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# SIMULTANEOUS APPROXIMATION BY A LINEAR COMBINATION OF BERNSTEIN-DURRMEYER TYPE POLYNOMIALS

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Abstract. The aim of the present paper is to study some direct results in simultaneous approximation for a linear combination of Bernstein-Durrmeyer type polynomials.

## 1. Introduction

For $f \in L_B[0,1]$ (the space of bounded and Lebesgue integrable functions on $[0,1]$), the modified Bernstein type polynomial operators

$$P_n(f;x) = n\sum_{k=1}^{n} p_{n,k}(x) \int_0^1 p_{n-1,k-1}(t)f(t)\,dt + (1-x)^n f(0),$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}, \ \ 0 \leqslant x \leqslant 1,$$

were introduced by Gupta and Maheshwari [8] wherein they studied the approximation of functions of bounded variation by these operators. In [6], Gupta and Ispir studied the pointwise convergence and Voronovskaja type asymptotic results in simultaneous approximation. Gairola [5] derived direct, inverse and saturation results for an iterative combination of these operators in ordinary approximation. We [1] studied a direct theorem in the $L_p-$ norm for these combinations of the operator $P_n$.
The operators $P_n(f;x)$ can be expressed as

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$$P_n(f;x) = \int_0^1 W_n(t,x) f(t)\,dt,$$

where the kernel of the operators is given by

$$W_n(t,x) = n\sum_{k=1}^{n} p_{n,k}(x)\,p_{n-1,k-1}(t) + (1-x)^n\delta(t),$$

$\delta(t)$ being the Dirac-delta function.

It turns out that the order of approximation by these operators is at best $O(n^{-1})$, however smooth the function may be.

Following the technique of linear combination described in [3] to improve the order of approximation, we define

$$P_n(f,k,x) = \sum_{j=0}^{k} C(j,k) P_{d_j n}(f,x),$$

where

$$C(j,k) = \prod_{i=0, i\neq j}^{k} \frac{d_j}{d_j - d_i}, k \neq 0 \text{ and } C(0,0) = 1, \tag{1.1}$$

$d_0, d_1, ...d_k$ being $(k+1)$ arbitrary but fixed distinct positive integers.

The object of the present paper is to investigate some direct results in the simultaneous approximation by the operators $P_n(.,k,x)$. First we establish a Voronovskaja type asymptotic formula and then obtain an error estimate in terms of local modulus of continuity of the function involved for the operator $P_n^{(r)}(.,k,x)$.

## 2. Auxiliary Results

In the sequel we shall require the following results:

**Lemma 2.1.** [6] *For the function $u_{n,m}(x), m \in \mathbb{N}^0$ (the set of non-negative integers) defined as*

$$u_{n,m}(x) = \sum_{\nu=0}^{n} p_{n,\nu}(x)\left(\frac{\nu}{n} - x\right)^m,$$

*we have $u_{n,0}(x) = 1$ and $u_{n,1}(x) = 0$. Further, there holds the recurrence relation*

$$nu_{n,m+1}(x) = x\left[u'_{n,m}(x) + mu_{n,m-1}(x)\right], m = 1, 2, 3, ...$$

*Consequently,*
*(i) $u_{n,m}(x)$ is a polynomial in $x$ of degree $[m/2]$, where $[\alpha]$ denotes the integral part of $\alpha$;*
*(ii) for every $x \in [0,1], u_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$.*

**Remark 1.** *From the above lemma, we have*

$$\sum_{\nu=0}^{n} p_{n,\nu}(x)\,(\nu - nx)^{2j} = O(n^j) \tag{2.1}$$

For $m \in \mathbb{N}^0$ (the set of non-negative integers), the $m$th order moment for the operators $P_n$ is defined as

$$\mu_{n,m}(x) = P_n\left((t-x)^m; x\right).$$

**Lemma 2.2.** [1] *For the function $\mu_{n,m}(x)$, we have $\mu_{n,0}(x) = 1$, $\mu_{n,1}(x) = \frac{(-x)}{(n+1)}$, and there holds the recurrence relation*

$$(n+m+1)\mu_{n,m+1}(x) = x(1-x)\left\{\mu'_{n,m}(x) + 2m\mu_{n,m-1}(x)\right\} + (m(1-2x)-x)\mu_{n,m}(x),$$

*for $m \geq 1$.*
*Consequently, we have*
(i) *$\mu_{n,m}(x)$ is a polynomial in $x$ of degree $m$;*
(ii) *for every $x \in [0,1], \mu_{n,m}(x) = \mathcal{O}\left(n^{-[(m+1)/2]}\right)$.*

