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HYPOCYCLOID OF $n+1$ CUSPS HARMONIC FUNCTION

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Abstract. For a harmonic function $h(z) = f(z) + \overline{g(z)}$ in the open unit disk $\mathbb{U}$ with holomorphic functions $f(z)$ and $g(z)$ satisfying $g'(z) = z^{n-1}f'(z)$ $(n = 2, 3, 4, \cdots)$, a sufficient condition on $f(z)$ for $h(z)$ to be univalent in $\mathbb{U}$ and the image of $\mathbb{U}$ by $h(z)$ to be a hypocycloid of $n+1$ cusps are discussed.

1. Introduction

For holomorphic functions $f(z)$ and $g(z)$ in a simply connected domain $\mathbb{D}$, a complex-valued harmonic function $h(z)$ is given by $h(z) = f(z) + \overline{g(z)}$. The theory and applications of harmonic mappings are discussed by Duren [1]. Mocanu [3] has shown the following result for the univalence of harmonic functions.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be holomorphic functions in a domain $\mathbb{D}$. If the function $f(z)$ is convex and $|g'(z)| < |f'(z)|$ for $z \in \mathbb{D}$, then the harmonic function $h(z) = f(z) + \overline{g(z)}$ is univalent and sense preserving in $\mathbb{D}$.*

In fact, considering the harmonic function

$$h(z) = f(z) + \overline{g(z)} = z + \frac{1}{n}\overline{z}^n \qquad (n = 2, 3, 4, \cdots)$$

for all $z \in \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, it is clear that $f(z) = z$ is convex, $|g'(z)| < |f'(z)|$ $(z \in \mathbb{U})$ and $h(z)$ is univalent and sense preserving in $\mathbb{U}$. For this harmonic function $h(z)$ and

$$Dh(z) = z\frac{\partial h(z)}{\partial z} - \overline{z}\frac{\partial h(z)}{\partial \overline{z}},$$

it follows that

$$\begin{aligned} \mathrm{Re}\left(\frac{Dh(z)}{h(z)}\right) &= \mathrm{Re}\left(\frac{re^{it} - r^n e^{-int}}{re^{it} + \frac{r^n}{n}e^{-int}}\right) \qquad (z = re^{it}) \\ &\geqq \frac{n(1 - r^{n-1})}{n + r^{n-1}} > 0 \end{aligned}$$

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which shows that $h(z)$ is also starlike in $\mathbb{U}$ (see, [2]). Furthermore, $h(z)$ maps $\mathbb{U}$ onto the region inside a hypocycloid of $n+1$ cusps (for detail, [1, p. 115]).

This work is motivated by the following theorem due to Mocanu [4].

Theorem 1.2. *Let $f(z)$ be holomorphic in the closed unit disk $\overline{\mathbb{U}}$, with $f'(z) \neq 0$ for $z \in \overline{\mathbb{U}}$ and let*

$$F(t) = 3t + 2\arg\left(f'(e^{it})\right) \quad (-\pi \leqq t \leqq \pi).$$

If for each $k \in K = \{0, \pm 1, \pm 2\}$ the equation

$$F(t) = 2k\pi$$

has at most a single root in $[-\pi, \pi]$ and for all $k \in K$ there exist three such roots in $[-\pi, \pi]$, then the harmonic function $h(z) = f(z) + \overline{g(z)}$, with $g'(z) = zf'(z)$ is univalent in $\mathbb{U}$, sense preserving and the image of $\mathbb{U}$ by $h(z)$ is a "three-cornered hat" domain.

We obtain an extension result of the above theorem for the following generalized class of harmonic functions $h(z)$ in $\mathbb{U}$ of the form

$$h(z) = f(z) + \overline{g(z)}$$

where $f(z)$ and $g(z)$ are holomorphic functions in $\mathbb{U}$ and satisfy $g'(z) = z^{n-1}f'(z)$ $(n = 2, 3, 4, \cdots)$. This shows that the harmonic function $h(z)$ is well defined if a holomorphic function $f(z)$ in $\mathbb{U}$ is given.

