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ON SOME PROPERTIES OF ANALYTIC AND MEROMORPHIC FUNCTIONS

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Abstract. In the present paper, we obtain some new properties of analytic and meromorphic functions

1. Introduction

Let $p(z)$ be analytic in the open unit disc $\mathcal{U} = \{z : z \in C;\ |z| < 1\}$ with $p(0) = 1$ and suppose that there exixts a point z_0 $(|z_0| < 1)$ such that
(i) $|p(z)| > a \quad (|z| < |z_0|; 0 < a < 1)$ and $|p(z_0)| = a$
or
(ii) $\text{Re}\{p(z)\} < M \quad (|z| < |z_0|; M > 1)$ and $\text{Re}\{p(z_0)\} = M$.
Also let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

be analytic in $\mathcal{U}$ and

$$F(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n \tag{1.2}$$

be meromorphic in $\mathcal{U}$.
In this paper, we shall obtain some basic results for $z_0 p'(z_0)/p(z_0)$ under the condition (i) or (ii). Our main results will be applied to get some properties of $z_0 f'(z_0)/f(z_0)$ or $-z_0 F'(z_0)/F(z_0)$.

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2. Main Results

Lemma 2.1. *(see [1]). Let $w(z)$ be regular in $\mathcal{U}$ with $w(0) = 0$. If $w(z)$ attains its maximum value on the circle $|z| = r$ at a given point $z_0 \in \mathcal{U}$, then $z_0 w'(z_0) = k\,w(z_0)$, where k is a a real number and $k \geq 1$.*

Theorem 2.2. *Let $p(z)$ be analytic in $\mathcal{U}$ with $p(0) = 1$ and suppose that there exists a point z_0, $|z_0| < 1$ such that $|p(z)| > a$ $(|z| < |z_0|)$ and $|p(z_0)| = a$, where $0 < a < 1$.*
Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = \mathrm{Re}\left(\frac{z_0 p'(z_0)}{p(z_0)}\right) \leq -\frac{1-a}{1+a}.$$

Proof. Let us put

$$p(z) = \frac{a(1+\psi(z))}{1-\psi(z)}$$

or

$$\psi(z) = \frac{p(z)-a}{p(z)+a} \quad (|z| < |z_0|)\,.$$

Then $\psi(z)$ is analytic in $|z| < |z_0|$ and

$$0 < \psi(0) = \frac{1-a}{1+a} < 1.$$

By the hypothesis of the theorem, we have

$$|p(z)| = a\left|\frac{1+\psi(z)}{1-\psi(z)}\right| > a \quad (|z| < |z_0|)\,.$$

Thus

$$\left|\frac{1+\psi(z)}{1-\psi(z)}\right| > 1 \quad (|z| < |z_0|)\,.$$

It shows that

$$\mathrm{Re}\,\{\psi(z)\} > 0 \quad (|z| < |z_0|)\,.$$

On the other hand, we have

$$|p(z_0)| = a\left|\frac{1+\psi(z_0)}{1-\psi(z_0)}\right| = a,$$

this shows that

$$\mathrm{Re}\,\{\psi(z_0)\} = 0.$$

Putting

$$\Phi(z) = \frac{\psi(0)-\psi(z)}{\psi(0)+\psi(z)},\ \ \Phi(0) = 0,$$

then we have $\Phi(z)$ is analytic in $\mathcal{U}$, $|\Phi(z)| < 1$ for $|z| < |z_0|$ and $|\Phi(z_0)| = 1$. Therefore, applying Lemma, we have that

$$\frac{z_0\,\Phi'(z_0)}{\Phi(z_0)} = -\frac{2\psi(0) z_0\,\psi'(z_0)}{(\psi(0))^2 + |\psi(z_0)|^2} = k \geq 1.$$

This means that $z_0\ \psi'(z_0)$ is real negative because $0 < \psi(0) < 1$. Then we say that

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{z_0 \psi'(z_0)}{1+\psi(z_0)} + \frac{z_0 \psi'(z_0)}{1-\psi(z_0)}$$

$$
\begin{aligned}
&= \frac{2z_0\psi'(z_0)}{1-(\psi(z_0))^2} \\
&= \frac{2z_0\psi'(z_0)}{1+|\psi(z_0)|^2} \\
&= -\frac{k}{\psi(0)}\left(\frac{(\psi(0))^2+|\psi(z_0)|^2}{1+|\psi(z_0)|^2}\right) \\
&\leq -\psi(0)\left(\frac{(\psi(0))^2+|\psi(z_0)|^2}{(\psi(0))^2+(\psi(0))^2|\psi(z_0)|^2}\right) \\
&< -\psi(0) = -\frac{1-a}{1+a}.
\end{aligned}
$$

This completes the proof.