**Remark 2.** *From the above lemma, it follows that for each $x \in (0,1)$,*

$$n\sum_{k=1}^{n} p_{n,k}(x)\int_0^1 p_{n-1,k-1}(t)(t-x)^m\,dt = \mathcal{O}\left(n^{-[(m+1)/2]}\right),\ m \in \mathbb{N}^0.$$

**Lemma 2.3.** *If $C(j,k), j = 0,1,2,......., k$ is defined as in 1.1, then*

$$\sum_{j=0}^{k} C(j,k)d_j^{-m} = \begin{cases} 1, & m = 0 \\ 0, & m = 1,2,3,4........ \end{cases}$$

**Lemma 2.4.** *For $p \in \mathbb{N}, P_n((t-x)^p, k, x) = n^{-(k+1)}\{Q(p,k,x) + o(1)\}$ where $Q(p,k,x)$ are certain polynomials in $x$ of degree at most $p$.*

From Lemma 2.2 and Lemma 2.3 the above lemma easily follows hence the details are omitted.

Throughout this paper, we assume $0 < a < b < 1$, $I = [a,b]$, $0 < a_1 < a_2 < b_2 < b_1 < 1$, $I_i = [a_i, b_i], i = 1,2$, $[a_1, b_1] \subset (a,b)$, $\|.\|_{C(I)}$ the sup- norm on the interval $I$ and $C$ a constant not necessarily the same at each occurrence.
Let $f \in C[a,b]$. Then, for a sufficiently small $\eta > 0$, the Steklov mean $f_{\eta,m}$ of $m-$th order corresponding to $f$ is defined as follows:

$$f_{\eta,m}(t) = \eta^{-m}\int_{-\eta/2}^{\eta/2}\cdots\int_{-\eta/2}^{\eta/2}\left(f(t) + (-1)^{m-1}\Delta^m_{\sum_{i=1}^m t_i} f(t)\right)\prod_{i=1}^{m} dt_i,\ ,\ t \in I_1,$$

where $\Delta_h^m$ is the $m-$th forward difference operator with step length $h$.

**Lemma 2.5.** *Let $f \in C[a,b]$. Then, for the function $f_{\eta,m}$, we have*
(a) *$f_{\eta,m}$ has derivatives up to order $m$ over $I_1$;*
(b) *$\|f^{(r)}_{\eta,m}\|_{C(I_1)} \leqslant C_r\ \omega_r(f,\eta,[a,b]), r = 1,2,...,m$;*
(c) *$\|f - f_{\eta,m}\|_{C(I_1)} \leqslant C_{m+1}\ \omega_m(f,\eta,[a,b])$;*
(d) *$\|f_{\eta,m}\|_{C(I_1)} \leqslant C_{m+2}\ \eta^{-m}\|f\|_{C[a,b]}$;*
(e) *$\|f^{(r)}_{\eta,m}\|_{C(I_1)} \leqslant C_{m+3}\ \|f\|_{C[a,b]}$,*

*where $C_i's$ are certain constants that depend on $i$ but are independent of $f$ and $\eta$.*

Following ([9], Theorem 18.17) or ([10], pp.163-165), the proof of the above lemma easily follows hence the details are omitted.

**Lemma 2.6.** [1] *For the function $p_{n,k}(x)$, there holds the result*

$$x^r(1-x)^r\frac{d^r p_{n,k}(x)}{dx^r}=\sum_{\substack{2i+j\leqslant r\\ i,j\geq 0}} n^i(k-nx)^j q_{i,j,r}(x)p_{n,k}(x),$$

*where $q_{i,j,r}(x)$ are certain polynomials in $x$ independent of $n$ and $k$.*

**Theorem 2.7.** *Let $f\in L_B[0,1]$ admitting a derivative of order $2k+2$ at a point $x\in[0,1]$ then we have*

$$\lim_{n\to\infty} n^{k+1}[P_n(f,k,x)-f(x)]=\sum_{\nu=1}^{2k+2}\frac{f^{(\nu)}(x)}{\nu!}Q(\nu,k,x) \tag{2.2}$$

*and*

$$\lim_{n\to\infty} n^{k+1}[P_n(f,k+1,x)-f(x)]=0, \tag{2.3}$$

*where $Q(\nu,k,x)$ are certain polynomials in $x$ of degree $\nu$. Further, the limits in (2.2) and (2.3) hold uniformly in $[a,b]$ if $f^{(2k+2)}$ is continuous on $(a-\eta,b+\eta)\subset(0,1)$, $\eta>0$.*

Proceeding along the lines of the proof of (Thm., [2]), the above theorem easily follows. Hence the details are omitted.