2. Main result

Our first result is contained in

Theorem 2.1. *Let $f(z)$ be holomorphic in the closed unit disk $\overline{\mathbb{U}}$, with $f'(z) \neq 0$ for $z \in \overline{\mathbb{U}}$ and let*

$$F(t) = (n+1)t + 2\arg(f'(e^{it})) \quad (-\pi \leqq t < \pi), \quad n = 2, 3, 4, \cdots. \tag{2.1}$$

If for each $k \in K = \left\{0, \pm 1, \pm 2, \cdots, \pm\left[\frac{n+3}{2}\right]\right\}$ where $[\ ]$ denotes the Gauss symbol, the equation

$$F(t) = 2k\pi \tag{2.2}$$

has at most a single root in $[-\pi, \pi)$ and for any $k \in K$ there exist exactly $(n+1)$ such roots in $[-\pi, \pi)$, then the harmonic function

$$h(z) = f(z) + \overline{g(z)}$$

with $g'(z) = z^{n-1}f'(z)$ is univalent in $\mathbb{U}$, sense preserving and the image of $\mathbb{U}$ by $h(z)$ is a hypocycloid of $n+1$ cusps.

Proof. Let us define the function $w(t)$ by

$$w(t) = h(e^{it}) = f(e^{it}) + \overline{g(e^{it})} \quad (z = e^{it}).$$

Supposing that

$$w'(t) = i(zf'(z) - \overline{z}\overline{g'(z)}) = i(zf'(z) - \overline{z}^n\overline{f'(z)}) = 0,$$

then we need the equation

$$z^{n+1}\frac{f'(z)}{\overline{f'(z)}} = 1.$$

Therefore, it follows that

$$\left|z^{n+1}\frac{f'(z)}{\overline{f'(z)}}\right| = 1 \quad \text{and} \quad \arg\left(z^{n+1}\frac{f'(z)}{\overline{f'(z)}}\right) = 2k\pi$$

that is, that

$$(n+1)t + 2\arg(f'(e^{it})) = 2k\pi \quad (-\pi \leqq t < \pi)$$

for $z = e^{it}$ which gives us the equation (2.2). By the assumption of the theorem, there exist $(n+1)$ distinct roots on the unit circle $|z| = 1$ and they divide the unit circle onto $(n+1)$ arcs.

Since $g''(z) = (n-1)z^{n-2}f'(z) + z^{n-1}f''(z)$, we have

$$\begin{aligned} w''(t) &= -\left(zf'(z) + z^2f''(z) + \overline{z}\overline{g'(z)} + \overline{z}^2\overline{g''(z)}\right) \\ &= -\left(zf'(z) + z^2f''(z) + n\overline{z}^n\overline{f'(z)} + \overline{z}^{n+1}\overline{f''(z)}\right) \end{aligned}$$

and therefore, we obtain that

$$\begin{aligned} &w''(t)\overline{w'(t)} = -\left(zf'(z) + z^2f''(z) + n\overline{z}^n\overline{f'(z)} + \overline{z}^{n+1}\overline{f''(z)}\right)(-i)\left(\overline{z}\overline{f'(z)} - z^nf'(z)\right) \\ &= i\Big(-(n-1)|f'(z)|^2 + z\overline{f'(z)}f''(z) - \overline{z\overline{f'(z)}f''(z)} - z^{n+1}f'(z)^2 + \overline{z^{n+1}f'(z)^2} \\ &\qquad + (n-1)\overline{z}^{n+1}\overline{f'(z)}^2 - z^{n+2}f'(z)f''(z) + \overline{z^{n+2}f'(z)f''(z)}\Big) \\ &= i\left((n-1)\left(\overline{z}^{n+1}\overline{f'(z)}^2 - |f'(z)|^2\right)\right. \\ &\qquad \left. + 2i\,\mathrm{Im}\left(z\overline{f'(z)}f''(z) - z^{n+1}f'(z)^2 - z^{n+2}f'(z)f''(z)\right)\right). \end{aligned}$$

