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GLOBAL SOLVABILITY AND MANN ITERATION METHOD WITH ERROR FOR A THIRD ORDER NONLINEAR NEUTRAL DELAY DIFFERENTIAL EQUATION

(COMMUNICATED BY AGACIK ZAFER)

MIN LIU, ZHENYU GUO

Abstract. This paper intends to investigate the existence of uncountably many bounded positive solutions of a third order nonlinear neutral delay differential equation

$$\frac{d}{dt}\Big\{r_1(t)\frac{d}{dt}\Big[r_2(t)\frac{d}{dt}\Big(x(t)-f(t,x(t-\sigma))\Big)\Big]\Big\}$$
$$+\frac{d}{dt}\Big[r_1(t)\frac{d}{dt}g(t,x(p(t)))\Big]+\frac{d}{dt}h(t,x(q(t)))=l(t,x(\eta(t))),\quad t\ge t_0$$

in the following bounded closed and convex set

$$\Omega(a,b)=\big\{x(t)\in C([t_0,+\infty),\mathbb{R}):a(t)\le x(t)\le b(t),\ \forall t\ge t_0\big\},$$

where $\sigma>0, r_1,r_2,a,b\in C([t_0,+\infty),\mathbb{R}^+), f,g,h,l\in C([t_0,+\infty)\times\mathbb{R},\mathbb{R}), p,q,\eta\in C([t_0,+\infty),[t_0,+\infty))$. By using the Krasnoselskii fixed point theorem, the Schauder fixed point theorem, the Sadovskii fixed point theorem and the Banach contraction principle, four existence results of uncountably many bounded positive solutions of the differential equation are established. Moreover, a perturbed Mann iteration method with error is constructed for approximating the solution of the third order differential equation, and the convergence and stability of the iterative sequence generated by the algorithm are discussed.

1. Introduction and preliminaries

In recent years, it undergoes a rapid development for the theory of neutral delay differential equations and systems, especially for the existence of nonoscillatory solutions of second-order and higher order neutral delay differential equations, refer to [1, 3-5, 9-11, 13-16] and the references therein.

In 2005, Zhang, Feng, Yan and Song [15] studied the existence of nonoscillatory solutions of the first-order neutral delay differential equations with variable

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coefficients and delays

$$\frac{d}{dt}\Big[x(t)+p(t)x(t-\tau)\Big]+Q_1(t)x(t-\sigma_1)-Q_2(t)x(t-\sigma_2)=0,\quad t\geq t_0, \tag{1.1}$$

where $p\in C([t_0,+\infty),\mathbb{R}),\tau>0,\sigma_1,\sigma_2\geq 0$ and $Q_1,Q_2\in C([t_0,+\infty),\mathbb{R}^+)$ with $\int_{t_0}^{+\infty}Q_i(s)ds<+\infty$ for $i\in\{1,2\}$, and

$$\frac{d}{dt}\Big[x(t)+p(t)x(t-\tau)\Big]+\sum_{i=1}^{m}A_i(t)x(t-\sigma_i)-\sum_{i=m+1}^{n}A_i(t)x(t-\sigma_i)=0,\quad t\geq t_0, \tag{1.2}$$

$$\frac{d}{dt}\Big[x(t)+p(t)x(t-\tau)\Big]+\sum_{i=1}^{n}B_i(t)x(t-\sigma_i)=0,\quad t\geq t_0, \tag{1.3}$$

where $p,B_i\in C([t_0,+\infty),\mathbb{R}),\tau>0,\sigma_i\geq 0,A_i\in C([t_0,+\infty),\mathbb{R}^+)$ with $\int_{t_0}^{+\infty}A_i(s)ds<+\infty$ and $\int_{t_0}^{+\infty}|B_i(s)|ds<+\infty$ for $i\in\{1,2,\dots,n\}$.

In 2005, Lin [10] got some sufficient conditions for oscillation and nonoscillation for the second-order nonlinear neutral differential equation

$$\frac{d^2}{dt^2}\Big[x(t)-p(t)x(t-\tau)\Big]+q(t)f(x(t-\sigma))=0,\quad t\geq 0, \tag{1.4}$$

where $\tau,\sigma>0,p,q\in C([0,+\infty),\mathbb{R}),f\in C(\mathbb{R},\mathbb{R})$ with $q(t)\geq 0$ and $xf(x)>0$ for $t\in\mathbb{R},x\in\mathbb{R}/\{0\}$.

In 2007, Islam and Raffoul [5] employed Krasnoselskii fixed point theorem and the Banach contraction principle to discuss the existence of periodic solutions of the nonlinear neutral system of differential equations of the form

$$\frac{d}{dt}x(t)=A(t)x(t)+\frac{d}{dt}Q(t,x(t-g(t)))+G(t,x(t),x(t-g(t))), \tag{1.5}$$

where $A(t)$ is a nonsingular $n\times n$ matrix, $Q\in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n),G\in C(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)$.

In 2007, Zhou [14] used Krasnoselskii fixed point theorem to study the existence of nonoscillatory soluions of the following second-order nonlinear neutral differential equation

$$\frac{d}{dt}\Big[r(t)\frac{d}{dt}(x(t)+p(t)x(t-\tau))\Big]+\sum_{i=1}^{m}Q_i(t)f_i(x(t-\sigma_i))=0,\quad t\geq t_0, \tag{1.6}$$

where $m\geq 1$ is an integer, $\tau>0,\sigma_i\geq 0,r,p,Q_i\in C([t_0,+\infty),\mathbb{R})$ and $f_i\in C(\mathbb{R},\mathbb{R})$ for $i\in\{1,2,\dots,m\}$.