Theorem 2.3. *Let $p(z)$ be analytic in $\mathcal{U}$ with $p(0)=1$, suppose that there exists a point z_0 $(|z_0|<1)$ such that $\operatorname{Re}\{p(z)\}<M$, $p(z)\neq 0$ for $|z|<|z_0|$, and $\operatorname{Re}\{p(z_0)\}=M$, where $M>1$.*
Then we have

$$\operatorname{Re}\{z_0\,p'(z_0)\}\geq\frac{M-1}{2}.$$

Proof. Let us put

$$g(z)=\frac{1}{2M-1}\left(\frac{2M-p(z)}{p(z)}\right),\quad g(0)=1.$$

Since $\operatorname{Re}\{p(z)\}<M$ for $|z|<|z_0|$, we see that

$$\left|\frac{2M-p(z)}{p(z)}\right|>1$$

for $|z|<|z_0|$. This gives us that

$$|g(z)|\;>\;\frac{1}{2M-1}\quad(|z|<|z_0|)$$

and

$$|g(z_0)|\;=\;\frac{1}{2M-1}\;.$$

Now

$$\operatorname{Re}\left(\frac{z_0g'(z_0)}{g(z_0)}\right)=\operatorname{Re}\left(-\frac{z_0p'(z_0)}{p(z_0)}-\frac{z_0p'(z_0)}{2M-p(z_0)}\right) \tag{2.1}$$

Applying Theorem (2.2), we have

$$\operatorname{Re}\left(\frac{z_0g'(z_0)}{g(z_0)}\right)\leq-\frac{1-\frac{1}{2M-1}}{1+\frac{1}{2M-1}}=-\frac{M-1}{M}. \tag{2.2}$$

Putting $p(z_0)=M+ia$, where a is a real number, we have

$$
\begin{aligned}
-\operatorname{Re}\left(\frac{z_0p'(z_0)}{p(z_0)}+\frac{z_0p'(z_0)}{2M-p(z_0)}\right)&=-\operatorname{Re}\left(\frac{2M}{M^2+a^2}z_0p'(z_0)\right)\\
&=-\frac{2M}{M^2+a^2}\operatorname{Re}\{z_0p'(z_0)\}.
\end{aligned} \tag{2.3}
$$

Now (2.1) in conjunction with (2.2) and (2.3) gives

$$-\frac{2M}{M^2+a^2}\operatorname{Re}\{z_0p'(z_0)\} \leq -\frac{M-1}{M}.$$

This shows that

$$\operatorname{Re}\{z_0p'(z_0)\} \geq \left(\frac{M^2+a^2}{2M}\right)\left(\frac{M-1}{M}\right) \geq \frac{M-1}{2}. \tag{2.4}$$

It completes the proof.

Corollary 2.4. *Let $p(z)$ be analytic in U with $p(0)=1$ and suppose that*

$$\operatorname{Re}\left(p(z)+\frac{zp'(z)}{p(z)}\right) > \frac{a^2+2a-1}{a+1} \quad (|z|<1; 0<a<1). \tag{2.5}$$

Then we have $|p(z)|>a$ in $\mathcal{U}$.

Proof. Suppose that there exists a point z_0 ($|z_0|<1$) such that $a<|p(z)|$ in $|z|<|z_0|$ and $|p(z_0)|=a$, then Theorem (2.2) gives

$$\frac{z_0p'(z_0)}{p(z_0)} = \operatorname{Re}\left(\frac{z_0p'(z_0)}{p(z_0)}\right) \leq -\frac{1-a}{1+a}.$$

Thus it follows that

$$\operatorname{Re}\left(\frac{z_0p'(z_0)}{p(z_0)}+p(z_0)\right) \leq -\frac{1-a}{1+a}+a = \frac{a^2+2a-1}{a+1}.$$

It contradicts the hypothesis (2.5) and it completes the proof.

Corollary 2.5. *Let $f(z)$ defined by (1.1) be analytic and $f'(z)\neq 0$ in $\mathcal{U}$, suppose that*

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \frac{a^2+2a-1}{a+1} \quad (z\in U;\ 0<a<1).$$

Then we have

$$\left|\frac{zf'(z)}{f(z)}\right| > a \text{ in } \mathcal{U}.$$

Proof. Let us put

$$p(z)=\frac{zf'(z)}{f(z)}, \quad p(0)=1.$$

Then we have

$$1+\frac{zf''(z)}{f'(z)} = p(z)+\frac{zp'(z)}{p(z)}.$$

Applying Corollary 2.4, we have Corollary 2.5.