This implies that

$$\mathrm{Im}\left(w''(t)\overline{w'(t)}\right) = (n-1)|f'(z)|^2\left[\mathrm{Re}\left(z^{n+1}\left(\frac{f'(z)}{|f'(z)|}\right)^2 - 1\right)\right] \leqq 0.$$

Thus, we derive

$$\mathrm{Im}\left(\frac{w''(t)}{w'(t)}\right) = \frac{1}{|w'(t)|^2}\mathrm{Im}\left(w''(t)\overline{w'(t)}\right) \leqq 0.$$

This shows that the image by $w(t)$ is concave. By the help of a simple geometrical observation, we know that the image of the unit circle, as a union of the $(n+1)$ concave arcs, is a simple curve. Namely, $h(z)$ is univalent and the domain $h(\mathbb{U})$ is a hypocycloid of $n+1$ cusps. □

3. Some illustrative examples and image domains

In this section, we enumerate several illustrative examples and the image domains for the harmonic functions $h(z) = f(z) + \overline{g(z)}$ satisfying the condition of Theorem 2.1.

Example 3.1. *Let $f(z) = z$. Then, we immediately obtain*

$$h(z) = f(z) + \overline{g(z)} = z + \frac{1}{n}\overline{z}^n$$

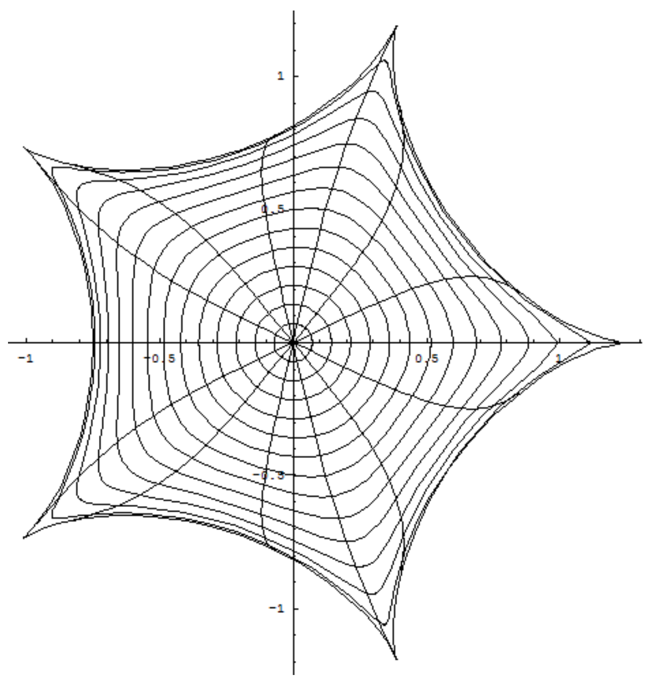
and the equation (2.2) *becomes*

$$(n+1)t = 2k\pi \quad \left(k = 0, \pm 1, \pm 2, \cdots, \pm \left[\frac{n+3}{2}\right]\right)$$

which has at most a single root in $[-\pi, \pi)$ and for any k, just $(n+1)$ roots in $[-\pi, \pi)$. Hence, $h(z)$ is univalent in $\mathbb{U}$ and $h(\mathbb{U})$ is a hypocycloid of $n+1$ cusps. For example, setting $n = 4$, the following hypocycloid of five cusps as the image of $\mathbb{U}$ by the harmonic function

$$h(z) = z + \frac{1}{4}\overline{z}^4$$

is obtained.



The image of $\mathbb{U}$ by $h(z) = z + \frac{1}{4}\overline{z}^4$.