## 3. Main Results

**Theorem 3.1.** *Let $f\in L_B[0,1]$ admitting a derivative of order $2k+r+2$ at a point $x\in(0,1)$ then we have*

$$\lim_{n\to\infty} n^{k+1}[P_n^{(r)}(f,k,x)-f^{(r)}(x)]=\sum_{\nu=r}^{2k+r+2}\frac{f^{(\nu)}(x)}{\nu!}Q_1(\nu,k,r,x) \tag{3.1}$$

*and*

$$\lim_{n\to\infty} n^{k+1}[P_n^{(r)}(f,k+1,x)-f^{(r)}(x)]=0, \tag{3.2}$$

*where $Q_1(\nu,k,r,x)$ are certain polynomials in $x$. Further, the limits in (3.1) and (3.2) hold uniformly in $[a,b]$ if $f^{(2k+r+2)}$ is continuous on $(a-\eta,b+\eta)\subset(0,1)$, $\eta>0$.*

*Proof.* By a partial Taylor's expansion of $f$, we have

$$f(t)=\sum_{\nu=0}^{2k+r+2}\frac{f^{(\nu)}(x)}{\nu!}(t-x)^\nu+\epsilon(t,x)(t-x)^{2k+r+2},$$

where $\epsilon(t,x)\to 0$ as $t\to x$. Thus, we can write

$$\begin{aligned} n^{k+1}[P_n^{(r)}(f,k,x)-f^{(r)}(x)] &= n^{k+1}\left[\sum_{\nu=0}^{2k+2+r}\frac{f^{(\nu)}(x)}{\nu!}P_n^{(r)}((t-x)^\nu,k,x)-f^{(r)}(x)\right] \\ &+ n^{k+1}\sum_{j=0}^{k}C(j,k)P_{d_jn}^{(r)}(\epsilon(t,x)(t-x)^{2k+r+2};x) \\ &= \Sigma_1+\Sigma_2,\ \text{say.} \end{aligned}$$

On an application of Lemma 2.2 and Theorem 2.7 we obtain

$$\begin{aligned} \Sigma_1 &= n^{k+1}\left[\sum_{\nu=0}^{2k+r+2}\frac{f^{(\nu)}(x)}{\nu!}P_n^{(r)}((t-x)^\nu,k,x)-f^{(r)}(x)\right] \\ &= n^{k+1}\left[\sum_{\nu=r}^{2k+r+2}\frac{f^{(\nu)}(x)}{\nu!}\sum_{i=0}^{\nu}\binom{\nu}{i}(-x)^{\nu-i}P_n^{(r)}(t^i,k,x)-f^{(r)}(x)\right] \\ &= n^{k+1}\left[\sum_{\nu=r}^{2k+r+2}\frac{f^{(\nu)}(x)}{\nu!}\sum_{i=0}^{\nu}\binom{\nu}{i}(-x)^{\nu-i}\times\right. \\ &\qquad \left.\left\{D^r x^i+n^{-(k+1)}\left[\sum_{j=1}^{2k+2}D^r\left(\frac{D^j x^i}{j!}Q(j,k,x)\right)+o(1)\right]\right\}-f^{(r)}(x)\right] \\ &= n^{k+1}\left[\sum_{\nu=r}^{2k+r+2}\frac{f^{(\nu)}(x)}{\nu!}r!\sum_{i=0}^{\nu}\binom{\nu}{i}\binom{i}{r}(-1)^{\nu-i}(x)^{\nu-r}-f^{(r)}(x)\right] \\ &+ \sum_{\nu=r}^{2k+r+2}Q_1(\nu,k,r,x)f^{(\nu)}(x)+o(1), \end{aligned}$$

where we have used the identity

$$\sum_{l=0}^{i}(-1)^l\binom{i}{l}\binom{l}{r}=\begin{cases}0, & i>r\\ (-1)^r, & i=r.\end{cases}$$

Thus, we get

$$\Sigma_1=\sum_{\nu=r}^{2k+r+2}Q_1(\nu,k,r,x)f^{(\nu)}(x)+o(1)$$

In order to prove the assertion 3.1, it is sufficient to show that

$$n^{k+1}P_n^{(r)}(\epsilon(t,x)(t-x)^{2k+r+2};x)\to 0 \text{ as } n\to\infty.$$

$$\begin{aligned} \Sigma &\equiv P_n^{(r)}(\epsilon(t,x)(t-x)^{2k+r+2};x) \\ &= n\sum_{k=1}^{n}p_{n,k}^{(r)}(x)\int_0^1 p_{n-1,k-1}(t)\epsilon(t,x)(t-x)^{2k+r+2}\,dt \\ &+ (-1)^r\frac{n!}{(n-r)!}(1-x)^{n-r}\epsilon(0,x)(-x)^{2k+r+2}. \end{aligned}$$

