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A SUBORDINATION THEOREM WITH APPLICATIONS TO ANALYTIC FUNCTIONS

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Abstract. In this paper, a subordination theorem is obtained and its applications to multivalent functions are discussed. It is shown that the results of this paper extend and unify some known results regarding p-valently close-to-convex functions. Some results for univalent close-to-convex functions are obtained as special cases.

1. Introduction

Let $\mathcal{A}$ be the class of all functions f which are analytic in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$ and normalized by the conditions that $f(0) = f'(0) - 1 = 0$. Thus, $f \in \mathcal{A}$ has the Taylor series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

A function f is said to be univalent in a domain $\mathbb{D}$ in the extended complex plane $\mathbb{C}$ if and only if it is analytic in $\mathbb{D}$ except for at most one simple pole and $f(z_1) \neq f(z_2)$ for $z_1 \neq z_2$ $(z_1, z_2 \in \mathbb{D})$. In this case, the equation $f(z) = w$ has at most one root in $\mathbb{D}$ for any complex number w. Such functions map $\mathbb{D}$ conformally onto a domain in the w-plane. Let $\mathcal{S}$ denote the class of all analytic univalent functions f defined on the unit disk $\mathbb{E}$ which are normalized by the conditions $f(0) = f'(0) - 1 = 0$. A function $f \in \mathcal{A}$ is said to be starlike in $\mathbb{E}$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0.$$

A function $f \in \mathcal{A}$ is said to be close-to-convex in $\mathbb{E}$ if there is a starlike function g (not necessarily normalized) such that

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0, \ z \in \mathbb{E}.$$

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It is well-known that every close-to-convex function is univalent. In 1934/35, Noshiro [3] and Warchawski [4] obtained a simple but interesting criterion for close-to-convexity of analytic functions. They proved that if an analytic function f satisfies the condition $\Re(f'(z)) > 0$ for all z in $\mathbb{E}$, then f is close-to-convex and hence univalent in $\mathbb{E}$.

The function, for which the equation $f(z) = w$ has p roots in $\mathbb{D}$ for every complex number w, is said to be p-valent (or multivalent) function. Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k,\ p \in \mathbb{N} = \{1, 2, \cdots\},$$

which are analytic and p-valent (or multivalent) in the open unit disk $\mathbb{E}$. Note that $\mathcal{A}_1 = \mathcal{A}$. A function $f \in \mathcal{A}_p$ is said to be p-valently starlike in $\mathbb{E}$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \mathbb{E}.$$

A function $f \in \mathcal{A}_p$ is said to be p-valently close-to-convex if there exists a p-valently starlike function $g \in \mathcal{A}_p$ such that

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0,\ z \in \mathbb{E}. \tag{1.1}$$

For $g(z) \equiv z^p$ in condition (1.1), we have that $f \in \mathcal{A}_p$ is p-valently close-to-convex if

$$\Re\left(\frac{f'(z)}{z^{p-1}}\right) > 0,\ z \in \mathbb{E}.$$

It is also well known that every p-valenty close-to-convex function is p-valent in $\mathbb{E}$ [1]. For two analytic functions f and g in the unit disk $\mathbb{E}$, we say that a function f is subordinate to a function g in $\mathbb{E}$ and write $f \prec g$ if there exists a Schwarz function w analytic in $\mathbb{E}$ with $w(0) = 0$ and $|w(z)| < 1, z \in \mathbb{E}$ such that $f(z) = g(w(z)), z \in \mathbb{E}$. In case the function g is univalent, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

Let $\psi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ and let h be univalent in $\mathbb{E}$. If p is analytic in $\mathbb{E}$ and satisfies the differential subordination

$$\psi(p(z), zp'(z); z) \prec h(z),\ \psi(p(0), 0; 0) = h(0), \tag{1.2}$$

then p is called a solution of the differential subordination (1.2). The univalent function q is called a dominant of the differential subordination (1.2) if $p \prec q$ for all p satisfying (1.2). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.2), is said to be the best dominant of (1.2).

The main objective of this paper is to extend and unify some known results regarding p-valently close-to-convex functions. For this purpose, a subordination theorem is obtained and its applications to multivalent functions are discussed. It is shown that the region of variability of some differential operators implying p-valently close-to-convex functions is extended. To prove the main theorem, we need the following lemma due to Miller and Mocanu.