Remark. The inequality $F'(t) \geqq 0$, with $F(t)$ defined by (2.1), is equivalent to

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geqq -\frac{n-1}{2} \quad (z = e^{it}). \tag{3.1}$$

Noting the above, we derive

Example 3.2. *Let* $f(z) = z + \frac{p}{m} z^m$ $(m = 2, 3, 4, \cdots)$. *Then, the equation* (3.1) *becomes*

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) = m - (m-1)\mathrm{Re}\left(\frac{1}{1+pz^{m-1}}\right) \geqq -\frac{n-1}{2}$$

and the function $F(t)$ *given by* (2.1) *satisfies*

$$F(-\pi) = -(n+1)\pi \quad \textit{and} \quad F(\pi) = (n+1)\pi.$$

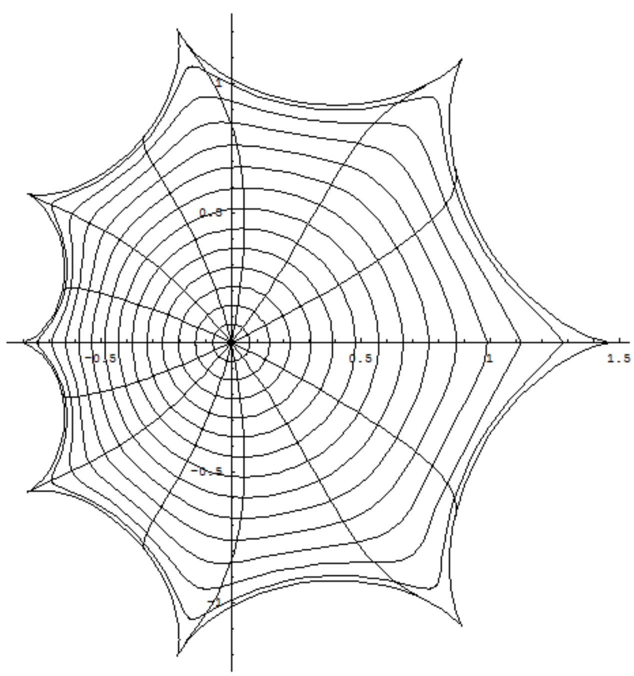
Therefore, if we take $-\frac{n+1}{n+2m-1} \leqq p \leqq \frac{n+1}{n+2m-1}$, *then the conditions of Theorem 2.1 are satisfied, so that the harmonic function*

$$h(z) = z + \frac{p}{m} z^m + \frac{1}{n}\overline{z}^n + \frac{p}{n+m-1}\overline{z}^{n+m-1}$$

is univalent in $\mathbb{U}$ *and* $h(\mathbb{U})$ *is a hypocycloid of* $n+1$ *cusps. In particular, considering* $h(z)$ *with* $n = 7$, $m = 2$ *and* $p = \frac{8}{15}$, *the following hypocycloid of eight cusps as the image of* $\mathbb{U}$ *by the harmonic function*

$$h(z) = z + \frac{4}{15} z^2 + \frac{1}{7}\overline{z}^7 + \frac{1}{15}\overline{z}^8$$

is obtained.



The image of $\mathbb{U}$ by $h(z) = z + \frac{4}{15} z^2 + \frac{1}{7}\overline{z}^7 + \frac{1}{15}\overline{z}^8$.

4. A problem for harmonic functions

A problem related to the elementary transform of harmonic functions is the following.

Problem 4.1. For each holomorphic function $f(z)$ in certain domain with $f(0) = f'(0) - 1 = 0$, can we find the largest domain $\mathbb{D}_c$, such that the harmonic function $h_c(z) = f_c(z) + \overline{g_c(z)}$, where $f_c(z) = \frac{1}{c}f(cz)$, with $g_c'(z) = z^{n-1}f_c'(z)$, is univalent for all $c \in \mathbb{D}_c$?

Let $c = re^{i\theta}$ and

$$F(t, r, \theta) = (n+1)t + 2\arg\left(f'(re^{i(t+\theta)})\right)$$

for $z = e^{it}$ $(-\pi \leqq t < \pi)$. Then, the boundary of the domain $\mathbb{D}_c$ is obtained by the elimination of t from the system

$$\begin{cases} F(t, r, \theta) = 2k\pi & \left(k = 0, \pm 1, \pm 2, \cdots, \pm\left[\dfrac{n+3}{2}\right]\right) \\ \dfrac{\partial F(t, r, \theta)}{\partial t} = 0. \end{cases}$$

where [ ] is the Gauss symbol.