However, the works on Eqs.(1.1)-(1.6) listed above and others are all concerning the existence of single nonoscillatory solution or at most infinitely many nonoscillatory solutions. As far as we are concerned, the existence of uncountably many nonoscillatory solutions of Eqs.(1.1)-(1.6) and other differential equations or systems has received much less attention until now.

In this paper, we are concerned with the following third order nonlinear neutral delay differential equation:

$$\begin{aligned}&\frac{d}{dt}\Big\{r_1(t)\frac{d}{dt}\Big[r_2(t)\frac{d}{dt}\Big(x(t)-f(t,x(t-\sigma))\Big)\Big]\Big\}\\&\quad+\frac{d}{dt}\Big[r_1(t)\frac{d}{dt}g(t,x(p(t)))\Big]+\frac{d}{dt}h(t,x(q(t)))=l(t,x(\eta(t))),\quad t\geq t_0,\end{aligned} \tag{1.7}$$

where $\sigma > 0, r_1, r_2 \in C([t_0, +\infty), \mathbb{R}^+), f, g, h, l \in C([t_0, +\infty) \times \mathbb{R}, \mathbb{R}), p, q, \eta \in C([t_0, +\infty), [t_0, +\infty))$ with

$$\lim_{t\to+\infty} p(t) = \lim_{t\to+\infty} q(t) = \lim_{t\to+\infty} \eta(t) = +\infty.$$

By applying the Krasnoselskii fixed point theorem, the Schauder fixed point theorem, the Sadovskii fixed point theorem and the Banach contraction principle, we obtain four existence results of uncountably many bounded positive solutions of Eq.(1.7). Furthermore, we construct a perturbed Mann iteration algorithm for approximating the solution of Eq.(1.7) and discuss the convergence and stability of the iterative sequence.

Throughout this paper, put $I = [t_0, +\infty)$ and $C(I, \mathbb{R})$ denote the Banach space of all continuous and bounded functions $x(t)$ on I with norm $\|x\| = \sup_{t\in I} |x(t)|$. For any $a, b \in C(I, \mathbb{R}^+)$, set $\overline{a} = \sup_{t\in I} a(t)$, $\underline{a} = \inf_{t\in I} a(t)$, $\overline{b} = \sup_{t\in I} b(t)$, $\underline{b} = \inf_{t\in I} b(t)$ and

$$\Omega(a, b) = \big\{x(t) \in C(I, \mathbb{R}) : a(t) \le x(t) \le b(t),\ \forall t \in I\big\}.$$

Obviously, $\Omega(a, b)$ is a bounded closed and convex subset of $C(I, \mathbb{R})$. For any $D \subseteq \Omega(a, b)$ and $t \in I$, let

$$D(t) = \sup\big\{|x(t) - y(t)| : x(t), y(t) \in D\big\};$$
$$\mathrm{diam} D = \sup\{\|x - y\| : x, y \in D\}.$$

It's assumed in the sequel that there exist functions $a, b, c, d, \alpha, \beta, \gamma, \lambda, \tau, \zeta \in C(I, \mathbb{R}^+)$ with $a(t) < b(t)$ for $t \in I$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

(i) $\int_{t_0}^{+\infty} \max\Big\{\frac{\alpha(s)}{r_2(s)}, \frac{\beta(s)}{r_1(s)}, \gamma(s), \frac{1}{r_1(s)}, \frac{1}{r_2(s)}\Big\} ds < +\infty$;

(ii) $|f(t, u)| \le c(t), \quad \forall t \in I,\ u \in [\underline{a}, \overline{b}]$;

(iii) $|f(t, u) - f(t, v)| \le d(t)|u - v|, \quad \forall t \in I,\ u, v \in [\underline{a}, \overline{b}]$;

(iv) $|g(t, u)| \le \alpha(t), |h(t, u)| \le \beta(t), |l(t, u)| \le \gamma(t), \quad \forall t \in I,\ u \in [\underline{a}, \overline{b}]$;

(v) $\int_{t_0}^{+\infty} \max\Big\{\frac{s\alpha(s)}{r_2(s)}, \frac{\beta(s)}{r_1(s)}, \gamma(s), \frac{1}{r_1(s)}, \frac{s}{r_2(s)}\Big\} ds < +\infty$;

(vi)

$$\begin{aligned}
&|f(t, x(t-\sigma)) - f(t, y(t-\sigma))| \\
&+ \int_t^{+\infty} \frac{|g(s, x(p(s))) - g(s, y(p(s)))|}{r_2(s)} ds \\
&+ \int_t^{+\infty}\int_s^{+\infty} \frac{|h(u, x(q(u))) - h(u, y(q(u)))|}{r_2(s) r_1(u)} du ds \\
&+ \int_t^{+\infty}\int_s^{+\infty}\int_u^{+\infty} \frac{|l(v, x(\eta(v))) - l(v, y(\eta(v)))|}{r_2(s) r_1(u)} dv du ds \\
&\le \varphi(D(t)), \quad \forall D \subseteq \Omega(a, b),\ x, y \in D,\ t \in I;
\end{aligned}$$

(vii) $|g(t, u) - g(t, v)| \le \lambda(t)|u - v|, \quad |h(t, u) - h(t, v)| \le \tau(t)|u - v|,$
$|l(t, u) - l(t, v)| \le \zeta(t)|u - v|, \quad \forall t \in I,\ u, v \in [\underline{a}, \overline{b}]$;

(viii) $\int_{t_0}^{+\infty} \max\Big\{\frac{\lambda(s)}{r_2(s)}, \frac{\tau(s)}{r_1(s)}, \zeta(s), \frac{1}{r_1(s)}, \frac{1}{r_2(s)}\Big\} ds < +\infty$.