Lemma 1.1. *([5], p.132, Theorem 3.4 h) Let q be univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ and suppose that either*
(i) h is convex, or

(ii) Q is starlike.
In addition, assume that
(iii) $\Re \dfrac{zh'(z)}{Q(z)} > 0, \ z \in \mathbb{E}$.
If p is analytic in $\mathbb{E}$, with $p(0) = q(0), p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)],$$

then $p \prec q$ and q is the best dominant.

2. Main Theorem

In what follows, all the powers taken, are the principle ones.

Theorem 2.1. *Let α, β, γ be complex numbers such that $\alpha \neq 0$. Let $q, \ q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that*
(i) $\Re \left[1 + \frac{zq''(z)}{q'(z)} + (\gamma - 1)\frac{zq'(z)}{q(z)}\right] > 0$ *and*
(ii) $\Re \left[1 + \frac{zq''(z)}{q'(z)} + (\gamma - 1)\frac{zq'(z)}{q(z)} + \frac{(1-\alpha)\beta}{\alpha} q^{\beta-\gamma}(z) + p\gamma\right] > 0.$
If $P, \ P(z) \neq 0, z \in \mathbb{E}$, satisfies the differential subordination

$$(1-\alpha)(P(z))^{\beta} + \alpha(P(z))^{\gamma}\left(p + \frac{zP'(z)}{P(z)}\right) \prec (1-\alpha)(q(z))^{\beta} + \alpha(q(z))^{\gamma}\left(p + \frac{zq'(z)}{q(z)}\right), \tag{2.1}$$

then $P(z) \prec q(z)$ and q is the best dominant.

Proof. Let us define the functions θ and ϕ as follows:

$$\theta(w) = (1-\alpha)w^{\beta} + \alpha p w^{\gamma},$$

and

$$\phi(w) = \alpha w^{\gamma-1}.$$

Obviously, the functions θ and ϕ are analytic in domain $\mathbb{D} = \mathbb{C} \setminus \{0\}$ and $\phi(w) \neq 0$ in $\mathbb{D}$. Now, define the functions Q and h as under:

$$Q(z) = zq'(z)\phi(q(z)) = \alpha z q'(z) q^{\gamma-1}(z) \ ,$$

and

$$h(z) = \theta(q(z)) + Q(z) = (1-\alpha)(q(z))^{\beta} + \alpha(q(z))^{\gamma}\left(p + \frac{zq'(z)}{q(z)}\right).$$

Then in view of conditions (i) and (ii), we have
(1) Q is starlike in $\mathbb{E}$ and
(2) $\Re \dfrac{z\ h'(z)}{Q(z)} > 0, \ z \in \mathbb{E}$.
Thus conditions (ii) and (iii) of Lemma 1.1, are satisfied. In view of (2.1), we have

$$\theta[P(z)] + zP'(z)\phi[P(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)].$$

Therefore, the proof, now, follows from Lemma 1.1. $\square$

3. Applications

Theorem 3.1. *Let α be a non zero complex number. Let $q,\ q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right) > \max\left\{0, \Re\left(\frac{\alpha-1}{\alpha}q(z)\right)\right\}.$$

If $f \in \mathcal{A}_p,\ \dfrac{f'(z)}{z^{p-1}} \neq 0, z \in \mathbb{E}$, satisfies the differential subordination

$$(1-\alpha)\frac{f'(z)}{z^{p-1}}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right) \prec (1-\alpha)q(z)+\alpha\left(p+\frac{zq'(z)}{q(z)}\right),\ z \in \mathbb{E},$$

then $\dfrac{f'(z)}{z^{p-1}} \prec q(z)$ and q is the best dominant.