For example, we consider this problem for the case $f(z) = e^z - 1$. Then, we know that

$$f_c(z) = \frac{e^{cz} - 1}{c} \quad (c = re^{i\theta})$$

and the equation (2.2) implies that

$$(n+1)t + 2r\sin(t+\theta) = 2k\pi.$$

Differentiating the both sides with respect to t, we have that

$$(n+1) + 2r\cos(t+\theta) = 0$$

or

$$t + \theta = \cos^{-1}\left(\frac{-(n+1)}{2r}\right) = \pi - \cos^{-1}\left(\frac{n+1}{2r}\right),$$

which means that

$$t = -\frac{2r}{n+1}\sin\left(\pi - \cos^{-1}\left(\frac{n+1}{2}\right)\right) + \frac{2k\pi}{n+1}.$$

This gives us that

$$\theta = \frac{2r}{n+1}\sin\left(\pi - \cos^{-1}\left(\frac{n+1}{2r}\right)\right) - \cos^{-1}\left(\frac{n+1}{2r}\right) + \pi - \frac{2k\pi}{n+1}. \tag{4.1}$$

Further, we have that

$$\max_{k,t} r^2 = \frac{1}{4}\left\{\left(4\left[\frac{n+3}{2}\right]\pi + (n+1)\pi\right)^2 + (n+1)^2\right\}$$

and

$$\min_{k,t} r^2 = \frac{(n+1)^2}{4}.$$

Letting Γ be the boundary of the domain $\mathbb{D}_c$, we have that the polar equations of Γ are given by

$$\Gamma = \begin{cases} x = r\cos\theta \\ \\ y = r\sin\theta \end{cases}$$

where r and θ satisfy the condition (4.1) with

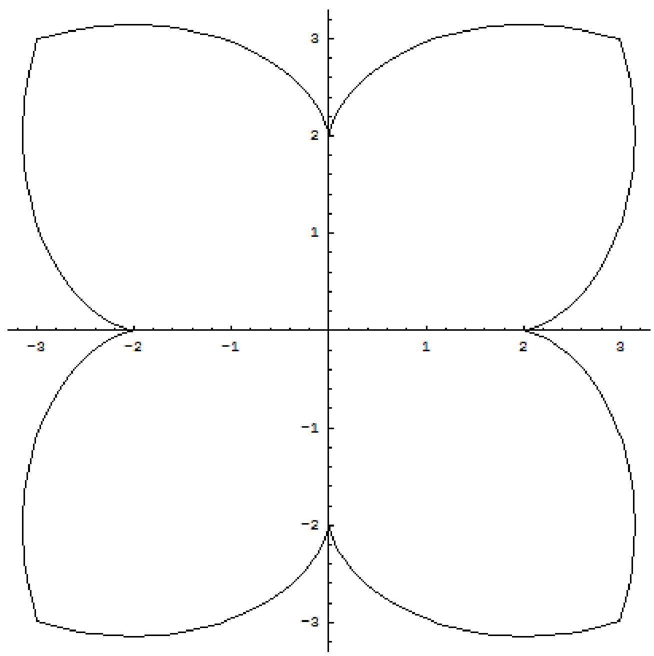
$$\begin{cases} k = 0, \pm 1, \pm 2, \cdots, \pm l & (n = 2l) \\ \\ k = 0, \pm 1, \pm 2, \cdots, \pm l, -(l+1) & (n = 2l+1). \end{cases}$$

Remark. Γ has a form of $(n+1)$-valently clover.

Example 4.1. *For the case $n = 3$, the harmonic function*

$$h_c(z) = \left(\frac{e^{cz}-1}{c}\right) - \overline{\frac{1}{c}\left(\frac{2}{c^2}(1-e^{cz}) + \frac{2}{c}ze^{cz} - z^2e^{cz}\right)}$$

is univalent in $\mathbb{U}$ where c is in the domain $\mathbb{D}_c$ as follows:



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