By a solution of Eq.(1.7), we mean a function x such that for some $t_1 \ge t_0$, $x \in C([t_1 - \sigma, +\infty), \mathbb{R})$, $x(t) - f(t, x(t-\sigma))$ is 3 times continuously differentiable on $[t_1, +\infty)$, $g(t, x(p(t)))$ is 2 times continuously differentiable on $[t_1, +\infty)$, $h(t, x(q(t)))$ is continuously differentiable on $[t_1, +\infty)$ and Eq.(1.7) holds for $t \ge t_1$.

The following four lemmas play significant roles in this paper.

Lemma 1.1. *(Krasnoselskii Fixed Point Theorem [2]) Let D be a nonempty bounded closed convex subset of a Banach space X and $S, Q : D \to X$ satisfy $Sx + Qy \in D$ for each $x, y \in D$. If Q is a contraction mapping and S is a completely continuous mapping, then the equation $Sx + Qx = x$ has at least one solution in D.*

Lemma 1.2. *(Schauder Fixed Point Theorem [2]) Let D be a nonempty closed convex subset of a Banach space X. Let $S : D \to D$ be a continuous mapping such that SD is a relatively compact subset of X. Then S has at least one fixed point in D.*

Lemma 1.3. *(Sadovskii Fixed Point Theorem [12]) Let D be a nonempty bounded closed convex subset of a Banach space X and $S : D \to D$ be a continuous condensing mapping. Then S has at least one fixed point in D.*

Lemma 1.4. *(Banach contraction principle) Let D be a closed subset of a completely metric space X and $S : D \to D$ be a contraction on D. Then S has at least one fixed point in D.*

2. Existence of uncountably many bounded positive solutions

In this section, we demonstrate the existence of uncountably many bounded positive solutions of Eq.(1.7). Let

$$c = \sup_{t\in I} c(t) \quad \text{and} \quad d = \sup_{t\in I} d(t).$$

Theorem 2.1. *Let $a, b \in C(I, \mathbb{R}^+)$ with $\overline{a} < \underline{b}$ and (i)-(iv) hold. If $d \in (0, 1)$ and $c < \frac{\underline{b}-\overline{a}}{2}$, then Eq.(1.7) possesses uncountably many bounded positive solutions in $\Omega(a, b)$.*

Proof. Set $L \in (\overline{a}+c, \underline{b}-c)$. According to (i), we deduce that there exists $T \geq t_0+\sigma$ large enough satisfying

$$\int_T^{+\infty} \frac{\alpha(s)}{r_2(s)} ds + \int_T^{+\infty}\int_s^{+\infty} \frac{\beta(u)}{r_2(s)r_1(u)} duds + \int_T^{+\infty}\int_s^{+\infty}\int_u^{+\infty} \frac{\gamma(v)}{r_2(s)r_1(u)} dvduds < \min\left\{\underline{b} - c - L, L - c - \overline{a}\right\}. \tag{2.1}$$

Define two mappings $Q_L, S_L : \Omega(a, b) \to C(I, \mathbb{R})$ by

$$(Q_Lx)(t) = \begin{cases} L + f(t, x(t-\sigma)), & t \geq T \\ (Q_Lx)(T), & t_0 \leq t < T \end{cases}$$
$$(S_Lx)(t) = \begin{cases} \int_t^{+\infty} \frac{g(s,x(p(s)))}{r_2(s)} ds - \int_t^{+\infty}\int_s^{+\infty} \frac{h(u,x(q(u)))}{r_2(s)r_1(u)} duds \\ \qquad - \int_t^{+\infty}\int_s^{+\infty}\int_u^{+\infty} \frac{l(v,x(\eta(v)))}{r_2(s)r_1(u)} dvduds, & t \geq T \\ (S_Lx)(T), & t_0 \leq t < T \end{cases} \tag{2.2}$$

for $x \in \Omega(a, b)$ and $t \in I$.

Firstly, we prove $Q_L x + S_L y \in \Omega(a,b)$ for all $x, y \in \Omega(a,b)$. Due to (ii), (iv), (2.1) and (2.2), we get that for each $x, y \in \Omega(a,b)$ and $t \geq T$,

$$\begin{aligned}
&(Q_L x + S_L y)(t) \\
&\leq L + c(t) + \int_T^{+\infty} \frac{\alpha(s)}{r_2(s)} ds + \int_T^{+\infty} \int_s^{+\infty} \frac{\beta(u)}{r_2(s) r_1(u)} duds \\
&\quad + \int_T^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{\gamma(v)}{r_2(s) r_1(u)} dvduds \\
&\leq L + c + (\underline{b} - c - L) \\
&\leq b(t)
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
&(Q_L x + S_L y)(t) \\
&\geq L - c(t) - \Bigg[\int_T^{+\infty} \frac{\alpha(s)}{r_2(s)} ds + \int_T^{+\infty} \int_s^{+\infty} \frac{\beta(u)}{r_2(s) r_1(u)} duds \\
&\quad + \int_T^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{\gamma(v)}{r_2(s) r_1(u)} dvduds \Bigg] \\
&\geq L - c - (L - c - \overline{a}) \\
&\geq a(t).
\end{aligned} \tag{2.4}$$

It follows from (2.3) and (2.4) that $Q_L \Omega(a,b) + S_L \Omega(a,b) \subseteq \Omega(a,b)$.