Therefore, by using Lemma 2.6 we have

$$\begin{aligned}
|\Sigma| &\leq n \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \frac{|q_{i,j,r}(x)|}{x^r(1-x)^r} \sum_{k=1}^{n} |k-nx|^j \, p_{n,k}(x) \times \\
&\qquad \int_0^1 p_{n-1,k-1}(t)|\epsilon(t,x)||t-x|^{2k+2+r}\, dt \\
&+ \frac{n!}{(n-r)!}(1-x)^{n-r}|\epsilon(0,x)|x^{2k+r+2} \\
&= J_1 + J_2, \text{ say.}
\end{aligned}$$

Since $\epsilon(t,x) \to 0$ as $t \to x$, for a given $\epsilon' > 0$ we can find a $\delta > 0$ such that $|\epsilon(t,x)| < \epsilon'$ whenever $0 < |t-x| < \delta$ and for $|t-x| \geq \delta$, $|\epsilon(t,x)| \leq K$ for some $K > 0$. Hence

$$\begin{aligned}
|J_1| &\leq nC_1 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \sum_{k=1}^{n} |k-nx|^j \, p_{n,k}(x) \times \\
&\qquad \left[\epsilon' \int_{|t-x|<\delta} p_{n-1,k-1}(t)|t-x|^{2k+2+r}\, dt + \right. \\
&\qquad \left. \frac{1}{\delta^2} \int_{|t-x|\geq\delta} p_{n-1,k-1}(t) K |t-x|^{2k+4+r}\, dt \right] \\
&= J_3 + J_4, \text{ say,}
\end{aligned}$$

where $C_1 = \sup_{\substack{2i+j\leq r\\ i,j\geq 0}} |q_{i,j,r}(x)|/x^r(1-x)^r$.
Applying Schwarz inequality for integration and then summation and Lemma 2.2, 2.1 we have

$$\begin{aligned}
|J_3| &\leq C_1\epsilon' n^{1/2} \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \left( \sum_{k=1}^{n} (k-nx)^{2j} \, p_{n,k}(x) \right)^{1/2} \times \left( \int_0^1 p_{n-1,k-1}(t) dt \right)^{1/2} \\
&\qquad \left( n \sum_{k=1}^{n} p_{n,k}(x) \int_0^1 p_{n-1,k-1}(t)(t-x)^{4k+4+2r}\, dt \right)^{1/2} \\
&\leq C_1\epsilon' \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i O(n^{j/2}) O(n^{-(2k+2+r)/2}), (\text{ in view of Remark 2}), \\
&= \epsilon' O(n^{-(k+1)}).
\end{aligned}$$

Next, again Applying Schwarz inequality for integration and then summation and Lemma 2.2, 2.1 we have

$$\begin{aligned}
|J_4| &\leq \frac{C_1}{\delta^2} n^{1/2} \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \left(\sum_{k=1}^{n} (k-nx)^{2j}\, p_{n,k}(x)\right)^{1/2} \times \\
&\quad \left(n\sum_{k=1}^{n} p_{n,k}(x) \int_0^1 p_{n-1,k-1}(t)(t-x)^{4k+8+2r} dt\right)^{1/2} \times \\
&\quad \left(\int_0^1 p_{n-1,k-1}(t)dt\right)^{1/2} \\
&\leq \frac{C_1}{\delta^2} \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i O(n^{j/2}) O(n^{-(2k+4+r)/2}) \\
&= C_2 O(n^{-(k+2)}).
\end{aligned}$$

Combining the estimates of $J_3$ and $J_4$, we get $J_1 = \epsilon' O(n^{-(k+1)})$. Clearly, $J_2 = o(n^{-(k+1)})$. Combining the estimates $J_1$ and $J_2$, due to the arbitrariness of $\epsilon' > 0$, it follows that $n^{k+1}\Sigma \to 0$ as $n \to \infty$. This completes the proof of the assertion 3.1. The assertion 3.2 can be proved along similar lines by noting that

$$M_n((t-x)^i, k+1, x) = O(n^{-(k+2)}),\ i = 1, 2, 3 \ldots$$

which follows from Lemma 2.4.
Uniformity assertion follows easily from the fact that $\delta(\epsilon)$ in the above proof can be chosen to be independent of $x \in [a, b]$ and all the other estimates hold uniformly on $[a, b]$. □

In the following theorem, we study an error estimate for $P_n^{(r)}(f, k, x)$.