Corollary 2.6. *Let $p(z)$ be analytic in $\mathcal{U}$ with $p(0)=1$ and suppose that there exists a point z_0 ($|z_0|<1$) such that* $\operatorname{Re}\{p(z)\}<M$, $p(z)\neq 0$ *for* $|z|<|z_0|$, $\operatorname{Re}\{p(z_0)\}=M$ where *where $M>1$.*
Then we have

$$\operatorname{Re}\left(\frac{z_0p'(z_0)}{p(z_0)}\right) \geq \frac{M-1}{2M}.$$

Proof. Applying (2.4) of Theorem (2.3), we have

$$\mathrm{Re}\left\{z_0 p'(z_0)\right\} \geq \left(\frac{M^2+a^2}{2M}\right)\left(\frac{M-1}{M}\right)$$

where $p(z_0) = M + ia$, a is a real number.
Then it follows that

$$\begin{aligned}
\mathrm{Re}\left(\frac{z_0\, p'(z_0)}{p(z_0)}\right) &= \mathrm{Re}\left(\frac{z_0\, p'(z_0)}{M+ia}\right) \\
&= \mathrm{Re}\left\{\left(\frac{M-ia}{M^2+a^2}\right) z_0 p'(z_0)\right\} \\
&= \left(\frac{M}{M^2+a^2}\right) z_0 p'(z_0) \\
&\geq \left(\frac{M}{M^2+a^2}\right)\left(\frac{M^2+a^2}{2M^2}\right)(M-1) \\
&= \frac{M-1}{2M}.
\end{aligned}$$

It completes the proof.

Applying the same method as the proof of Corollary (2.5) and (2.6), we have the following result.

Corollary 2.7. *Let $f(z)$ defined by (1.1) be analytic and in $f'(z) \neq 0$ in $\mathcal{U}$. Also suppose that*

$$1 + \mathrm{Re}\left(\frac{z\, f''(z)}{f'(z)}\right) < \frac{2M^2+M-1}{2M} \quad (|z| < 1\,; M > 1)\,.$$

Then we have

$$\mathrm{Re}\left(\frac{z\, f'(z)}{f(z)}\right) < M \quad in\, \mathcal{U}.$$

Corollary **??** is equivalent to Corollary 2.7.

Corollary 2.8. *Let $f(z)$ defined by (1.1) be analytic and $f'(z) \neq 0$ in $\mathcal{U}$. Also suppose that*

$$1 + \mathrm{Re}\left(\frac{z\, f''(z)}{f'(z)}\right) < \beta \quad (|z| < 1\,; \beta > 1)\,.$$

Then we have

$$\mathrm{Re}\left(\frac{z\, f'(z)}{f(z)}\right) < \frac{2\beta - 1 + \sqrt{4\beta^2 - 4\beta + 9}}{4} \quad in\, \mathcal{U}.$$

Corollary 2.9. *Let $F(z)$ defined by (1.2) be meromorphic and $F'(z) \neq 0$ in $\mathcal{U}$. Also suppose that*

$$-\left\{1 + \mathrm{Re}\left(\frac{z\, F''(z)}{F'(z)}\right)\right\} > \frac{2M^2-M+1}{2M} \quad (|z| < 1\,; M > 1)\,. \tag{2.6}$$

Then we have

$$-\mathrm{Re}\left(\frac{z\, F'(z)}{F(z)}\right) < M \quad in\, \mathcal{U}.$$

Proof. Let us put

$$p(z) = -\frac{zF'(z)}{F(z)}, \quad p(0) = 1,$$

then $p(z)$ is analytic in $\mathcal{U}$ and it follows that

$$-\left(1 + \frac{zF''(z)}{F'(z)}\right) = p(z) - \frac{zp'(z)}{p(z)}. \tag{2.7}$$

If there exists a point z_0 ($|z_0| < 1$) such that $\text{Re}\{p(z)\} < M$ for $|z| < |z_0|$ and $Re\{p(z_0)\} = M$, then from Corollary 2.5, we have

$$\text{Re}\left(\frac{z_0 p'(z_0)}{p(z_0)}\right) \geq \frac{M-1}{2M}.$$

It follows that

$$-\left\{1 + \text{Re}\left(\frac{z_0 F''(z_0)}{F'(z_0)}\right)\right\} = \text{Re}\left(p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right)$$
$$\leq M - \frac{M-1}{2M} = \frac{2M^2 - M + 1}{2M}.$$

This contradicts (2.6) and it completes the proof of Corollary 2.9.

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