Secondly, we demonstrate that Q_L is a contraction mapping. According to (2.2) and (iii), we derive that

$$\begin{aligned}
|(Q_L x)(t) - (Q_L y)(t)| &= |f(t, x(t-\sigma)) - f(t, y(t-\sigma))| \\
&\leq d(t) |x(t-\sigma) - y(t-\sigma)| \\
&\leq d \|x - y\|, \quad \forall x, y \in \Omega(a,b),\ t \geq T,
\end{aligned}$$

which infers that

$$\|Q_L x - Q_L y\| \leq d \|x - y\|, \quad \forall x, y \in \Omega(a,b).$$

That is, Q_L is a contraction mapping by $d \in (0,1)$.

Thirdly, we show that S_L is completely continuous. Now we demonstrate S_L is continuous in $\Omega(a,b)$. Let $x_0 \in \Omega(a,b)$ and $\{x_k\}_{k \geq 0} \subset \Omega(a,b)$ with $x_k \to x_0$ as

$k \to +\infty$. (2.2) yields that

$$
\begin{aligned}
&\|S_L x_k - S_L x_0\| \\
&\quad = \sup_{t\in I} |(S_L x_k)(t) - (S_L x_0)(t)| \\
&\quad \le \sup_{t\ge T} \Big\{ \int_t^{+\infty} \frac{|g(s, x_k(p(s))) - g(s, x_0(p(s)))|}{r_2(s)} ds \\
&\qquad\qquad + \int_t^{+\infty} \int_s^{+\infty} \frac{|h(u, x_k(q(u))) - h(u, x_0(q(u)))|}{r_2(s) r_1(u)} duds \\
&\qquad\qquad + \int_t^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{|l(v, x_k(\eta(v))) - l(v, x_0(\eta(v)))|}{r_2(s) r_1(u)} dvduds \Big\} \qquad (2.5) \\
&\quad \le \int_T^{+\infty} \frac{|g(s, x_k(p(s))) - g(s, x_0(p(s)))|}{r_2(s)} ds \\
&\qquad + \int_T^{+\infty} \int_s^{+\infty} \frac{|h(u, x_k(q(u))) - h(u, x_0(q(u)))|}{r_2(s) r_1(u)} duds \\
&\qquad + \int_T^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{|l(v, x_k(\eta(v))) - l(v, x_0(\eta(v)))|}{r_2(s) r_1(u)} dvduds.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&|g(s, x_k(p(s))) - g(s, x_0(p(s)))| \le 2\alpha(s), \\
&|h(u, x_k(q(u))) - h(u, x_0(q(u)))| \le 2\beta(u), \qquad (2.6) \\
&|l(v, x_k(\eta(v))) - l(v, x_0(\eta(v)))| \le 2\gamma(v),
\end{aligned}
$$

$$
\begin{aligned}
&|g(s, x_k(p(s))) - g(s, x_0(p(s)))| \to 0, \\
&|h(u, x_k(q(u))) - h(u, x_0(q(u)))| \to 0, \qquad (2.7) \\
&|l(v, x_k(\eta(v))) - l(v, x_0(\eta(v)))| \to 0
\end{aligned}
$$

as $k \to +\infty$ and $s, u, v \in [T, +\infty)$. It follows from (2.5), (2.6), (2.7) and Lebesgue dominated convergence theorem that $\|S_L x_k - S_L x_0\| \to 0$ as $k \to +\infty$. Hence S_L is continuous in $\Omega(a, b)$. Now we prove that $S_L \Omega(a, b)$ is relatively compact. In view of (i), (iv) and (2.2), we deduce that

$$
\begin{aligned}
\|S_L x\| &= \sup_{t\in I} |(S_L x)(t)| \\
&\le \int_T^{+\infty} \frac{\alpha(s)}{r_2(s)} ds + \int_T^{+\infty} \int_s^{+\infty} \frac{\beta(u)}{r_2(s) r_1(u)} duds \\
&\quad + \int_T^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{\gamma(v)}{r_2(s) r_1(u)} dvduds, \quad \forall x \in \Omega(a, b).
\end{aligned}
$$

That is, $S_L \Omega(a, b)$ is uniformly bounded. For the equicontinuity of $S_L \Omega(a, b)$ on I, according to Levitans result [6], it suffices to prove that for any given $\epsilon > 0$, I can be decomposed into finite subintervals in such a way that on each subinterval all functions of the family have change of amplitude less than ϵ. Let $\epsilon > 0$. By (i), there exists $T_* > T$ such that

$$
\begin{aligned}
&\int_{T_*}^{+\infty} \frac{\alpha(s)}{r_2(s)} ds + \int_{T_*}^{+\infty} \int_s^{+\infty} \frac{\beta(u)}{r_2(s) r_1(u)} duds \\
&\quad + \int_{T_*}^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{\gamma(v)}{r_2(s) r_1(u)} dvduds < \frac{\epsilon}{2}.
\end{aligned}
\qquad (2.8)
$$