**Theorem 3.2.** *Let $p \in \mathbb{N}, 1 \leq p \leq 2k+2$ and $f \in L_B[0, 1]$. If $f^{(p+r)}$ exists and is continuous on $(a-\eta, b+\eta) \subset [0, 1]$, $\eta > 0$ then*

$$\left\|P_n^{(r)}(f, k, .) - f^{(r)}\right\| \leq \max\left\{C_1 n^{-p/2} \omega\left(f^{(p+r)}, n^{-1/2}\right), C_2 n^{-(k+1)}\right\}, \qquad (3.3)$$

*where $C_1 = C_1(k, p, r)$, $C_2 = C_2(k, p, r, f)$ and $\omega\left(f^{(p+r)}, \delta\right)$ is the modulus of continuity of $f^{(p+r)}$ on $(a-\eta,\, b+\eta)$.*

*Proof.* By our hypothesis, we may write for all $t \in [0, 1]$ and $x \in [a, b]$

$$\begin{aligned}
f(t) &= \sum_{\nu=0}^{p+r} \frac{f^{(\nu)}(x)}{\nu!}(t-x)^\nu + \frac{f^{(p+r)}(\xi) - f^{(p+r)}(x)}{(p+r)!}(t-x)^{p+r}\chi(t) \\
&\qquad + F(t, x)(1-\chi(t)), \qquad (3.4)
\end{aligned}$$

where $\chi(t)$ is the characteristic function of $(a-\eta,\, b+\eta)$, $\xi$ lies between $t$ and $x$ and $F(t, x)$ is defined as

$$F(t, x) = f(t) - \sum_{\nu=0}^{p+r} \frac{f^{(\nu)}(x)}{\nu!}(t-x)^\nu,\ \forall\, t \in [0, 1] \setminus (a-\eta,\, b+\eta) \text{ and } x \in [a, b].$$

Now operating by $P_n^{(r)}(.,k,x)$ on both sides of (3.4) and breaking the right hand side into three parts $I_1, I_2$ and $I_3$ say, corresponding to the three terms on the right hand side of (3.4), we get

$$P_n^{(r)}(f,k,x) - f^{(r)}(x) = I_1 + I_2 + I_3.$$

To estimate

$$I_1 = \sum_{\nu=0}^{p+r} \frac{f^{(\nu)}(x)}{\nu!} P_n^{(r)}((t-x)^\nu, k, x),$$

proceeding as in the estimate of $\Sigma_1$ of Theorem 3.1, we obtain

$$I_1 = O(n^{-(k+1)}),$$

uniformly in $x \in [a,b]$.
For every $\delta > 0$, we have

$$|f^{(p+r)}(\xi) - f^{(p+r)}(x)| \le \omega_{f^{(p+r)}}(|\xi - x|) \le \omega_{f^{(p+r)}}(|t-x|) \le \left(1 + \frac{|t-x|}{\delta}\right)\omega_{f^{(p+r)}}(\delta).$$

Consequently,

$$\begin{aligned} |I_2| &\le \frac{\omega\left(f^{(p+r)},\delta\right)}{(p+r)!} \sum_{j=0}^{k} |C(j,k)| \Bigg[ d_j n \sum_{\nu=1}^{d_j n} |p_{d_j n,\nu}^{(r)}(x)| \\ &\qquad \times \int_0^1 p_{d_j n-1,\nu-1}(t)|t-x|^{p+r}(1+|t-x|\delta^{-1})dt \\ &+ \frac{d_j n!}{(d_j n - r)!}(1-x)^{d_j n - r}\left(|x|^{p+r} + \delta^{-1}|x|^{p+r+1}\right)\Bigg] \\ &= I_4 + I_5, \text{say.} \end{aligned}$$