Proof. By setting $\beta = 1,\ \gamma = 0$ and $P(z) = \dfrac{f'(z)}{z^{p-1}}$ in Theorem 2.1, we obtain this result. □

Theorem 3.2. *Let α be a non zero complex number. Let $q,\ q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right) > \max\left\{0, \Re\left(\frac{\alpha-1}{\alpha}\right)-p\right\}.$$

If $f \in \mathcal{A}_p,\ \dfrac{f'(z)}{pz^{p-1}} \neq 0, z \in \mathbb{E}$, satisfies the differential subordination

$$(1-\alpha)\frac{f'(z)}{pz^{p-1}}+\frac{\alpha f'(z)}{pz^{p-1}}\left(1+\frac{zf''(z)}{f'(z)}\right) \prec (1-\alpha)q(z)+\alpha q(z)\left(p+\frac{zq'(z)}{q(z)}\right),\ z \in \mathbb{E},$$

then $\dfrac{f'(z)}{pz^{p-1}} \prec q(z)$ and q is the best dominant.

Proof. Select $\beta = 1,\ \gamma = 1$ and $P(z) = \dfrac{f'(z)}{pz^{p-1}}$ in Theorem 2.1 to get this result. □

Theorem 3.3. *Let α be a non zero complex number. Let $q,\ q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right) > \max\left\{0, \Re\left(\frac{2(\alpha-1)}{\alpha}q(z)\right)-p\right\}.$$

If $f \in \mathcal{A}_p,\ \dfrac{f'(z)}{pz^{p-1}} \neq 0, z \in \mathbb{E}$, satisfies the differential subordination

$$(1-\alpha)\left(\frac{f'(z)}{pz^{p-1}}\right)^2+\frac{\alpha f'(z)}{pz^{p-1}}\left(1+\frac{zf''(z)}{f'(z)}\right) \prec (1-\alpha)q^2(z)+\alpha q(z)\left(p+\frac{zq'(z)}{q(z)}\right),$$

then $\dfrac{f'(z)}{pz^{p-1}} \prec q(z)$ and q is the best dominant.

Proof. Writing $\beta = 2,\ \gamma = 1$ and $P(z) = \dfrac{f'(z)}{pz^{p-1}}$ in Theorem 2.1, we get this result. □

Remark 3.4. *When we select the dominant* $q(z) = \dfrac{p(1+z)}{1-z}$, $z \in \mathbb{E}$. *Then*

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{1+z^2}{1-z^2},$$

and hence

$$\Re\left[1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right] > 0, \ z \in \mathbb{E}.$$

We also have

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{1-\alpha}{\alpha}q(z) = \frac{1+z^2}{1-z^2} + \frac{p(1-\alpha)}{\alpha}\frac{1+z}{1-z},$$

and therefore for $0 < \alpha \leq \dfrac{2p}{2p-1}$,

$$\Re\left[1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{1-\alpha}{\alpha}q(z)\right] > 0 \ z \in \mathbb{E}.$$

Hence $q(z) = \dfrac{p(1+z)}{1-z}$ *satisfies the conditions of Theorem 3.1 and we immediately get the following result.*

Corollary 3.5. *Let* α *be a real number such that* $0 < \alpha \leq \dfrac{2p}{2p-1}$. *Let* $f \in \mathcal{A}_p$, $\dfrac{f'(z)}{z^{p-1}} \neq 0, z \in \mathbb{E}$, *satisfy the differential subordination*

$$(1-\alpha)\frac{f'(z)}{z^{p-1}} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec (1-\alpha)\frac{p(1+z)}{1-z} + \alpha\left(p + \frac{2z}{1-z^2}\right) = F(z),$$

then $\dfrac{f'(z)}{z^{p-1}} \prec \dfrac{p(1+z)}{1-z}$, $z \in \mathbb{E}$. *Hence,* f *is p-valently close-to-convex.*

In view of above corollary, we obtain the following result.

Corollary 3.6. *Let* α *be a real number such that* $0 < \alpha \leq \dfrac{2p}{2p-1}$. *Let* $f \in \mathcal{A}_p$, $\dfrac{f'(z)}{z^{p-1}} \neq 0, z \in \mathbb{E}$, *satisfy the differential subordination*

$$(1-\alpha)\frac{f'(z)}{z^{p-1}} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec F(z),$$

where F *is a conformal mapping of the unit disk* $\mathbb{E}$ *with* $F(0) = p$ *and*

$$F(\mathbb{E}) = \mathbb{C} \setminus \left\{w \in \mathbb{C} : \Re\, w = \alpha p, \ |\Im\, w| \geq \sqrt{\alpha[\alpha + 2(1-\alpha)p]}\right\},$$

then $\dfrac{f'(z)}{z^{p-1}} \prec \dfrac{p(1+z)}{1-z}$ *and therefore,* f *is p-valently close-to-convex.*