It follows from (iv), (2.2) and (2.8) that for all $x \in \Omega(a,b)$ and $t_2 \geq t_1 \geq T_*$,

$$\begin{aligned}|(S_Lx)(t_1) - (S_Lx)(t_2)| &\leq |(S_Lx)(t_1)| + |(S_Lx)(t_2)| \\ &\leq \int_{t_1}^{+\infty} \frac{\alpha(s)}{r_2(s)}ds + \int_{t_1}^{+\infty}\int_s^{+\infty} \frac{\beta(u)}{r_2(s)r_1(u)}duds \\ &\quad + \int_{t_1}^{+\infty}\int_s^{+\infty}\int_u^{+\infty} \frac{\gamma(v)}{r_2(s)r_1(u)}dvduds \\ &\quad + \int_{t_2}^{+\infty} \frac{\alpha(s)}{r_2(s)}ds + \int_{t_2}^{+\infty}\int_s^{+\infty} \frac{\beta(u)}{r_2(s)r_1(u)}duds \\ &\quad + \int_{t_2}^{+\infty}\int_s^{+\infty}\int_u^{+\infty} \frac{\gamma(v)}{r_2(s)r_1(u)}dvduds \\ &\leq 2\bigg[\int_{T_*}^{+\infty} \frac{\alpha(s)}{r_2(s)}ds + \int_{T_*}^{+\infty}\int_s^{+\infty} \frac{\beta(u)}{r_2(s)r_1(u)}duds \\ &\quad + \int_{T_*}^{+\infty}\int_s^{+\infty}\int_u^{+\infty} \frac{\gamma(v)}{r_2(s)r_1(u)}dvduds\bigg] \\ &< \epsilon;\end{aligned}$$

For each $x \in \Omega(a,b)$ and $T \leq t_1 \leq t_2 \leq T_*$, by (iv) and (2.2), we infer that

$$\begin{aligned}&|(S_Lx)(t_1) - (S_Lx)(t_2)| \\ &\leq \int_{t_1}^{t_2} \frac{\alpha(s)}{r_2(s)}ds + \int_{t_1}^{t_2}\int_s^{+\infty} \frac{\beta(u)}{r_2(s)r_1(u)}duds \\ &\quad + \int_{t_1}^{t_2}\int_s^{+\infty}\int_u^{+\infty} \frac{\gamma(v)}{r_2(s)r_1(u)}dvduds \\ &\leq M|t_1 - t_2|,\end{aligned} \tag{2.9}$$

where

$$M = \max_{T \leq s \leq T_*}\left\{\frac{\alpha(s)}{r_2(s)} + \int_s^{+\infty} \frac{\beta(u)}{r_2(s)r_1(u)}du + \int_s^{+\infty}\int_u^{+\infty} \frac{\gamma(v)}{r_2(s)r_1(u)}dvdu\right\}.$$

(2.9) implies that there exists $\delta = \frac{\epsilon}{1+M} > 0$ such that $|(S_Lx)(t_1) - (S_Lx)(t_2)| < \epsilon$ for any $t_1, t_2 \in [T, T_*]$ with $|t_1 - t_2| < \delta$ and $x \in \Omega(a,b)$;

For $x \in \Omega(a,b), t_0 \leq t_1 \leq t_2 \leq T$, due to (2.2), we achieve that

$$|(S_Lx)(t_1) - (S_Lx)(t_2)| = 0.$$

Hence Lemma 1.1 ensures that there exists $x \in \Omega(a,b)$ with $Q_Lx + S_Lx = x$. It is easy to see that x is a bounded positive solution of Eq.(1.7).

Finally, we investigate that Eq.(1.7) possesses uncountably many bounded positive solutions. Let $L_1, L_2 \in (\overline{a} + c, \underline{b} - c)$ and $L_1 \neq L_2$. For each $j \in \{1,2\}$, we choose a constant $T_j > t_0 + \sigma$ and two mappings Q_{L_j} and S_{L_j} satisfying (2.1) and (2.2), where L and T are replaced by L_j and T_j, respectively, and

$$\begin{aligned}&\int_{T_3}^{+\infty} \frac{\alpha(s)}{r_2(s)}ds + \int_{T_3}^{+\infty}\int_s^{+\infty} \frac{\beta(u)}{r_2(s)r_1(u)}duds \\ &\quad + \int_{T_3}^{+\infty}\int_s^{+\infty}\int_u^{+\infty} \frac{\gamma(v)}{r_2(s)r_1(u)}dvduds < \frac{|L_1 - L_2|}{2}\end{aligned} \tag{2.10}$$

for some $T_3 > \max\{T_1, T_2\}$. Obviously, the mappings $Q_{L_1} + S_{L_1}$ and $Q_{L_2} + S_{L_2}$ have the fixed points $x, y \in \Omega(a, b)$, respectively. That is, x and y are bounded positive solutions of Eq.(1.7) in $\Omega(a, b)$. In order to show that Eq.(1.7) possesses uncountably many bounded positive solutions in $\Omega(a, b)$, we need only to prove that $x \neq y$. Indeed, by (2.2) we gain that for $t \geq T_3$,

$$\begin{aligned} x(t) = L_1 + f(t, x(t-\sigma)) + \int_t^{+\infty} \frac{g(s, x(p(s)))}{r_2(s)} ds \\ - \int_t^{+\infty} \int_s^{+\infty} \frac{h(u, x(q(u)))}{r_2(s) r_1(u)} du ds - \int_t^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{l(v, x(\eta(v)))}{r_2(s) r_1(u)} dv du ds \end{aligned}$$