In order to estimate $I_2$, we proceed as follows:
Using Lemma 2.6 and Schwarz inequality for integration and then for summation we have

$$\begin{aligned} & n\sum_{\nu=1}^{n} |p_{n,\nu}^{(r)}(x)| \int_0^1 p_{n-1,\nu-1}(t)|t-x|^s\,dt \\ &\le n \sum_{\nu=1}^{n} \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i |\nu - nx|^j \frac{|q_{i,j,r}(x)|}{x^r(1-x)^r} p_{n,\nu}(x) \int_0^1 p_{n-1,\nu-1}(t)|t-x|^s\,dt \\ &\le K \sum_{\substack{2i+j\le p\\ i,j\ge 0}} n^i \left[ n\sum_{\nu=1}^{n} p_{n,\nu}(x)|\nu-nx|^j \int_0^1 p_{n-1,\nu-1}(t)|t-x|^s\,dt \right] \\ &= \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i O(n^{(j-s)/2}) = O(n^{(r-s)/2}), \end{aligned} \tag{3.5}$$

uniformly in $x \in [a,b]$, where $K = \sup_{\substack{2i+j\le r\\ i,j\ge 0}} \sup_{x\in[a,b]} \frac{|q_{i,j,r}(x)|}{x^r(1-x)^r}$.
Choosing $\delta = n^{-1/2}$ and using 3.5 we are lead to,

$$\begin{aligned}|I_4| &\leq \frac{\omega\left(f^{(p+r)},n^{-1/2}\right)}{(p+r)!}\left[O\left(n^{-p/2}\right)+n^{1/2}O\left(n^{-(p+1)/2}\right)\right]\\ &= \omega\left(f^{(p+r)},n^{-1/2}\right)O\left(n^{-p/2}\right),\text{ uniformly for all } x\in[a,b].\end{aligned}$$

Now, $I_5 = O\left(n^{-s}\right)$, for any $s > 0$, uniformly for all $x \in [a, b]$. Choosing $s > k + 1$, $I_5 = o\left(n^{-(k+1)}\right)$, uniformly for all $x \in [a, b]$.
To estimate $I_3$, we note that $t \in [0, 1] \setminus (a - \eta, b + \eta)$, we can choose $\delta > 0$ in such a way that $|t - x| \geq \delta$ for all $x \in [a, b]$.
Thus, by Lemma 2.6, we obtain

$$\begin{aligned}|I_3| \leq & \sum_{j=0}^{k}|C(j,k)|\Bigg[d_jn\sum_{\nu=1}^{d_jn}|p^{(r)}_{d_jn,\nu}(x)|\\ &\times\int\limits_{|t-x|\geq\delta}p_{d_jn-1,\nu-1}(t)|F(t,x)|\,dt+\frac{d_jn!}{(d_jn-r)!}(1-x)^{n-r}|F(0,x)|\Bigg]\end{aligned}$$

For $|t - x| \geq \delta$, we can find a constant $C > 0$ such that $|F(t, x)| \leq C$, therefore using 3.5 it easily follows that $I_3 = O\left(n^{-s}\right)$ for any $s > 0$, uniformly on $[a, b]$. Choosing $s > k+1$ we obtain $I_3 = o(n^{-(k+1)})$, uniformly on $[a, b]$. Now combining the estimates of $I_1, I_2, I_3$, the required result is immediate.
This completes the proof. □

In the following theorem, we study an error estimate for $P_n^{(r)}(f, k, x)$ in terms of higher order modulus of continuity in simultaneous approximation.

**Theorem 3.3.** *Let $f \in L_B[0, 1]$. If $f^{(r)}$ exists and is continuous on $I_1$, then for sufficiently large $n$,*

$$\left\|P_n^{(r)}(f,k,.)-f^{(r)}(.)\right\|_{C(I_2)}\leqslant C\left\{n^{-k}\|f\|_{L_B[0,1]}+\omega_{2k+2}\big(f^{(r)};n^{-1/2};I_1\big)\right\},$$

*where $C$ is independent of $f$ and $n$.*

*Proof.* We can write

$$\begin{aligned}I &= \left\|P_n^{(r)}(f,k,.)-f^{(r)}\right\|_{C(I_2)}\\ &\leqslant \left\|P_n^{(r)}(f-f_{\eta,2k+2},k,.)\right\|_{C(I_2)}+\left\|P_n^{(r)}(f_{\eta,2k+2},k,.)-f^{(r)}_{\eta,2k+2}\right\|_{C(I_2)}\\ &+ \left\|f^{(r)}(x)-f^{(r)}_{\eta,2k+2}(x)\right\|_{C(I_2)}\\ &:= E_1+E_2+E_3,\ \text{say}.\end{aligned}$$