Remark 3.7. *The result in Corollary 3.6 extends the region of variability of the differential operator* $(1-\alpha)\dfrac{f'(z)}{z^{p-1}} + \alpha\left(1 + \dfrac{zf''(z)}{f'(z)}\right)$ *over the result of Patel and Cho [2, Corollary 1] for implying p-valently close-to-convexity of* $f \in \mathcal{A}_p$. *According to the result of Corollary 3.6, the differential operator* $(1-\alpha)\dfrac{f'(z)}{z^{p-1}} + \alpha\left(1 + \dfrac{zf''(z)}{f'(z)}\right)$ *can vary over the entire complex plane except two slits parallel to imaginary axis* *for the implication of p-valently close-to-convexity of $f \in \mathcal{A}_p$ whereas by the result of Patel and Cho [2, Corollary 1], the same operator varies over the portion of the complex plane right to the slits parallel to the imaginary axis. Thus our result extends the result of Patel and Cho [2]. The above claim is shown below pictorially in Figure 3.1. In a special case when $\alpha = 1/2,\ p = 2$, the image of the unit disk $\mathbb{E}$ under function F (given in Corollary 3.5) is the entire complex plane except two slits $\left\{\Re\ z = 1,\ |\Im\ z| \geq \dfrac{\sqrt{5}}{2}\right\}$. This justify our claim.*

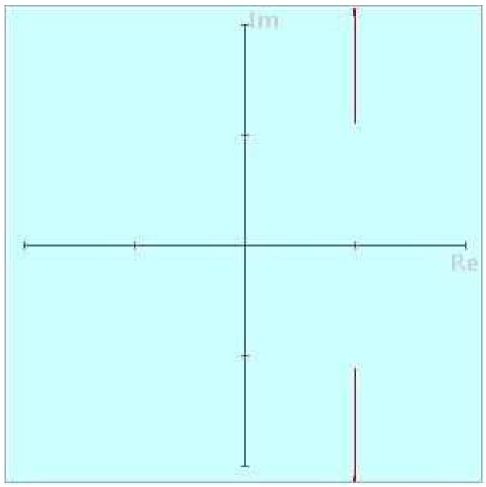

Figure 3.1 (when $\alpha = 1/2,\ p = 2$)

Remark 3.8. *For the dominant $q(z) = \dfrac{1+z}{1-z},\ z \in \mathbb{E}$, we have*

$$1 + \frac{zq''(z)}{q'(z)} = \frac{1+z}{1-z},$$

and hence

$$\Re\left[1 + \frac{zq''(z)}{q'(z)}\right] > 0,\ z \in \mathbb{E}.$$

Therefore, from Theorem 3.2, we obtain the following result.

Corollary 3.9. *Let α be a non zero complex number such that $\Re\left(\dfrac{(\alpha-1)\overline{\alpha}}{|\alpha|^2}\right) < p$. Let $f \in \mathcal{A}_p,\ \dfrac{f'(z)}{pz^{p-1}} \neq 0, z \in \mathbb{E}$, satisfy the differential subordination*

$$(1-\alpha)\frac{f'(z)}{pz^{p-1}} + \frac{\alpha f'(z)}{pz^{p-1}}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec (1 - \alpha + \alpha p)\frac{1+z}{1-z} + \frac{2\alpha z}{(1-z)^2},\ z \in \mathbb{E},$$

then $\dfrac{f'(z)}{pz^{p-1}} \prec \dfrac{1+z}{1-z},\ z \in \mathbb{E}$. Hence, f is p-valently close-to-convex.

In a special case when $p = 1$, the above corollary reduces to the following result.

Corollary 3.10. *Let α be a non zero complex number such that $\Re(\alpha) > 0$ and let $f \in \mathcal{A}$, $f'(z) \neq 0, z \in \mathbb{E}$, satisfy the differential subordination*

$$(1-\alpha)f'(z) + \alpha f'(z)\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{1+z}{1-z} + \frac{2\alpha z}{(1-z)^2} = G(z),$$

then $f'(z) \prec \dfrac{1+z}{1-z}$, i.e. $\Re(f'(z)) > 0$, $z \in \mathbb{E}$ and hence f is close-to-convex.