and

$$\begin{aligned} y(t) = L_2 + f(t, y(t-\sigma)) + \int_t^{+\infty} \frac{g(s, y(p(s)))}{r_2(s)} ds \\ - \int_t^{+\infty} \int_s^{+\infty} \frac{h(u, y(q(u)))}{r_2(s) r_1(u)} du ds - \int_t^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{l(v, y(\eta(v)))}{r_2(s) r_1(u)} dv du ds, \end{aligned}$$

which together with (iv) and (2.10) yield that

$$\begin{aligned} & \big| x(t) - y(t) - (f(t, x(t-\sigma)) - f(t, y(t-\sigma))) \big| \\ & \geq |L_1 - L_2| - 2 \Big[\int_{T_3}^{+\infty} \frac{\alpha(s)}{r_2(s)} ds + \int_{T_3}^{+\infty} \int_s^{+\infty} \frac{\beta(u)}{r_2(s) r_1(u)} du ds \\ & \quad + \int_{T_3}^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{\gamma(v)}{r_2(s) r_1(u)} dv du ds \Big] \\ & > 0, \quad \forall t \geq T_3, \end{aligned}$$

that is, $x \neq y$. This completes the proof. □

Theorem 2.2. *Let $a, b \in C(I, \mathbb{R}^+)$ with $\overline{a} < \underline{b}$ and (iv) and (v) hold. Then Eq.(1.7) with $f(t, u) = u$ possesses uncountably many bounded positive solutions in $\Omega(a, b)$.*

Proof. Due to (v), there exists $M_0 > 0$ such that

$$\max\Big\{ \int_{t_0}^{+\infty} \frac{\beta(u)}{r_1(u)} du, \int_{t_0}^{+\infty} \int_{t_0}^{+\infty} \frac{\gamma(v)}{r_1(u)} dv du \Big\} < M_0.$$

By the known result([2]), we gain that

$$\int_{t_0}^{+\infty} \frac{s\alpha(s)}{r_2(s)} ds < +\infty, \quad \int_{t_0}^{+\infty} \frac{s}{r_2(s)} ds < +\infty$$

are equivalent to

$$\sum_{j=0}^{+\infty} \int_{t_0 + j\sigma}^{+\infty} \frac{\alpha(s)}{r_2(s)} ds < +\infty, \quad \sum_{j=0}^{+\infty} \int_{t_0 + j\sigma}^{+\infty} \frac{1}{r_2(s)} ds < +\infty$$

respectively. Hence

$$\begin{aligned}
&\sum_{j=0}^{+\infty}\Big[\int_{t_0+j\sigma}^{+\infty}\frac{\alpha(s)}{r_2(s)}ds+\int_{t_0+j\sigma}^{+\infty}\int_s^{+\infty}\frac{\beta(u)}{r_2(s)r_1(u)}duds\\
&\qquad+\int_{t_0+j\sigma}^{+\infty}\int_s^{+\infty}\int_u^{+\infty}\frac{\gamma(v)}{r_2(s)r_1(u)}dvduds\Big]\\
&<\sum_{j=0}^{+\infty}\Big[\int_{t_0+j\sigma}^{+\infty}\frac{\alpha(s)}{r_2(s)}ds+\int_{t_0+j\sigma}^{+\infty}\int_{t_0}^{+\infty}\frac{\beta(u)}{r_2(s)r_1(u)}duds\\
&\qquad+\int_{t_0+j\sigma}^{+\infty}\int_{t_0}^{+\infty}\int_{t_0}^{+\infty}\frac{\gamma(v)}{r_2(s)r_1(u)}dvduds\Big]\\
&<\sum_{j=0}^{+\infty}\Big[\int_{t_0+j\sigma}^{+\infty}\frac{\alpha(s)}{r_2(s)}ds+2M_0\int_{t_0+j\sigma}^{+\infty}\frac{1}{r_2(s)}ds\Big]\\
&<+\infty.
\end{aligned}$$

Let $L\in(\overline{a},\underline{b})$. According to the above inequalities, we deduce that there exists $T\geq t_0+\sigma$ sufficiently large satisfying

$$\begin{aligned}
&\sum_{j=1}^{+\infty}\Big[\int_{T+j\sigma}^{+\infty}\frac{\alpha(s)}{r_2(s)}ds+\int_{T+j\sigma}^{+\infty}\int_s^{+\infty}\frac{\beta(u)}{r_2(s)r_1(u)}duds\\
&\qquad+\int_{T+j\sigma}^{+\infty}\int_s^{+\infty}\int_u^{+\infty}\frac{\gamma(v)}{r_2(s)r_1(u)}dvduds\Big]\\
&<\min\{\underline{b}-L,L-\overline{a}\}.
\end{aligned}\tag{2.11}$$

Define a mapping $Q_L:\Omega(a,b)\to C(I,\mathbb{R})$ by

$$(Q_Lx)(t)=\begin{cases}L-\sum_{j=1}^{+\infty}\Big[\int_{t+j\sigma}^{+\infty}\frac{g(s,x(p(s)))}{r_2(s)}ds-\int_{t+j\sigma}^{+\infty}\int_s^{+\infty}\frac{h(u,x(q(u)))}{r_2(s)r_1(u)}duds\\
\qquad-\int_{t+j\sigma}^{+\infty}\int_s^{+\infty}\int_u^{+\infty}\frac{l(v,x(\eta(v)))}{r_2(s)r_1(u)}dvduds\Big], & t\geq T\\
(Q_Lx)(T), & t_0\leq t<T.\end{cases}\tag{2.12}$$