Since $f^{(r)}_{\eta,2k+2} = \left(f^{(r)}\right)_{\eta,2k+2}$, by property (c) of the Steklov mean we get

$$E_3 \leqslant C\,\omega_{2k+2}\big(f^{(r)},\eta,I_1\big).$$

Next, applying Theorem 3.1 and the interpolation property [7], for each $m = r, r + 1, ..., 2k + 2 + r$, it follows that

$$\begin{aligned} E_2 &\leqslant C\,n^{-(k+1)} \sum_{m=r}^{2k+2+r} \left\| f_{\eta,2k+2}^{(m)} \right\|_{C(I_2)} \\ &\leqslant C\,n^{-(k+1)} \left( \left\| f_{\eta,2k+2} \right\|_{C(I_2)} + \left\| f_{\eta,2k+2}^{(2k+2+r)} \right\|_{C(I_2)} \right) \\ &\leqslant C\,n^{-(k+1)} \left( \left\| f_{\eta,2k+2} \right\|_{C(I_2)} + \left\| \left( f^{(r)} \right)_{\eta,2k+2}^{2k+2} \right\|_{C(I_2)} \right). \end{aligned}$$

Hence, by property (b) and (d) of the Steklov mean, we have

$$E_2 \leqslant C\,n^{-(k+1)} \left\{ \|f\|_{C(I_1)} + \eta^{-(2k+2)} \omega_{2k+2}\big(f^{(r)}, \eta, I_1\big) \right\}.$$

Let $f - f_{\eta,2k} = \mathcal{F}$. By our hypothesis, we can write

$$\begin{aligned} \mathcal{F}(t) &= \sum_{m=0}^{r} \frac{\mathcal{F}^{(m)}(x)}{m!}(t-x)^m + \frac{\mathcal{F}^{(r)}(\xi) - \mathcal{F}^{(r)}(x)}{r!}(t-x)^r \psi(t) \\ &+ h(t,x)\,(1-\psi(t))\,, \end{aligned}$$

where $\xi$ lies between $t$ and $x$, and $\psi$ is the characteristic function of the interval $I_1$. For $t \in I_1$ and $x \in I_2$, we get

$$\mathcal{F}(t) = \sum_{m=0}^{r} \frac{\mathcal{F}^{(m)}(x)}{m!}(t-x)^m + \frac{\mathcal{F}^{(r)}(\xi) - \mathcal{F}^{(r)}(x)}{r!}(t-x)^r,$$

and for $t \in [0,1] \setminus [a_1, b_1], x \in I_2$ we define

$$h(t,x) = \mathcal{F}(t) - \sum_{m=0}^{r} \frac{\mathcal{F}^{(m)}(x)}{m!}(t-x)^m.$$

Now,

$$\begin{aligned} P_n^{(r)}\big(\mathcal{F}(t), k, x\big) &= \sum_{m=0}^{r} \frac{\mathcal{F}^{(m)}(x)}{m!} P_n^{(r)}\big((t-x)^m, k, x\big) \\ &+ P_n^{(r)}\left( \frac{\mathcal{F}^{(r)}(\xi) - \mathcal{F}^{(r)}(x)}{r!}(t-x)^r \psi(t), k, x \right) \\ &+ P_n^{(r)}\big(h(t,x)\,(1-\psi(t))\,, k, x\big) := J_1 + J_2 + J_3, \quad \text{say.} \end{aligned}$$

In order to estimate $J_1$, in view of Lemma 2.2 we note that

$$\begin{aligned} \sum_{m=0}^{r} \frac{\mathcal{F}^{(m)}(x)}{m!} P_n^{(r)}\big((t-x)^m, x\big) &= \frac{\mathcal{F}^{(r)}(x)}{r!} P_n^{(r)}\big(t^r, x\big) \\ &= \frac{\mathcal{F}^{(r)}(x)}{r!} \left[ r! \frac{n^r}{\Pi_{j=1}^{r}(n+j)} \right]. \end{aligned}$$

By using Lemma 2.2, we get

$$\begin{aligned} J_1 &= \sum_{m=0}^{r} \frac{\mathcal{F}^{(m)}(x)}{m!} P_n^{(r)}\big((t-x)^m, k, x\big) \\ &= \sum_{m=0}^{r} \frac{\mathcal{F}^{(m)}(x)}{m!} \sum_{l=0}^{m} \binom{m}{l} (-x)^{m-l} P_n^{(r)}\big(t^l, k, x\big) \\ &\to \frac{(n+r-1)!}{n^r(n-1)!} \mathcal{F}^{(r)}(x). \end{aligned}$$

