given in Proposition 2.1 above.*

Now, let us complete the proof of Theorem 1.2. If $\{(u_\varepsilon, A_\varepsilon)\}$ are minimizers of J, we must have

$$\lim_{\varepsilon \longrightarrow 0} \frac{\min J}{H^2} = \min E_\lambda,$$

together with $j_0 = -\nabla^\perp[q(h_0 - 1)] = -\nabla^\perp[q(h_{\mu_*} - 1)]$. Hence, the vorticity of minimizers of J must converge, after extraction, to the unique minimizer μ_* of E_λ. The uniqueness of μ_* implies that whole of sequence μ_ε converges to μ_* in the sense of measures and $\frac{h_\varepsilon}{H}$ to h_{μ_*} weakly in H^1. Recall that $(u_\varepsilon, A_\varepsilon)$ is a critical point, so it solves the following

$$-\nabla^\perp[q(h_\varepsilon - H)] = (iu_\varepsilon, \nabla u_\varepsilon - iA_\varepsilon u_\varepsilon) =: j_\varepsilon \quad \text{in} \quad \Omega, \quad h_\varepsilon = H \quad \text{on} \quad \partial\Omega.$$

We can find this in [27]. Taking the curl, one can obtain that

$$-\Delta[q(h_\varepsilon - H)] + h_\varepsilon = \mu(u_\varepsilon, A_\varepsilon).$$

Since, $\mathcal{M}(\Omega)$ convergence is stronger than $W^{-1,p}(\Omega)$ convergence for $p < 2$, then we have from the last property of Proposition 2.2

$$\left| -\Delta[q(\frac{h_\varepsilon}{H} - h_{\mu_*})] + \frac{h_\varepsilon}{H} - h_{\mu_*} \right|_{W^{-1,p}_{p<2}(\Omega)} \longrightarrow 0.$$

By elliptic regularity, we deduce that $\frac{h_\varepsilon}{H}$ converges strongly in $W^{1,p}_{p<2}(\Omega)$ to h_{μ_*}. Now, following [26], the limiting vorticity measure μ_* can be expressed by means of the coincidence set w_λ,

$$w_\lambda = \{x \in \Omega \;:\; 1 - h_{\mu_*}(x) = \frac{1}{2q(x)\lambda}\}. \tag{4.2}$$

as follows,

$$\mu_* = \left(1 - \frac{1}{2q(x)\lambda}\right)\mathbf{1}_{w_\lambda}\mathrm{d}x\,, \tag{4.3}$$

where $\mathbf{1}_{w_\lambda}$ denotes the Lebesgue measure restricted to w_λ. Furthermore, h_{μ_*} (the minimizer of (1.5)) solves,

$$\begin{cases} -\Delta[q(h_{\mu_*}-1)] + h_{\mu_*} = 0 & \text{in } \Omega \setminus w_\lambda \\ h_{\mu_*} = 1 - \frac{1}{2q\lambda} & \text{in } w_\lambda \\ h_{\mu_*} = 1 & \text{on } \partial\Omega \end{cases} \tag{4.4}$$

According to the variante of the [26, Proposition 1.2], the vortex nucleation of the minimal energy with respect to the applied field H is given by the following
i) $\Omega \backslash w_\lambda$ is connected.
ii)

$$w_\lambda = \emptyset \Longleftrightarrow \lambda < \frac{1}{2\max_{x\in\Omega}\left(q(x)|\psi(x)|\right)}.$$

iii)

$$\mu_* \neq 0 \Longleftrightarrow \lambda > \frac{1}{2\max_{x\in\Omega}\left(q(x)|\psi(x)|\right)},$$

where ψ is the solution of

$$\begin{cases} -\Delta[q\psi] + \psi = -1 & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega \end{cases}$$

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