First of all, we prove $Q_L x \in \Omega(a,b)$ for all $x \in \Omega(a,b)$. Due to (iv) and (2.12), we derive that for each $x \in \Omega(a,b)$,

$$\begin{aligned}
&(Q_L x)(t)\\
&\le L + \sum_{j=1}^{+\infty}\Bigg[\int_{T+j\sigma}^{+\infty}\frac{\alpha(s)}{r_2(s)}ds + \int_{T+j\sigma}^{+\infty}\int_s^{+\infty}\frac{\beta(u)}{r_2(s)r_1(u)}duds\\
&\qquad\qquad + \int_{T+j\sigma}^{+\infty}\int_s^{+\infty}\int_u^{+\infty}\frac{\gamma(v)}{r_2(s)r_1(u)}dvduds\Bigg]\\
&\le L + (\underline{b} - L)\\
&\le b(t), \qquad t \ge T,\\
&(Q_L x)(t)\\
&\ge L - \sum_{j=1}^{+\infty}\Bigg[\int_{T+j\sigma}^{+\infty}\frac{\alpha(s)}{r_2(s)}ds + \int_{T+j\sigma}^{+\infty}\int_s^{+\infty}\frac{\beta(u)}{r_2(s)r_1(u)}duds\\
&\qquad\qquad + \int_{T+j\sigma}^{+\infty}\int_s^{+\infty}\int_u^{+\infty}\frac{\gamma(v)}{r_2(s)r_1(u)}dvduds\Bigg]\\
&\ge L - (L - \overline{a})\\
&\ge a(t), \qquad t \ge T.
\end{aligned}$$

Therefore, $Q_L\Omega(a,b) \subseteq \Omega(a,b)$.

Next, we demonstrate that Q_L is completely continuous. It's claimed that Q_L is continuous. Indeed, let $x_0 \in \Omega(a,b)$ and $\{x_k\}_{k\ge 0} \subset \Omega(a,b)$ with $x_k \to x_0$ as $k \to +\infty$. (2.12) yields that

$$\begin{aligned}
&\|Q_L x_k - Q_L x_0\|\\
&= \sup_{t\in I}|(Q_L x_k)(t) - (Q_L x_0)(t)|\\
&\le \sup_{t\in I}\Bigg\{\sum_{j=1}^{+\infty}\Bigg[\int_{t+j\sigma}^{+\infty}\frac{|g(s,x_k(p(s))) - g(s,x_0(p(s)))|}{r_2(s)}ds\\
&\qquad + \int_{t+j\sigma}^{+\infty}\int_s^{+\infty}\frac{|h(u,x_k(q(u))) - h(u,x_0(q(u)))|}{r_2(s)r_1(u)}duds\\
&\qquad + \int_{t+j\sigma}^{+\infty}\int_s^{+\infty}\int_u^{+\infty}\frac{|l(v,x_k(\eta(v))) - l(v,x_0(\eta(v)))|}{r_2(s)r_1(u)}dvduds\Bigg]\Bigg\} \qquad (2.13)\\
&\le \sum_{j=1}^{+\infty}\Bigg[\int_{T+j\sigma}^{+\infty}\frac{|g(s,x_k(p(s))) - g(s,x_0(p(s)))|}{r_2(s)}ds\\
&\qquad + \int_{T+j\sigma}^{+\infty}\int_s^{+\infty}\frac{|h(u,x_k(q(u))) - h(u,x_0(q(u)))|}{r_2(s)r_1(u)}duds\\
&\qquad + \int_{T+j\sigma}^{+\infty}\int_s^{+\infty}\int_u^{+\infty}\frac{|l(v,x_k(\eta(v))) - l(v,x_0(\eta(v)))|}{r_2(s)r_1(u)}dvduds\Bigg].
\end{aligned}$$

In light of (2.6), (2.7), (2.13) and Lebesgue dominated convergence theorem, we infer that $\|Q_L x_k - Q_L x_0\| \to 0$ as $k \to +\infty$, which means that Q_L is continuous. Now we show $Q_L\Omega(a,b)$ is relatively compact. On account of $Q_L\Omega(a,b) \subseteq \Omega(a,b)$,

Q_L is uniformly bounded. Because of (v) and for any $\epsilon > 0$, choose $T_* > T$ large enough such that

$$\sum_{j=1}^{+\infty}\Big[\int_{T_*+j\sigma}^{+\infty}\frac{\alpha(s)}{r_2(s)}ds+\int_{T_*+j\sigma}^{+\infty}\int_s^{+\infty}\frac{\beta(u)}{r_2(s)r_1(u)}duds \\ +\int_{T_*+j\sigma}^{+\infty}\int_s^{+\infty}\int_u^{+\infty}\frac{\gamma(v)}{r_2(s)r_1(u)}dvduds\Big]<\frac{\epsilon}{2}. \tag{2.14}$$