Hence, for sufficiently large $n$, we have

$$|J_1| \leqslant C\, \|f^{(r)} - f^{(r)}_{\eta,2k+2}\|_{C(I_2)}$$

Next, applying Schwarz inequality for integration and then for summation and using Remarks 1-2, we get

$$\begin{aligned} J_2 &\leqslant \frac{2}{r!}\|f^{(r)} - f^{(r)}_{\eta,2k+2}\|_{C(I_2)} P_n^{(r)}\big(\psi(t)|t-x|^r, k, x\big) \\ &\leqslant \frac{2}{r!}\|f^{(r)} - f^{(r)}_{\eta,2k+2}\|_{C(I_2)} \sum_{j=0}^{k} |C(j,k)| \sum_{\substack{2i+j\leqslant r \\ i,j\geqslant 0}} (d_j n)^i \frac{|q_{i,j,r}(x)|}{x^r(1-x)^r} d_j n \times \\ &\quad \times \sum_{\nu=1}^{d_j n} p_{d_j n,\nu}(x)|\nu - d_j n x|^j \int_0^1 p_{d_j n-1,\nu-1}(t)\psi(t)|t-x|^r\, dt \\ &\leqslant \frac{2}{r!}\|f^{(r)} - f^{(r)}_{\eta,2k+2}\|_{C(I_2)} \sum_{j=0}^{k} |C(j,k)| \sum_{\substack{2i+j\leqslant r \\ i,j\geqslant 0}} (d_j n)^{i+1} \frac{|q_{i,j,r}(x)|}{x^r(1-x)^r} \times \\ &\quad \times \sum_{\nu=1}^{d_j n} p_{d_j n,\nu}(x)|\nu - d_j n x|^j \left( \int_0^1 p_{d_j n-1,\nu-1}(t)\, dt \right)^{1/2} \times \\ &\quad \times \left( \int_0^1 p_{d_j n-1,\nu-1}(t)(t-x)^{2r}\, dt \right)^{1/2} \\ &\leqslant C\, \|f^{(r)} - f^{(r)}_{\eta,2k+2}\|_{C(I_2)} \sum_{j=0}^{k} |C(j,k)| \times \\ &\quad \times \sum_{\substack{2i+j\leqslant r \\ i,j\geqslant 0}} (d_j n)^i \left( \sum_{\nu=1}^{d_j n} p_{d_j n,\nu}(x)(\nu - d_j n x)^{2j} \right)^{1/2} \times \\ &\quad \times \left( d_j n \sum_{\nu=1}^{d_j n} p_{d_j n,\nu}(x) \int_0^1 p_{d_j n-1,\nu-1}(t)(t-x)^{2r} \right)^{1/2} \\ &\leqslant C\, \|f^{(r)} - f^{(r)}_{\eta,2k+2}\|_{C(I_2)} \sum_{j=0}^{k} |C(j,k)| \sum_{\substack{2i+j\leqslant r \\ i,j\geqslant 0}} (d_j n)^i O\big(n^{j/2}\big) O\big(n^{-r/2}\big) \end{aligned}$$

or

$$J_2 \leqslant C' \|f^{(r)} - f^{(r)}_{\eta,2k+2}\|_{C(I_2)}.$$

Since $t \in [0,1] \setminus I_1$, we can choose a $\delta > 0$ in such a way that $|t-x| \geqslant \delta$ for all $x \in I_2$. Thus, by Lemma 2.2, we obtain

$$\begin{aligned} |J_3| \quad \leqslant \quad & \sum_{j=0}^{k} |C(j,k)| \sum_{\substack{2i+j\leqslant r \\ i,j\geqslant 0}} (d_j n)^i \frac{|q_{i,j,r}(x)|}{x^r(1-x)^r} d_j n \sum_{\nu=1}^{d_j n} p_{d_j n,\nu}(x) |\nu - d_j n x|^j \times \\ & \int\limits_{|t-x|\geqslant\delta} p_{d_j n-1,\nu-1}(t) |h(t,x)|\, dt + \frac{d_j n!}{(d_j n - r)!}(1-x)^{d_j n - r} |h(0,x)| \end{aligned}$$

For $|t-x| \geqslant \delta$, we can find a constant $C > 0$ such that $|h(t,x)| \leqslant C$. Hence, proceeding as a manner similar to the estimate of $J_2$, it follows that $J_3 = O\big(n^{-s}\big)$ for any $s > 0$.
Combining the estimates of $J_1 - J_3$, we obtain

$$\begin{aligned} E_1 \quad & \leqslant \quad C \|f^{(r)} - f^{(r)}_{\eta,2k+2}\|_{C(I_2)} \\ & \leqslant \quad C\, \omega_{2k+2}\big(f^{(r)}, \eta, I_1\big) \text{ (in view of (c) of Steklov mean).} \end{aligned}$$

Therefore, with $\eta = n^{-1/2}$ the theorem follows. □

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