Remark 3.11. *For $p = 1$, $A = 1$, $B = -1$, Theorem 4 of Patel and Cho [2] gives the following result.*

If $f \in \mathcal{A}$, satisfies the differential subordination

$$(1-\alpha)f'(z) + \alpha f'(z)\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{1+z}{1-z}, \ \alpha > 0, \ z \in \mathbb{E},$$

then $f'(z) \prec \dfrac{1+z}{1-z}$, $z \in \mathbb{E}$.

Now we compare this result with the result of Corollary 3.10. According to the above result of Patel and Cho [2], the region of variability of the differential operator $(1-\alpha)f'(z)+\alpha f'(z)\left(1 + \dfrac{zf''(z)}{f'(z)}\right)$ for the required implication is the right half plane whereas the result of Corollary 3.10 extends the region of variability of the above operator for the same conclusion.

To justify the claim, we consider below a particular case when $\alpha = 1/2$ and show the extended region pictorially in Figure 3.2.

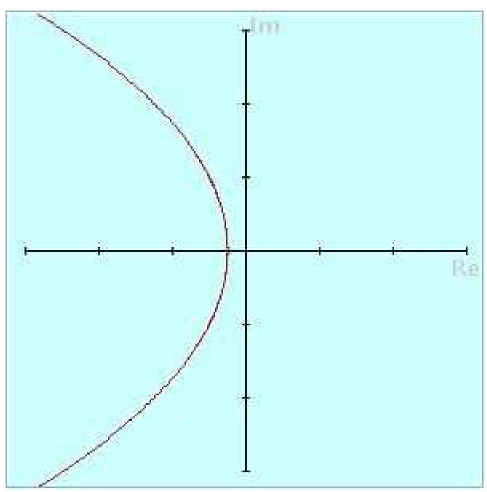

Figure 3.2 (when $\alpha = 1/2$)

The function G (given in Corollary 3.10) maps the unit disk $\mathbb{E}$ onto the portion of the plane right to the plotted curve (image of the unit circle under G). Therefore, according to Corollary 3.10, the differential operator $f'(z)\left(1 + \dfrac{1}{2}\dfrac{zf''(z)}{f'(z)}\right)$ can vary in the portion of the plane right to the plotted curve whereas by the above mentioned result of Patel and Cho [2], this differential operator varies in the portion of the *plane right to the imaginary axis. Thus the region bounded by the plotted curve and the imaginary axis is the claimed extension.*

Remark 3.12. *Consider the dominant* $q(z) = \dfrac{1+z}{1-z},\ z \in \mathbb{E}$. *We have*

$$1 + \frac{zq''(z)}{q'(z)} = \frac{1+z}{1-z},$$

and

$$1 + \frac{zq''(z)}{q'(z)} + \frac{2(1-\alpha)}{\alpha}q(z) + p = \frac{2-\alpha}{\alpha}\frac{1+z}{1-z} + p.$$

Obviously for $0 < \alpha \leq 2$, *the dominant* $q(z) = \dfrac{1+z}{1-z}$ *satisfies the conditions of Theorem 3.3 and we have the following result.*

Corollary 3.13. *Let* $\alpha(0 < \alpha \leq 2)$ *be a real number. Let* $f \in \mathcal{A}_p,\ \dfrac{f'(z)}{pz^{p-1}} \neq 0, z \in \mathbb{E}$, *satisfy the differential subordination*

$$(1-\alpha)\left(\frac{f'(z)}{pz^{p-1}}\right)^2 + \frac{\alpha f'(z)}{pz^{p-1}}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec (1-\alpha)\left(\frac{1+z}{1-z}\right)^2 + \alpha\frac{1+z}{1-z}\left(p + \frac{2z}{1-z^2}\right),$$

then $\dfrac{f'(z)}{pz^{p-1}} \prec \dfrac{1+z}{1-z},\ z \in \mathbb{E}$.

The result in Corollary 3.13 corresponds to Theorem 5 of Patel and Cho [2].

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