By (2.12) and (2.14), for $x \in \Omega(a, b), t_2 \geq t_1 \geq T_*$, we have

$$\begin{aligned}
&|(Q_Lx)(t_1)-(Q_Lx)(t_2)| \\
&\leq \sum_{j=1}^{+\infty}\Big[\int_{t_1+j\sigma}^{+\infty}\frac{\alpha(s)}{r_2(s)}ds+\int_{t_1+j\sigma}^{+\infty}\int_s^{+\infty}\frac{\beta(u)}{r_2(s)r_1(u)}duds \\
&\quad+\int_{t_1+j\sigma}^{+\infty}\int_s^{+\infty}\int_u^{+\infty}\frac{\gamma(v)}{r_2(s)r_1(u)}dvduds\Big] \\
&\quad+\sum_{j=1}^{+\infty}\Big[\int_{t_2+j\sigma}^{+\infty}\frac{\alpha(s)}{r_2(s)}ds+\int_{t_2+j\sigma}^{+\infty}\int_s^{+\infty}\frac{\beta(u)}{r_2(s)r_1(u)}duds \\
&\quad+\int_{t_2+j\sigma}^{+\infty}\int_s^{+\infty}\int_u^{+\infty}\frac{\gamma(v)}{r_2(s)r_1(u)}dvduds\Big] \\
&< \epsilon;
\end{aligned}$$

For $T \leq t_1 \leq t_2 \leq T_*$, choose a sufficiently large integer $w \geq 1$ satisfying $T+j\sigma \geq T_*$ with $j \geq w$. For $x \in \Omega(a,b)$, we get that

$$
\begin{aligned}
&|(Q_L x)(t_1) - (Q_L x)(t_2)| \\
&\leq \sum_{j=1}^{+\infty} \Big[\int_{t_1+j\sigma}^{t_2+j\sigma} \frac{\alpha(s)}{r_2(s)} ds + \int_{t_1+j\sigma}^{t_2+j\sigma} \int_s^{+\infty} \frac{\beta(u)}{r_2(s)r_1(u)} duds \\
&\quad + \int_{t_1+j\sigma}^{t_2+j\sigma} \int_s^{+\infty} \int_u^{+\infty} \frac{\gamma(v)}{r_2(s)r_1(u)} dvduds \Big] \\
&= \sum_{j=1}^{w} \Big[\int_{t_1+j\sigma}^{t_2+j\sigma} \frac{\alpha(s)}{r_2(s)} ds + \int_{t_1+j\sigma}^{t_2+j\sigma} \int_s^{+\infty} \frac{\beta(u)}{r_2(s)r_1(u)} duds \\
&\quad + \int_{t_1+j\sigma}^{t_2+j\sigma} \int_s^{+\infty} \int_u^{+\infty} \frac{\gamma(v)}{r_2(s)r_1(u)} dvduds \Big] \\
&\quad + \sum_{j=w+1}^{+\infty} \Big[\int_{t_1+j\sigma}^{t_2+j\sigma} \frac{\alpha(s)}{r_2(s)} ds + \int_{t_1+j\sigma}^{t_2+j\sigma} \int_s^{+\infty} \frac{\beta(u)}{r_2(s)r_1(u)} duds \\
&\quad + \int_{t_1+j\sigma}^{t_2+j\sigma} \int_s^{+\infty} \int_u^{+\infty} \frac{\gamma(v)}{r_2(s)r_1(u)} dvduds \Big] \\
&\leq \sum_{j=1}^{w} \Big[\int_{t_1+j\sigma}^{t_2+j\sigma} \frac{\alpha(s)}{r_2(s)} ds + \int_{t_1+j\sigma}^{t_2+j\sigma} \int_s^{+\infty} \frac{\beta(u)}{r_2(s)r_1(u)} duds \\
&\quad + \int_{t_1+j\sigma}^{t_2+j\sigma} \int_s^{+\infty} \int_u^{+\infty} \frac{\gamma(v)}{r_2(s)r_1(u)} dvduds \Big] \\
&\quad + \sum_{j=1}^{+\infty} \Big[\int_{T_*+j\sigma}^{+\infty} \frac{\alpha(s)}{r_2(s)} ds + \int_{T_*+j\sigma}^{+\infty} \int_s^{+\infty} \frac{\beta(u)}{r_2(s)r_1(u)} duds \\
&\quad + \int_{T_*+j\sigma}^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{\gamma(v)}{r_2(s)r_1(u)} dvduds \Big] \\
&< W|t_1 - t_2| + \frac{\epsilon}{2},
\end{aligned}
$$

where

$$
\begin{aligned}
W = \max_{T+\sigma \leq s \leq T_*+w\sigma} \Bigg\{ \sum_{j=1}^{w} \Big[\frac{\alpha(s)}{r_2(s)} + \int_s^{+\infty} \frac{\beta(u)}{r_2(s)r_1(u)} du \\
+ \int_s^{+\infty} \int_u^{+\infty} \frac{\gamma(v)}{r_2(s)r_1(u)} dvdu \Big] \Bigg\},
\end{aligned}
$$

which implies that there exists $\delta = \frac{\epsilon}{2(1+W)} > 0$ such that $|(Q_L x)(t_1) - (Q_L x)(t_2)| < \epsilon$ for any $t_1, t_2 \in [T, T_*]$ with $|t_1 - t_2| < \delta$ and $x \in \Omega(a,b)$;

For $x \in \Omega(a,b), t_0 \leq t_1 \leq t_2 \leq T$, it follows from (2.12) that

$$|(Q_L x)(t_1) - (Q_L x)(t_2)| = 0.$$

Thus Lemma 1.2 ensures that there exists $x \in \Omega(a,b)$ with $Q_L x = x$. That is,

$$x(t) = \begin{cases} L - \sum_{j=1}^{+\infty} \Big[\int_{t+j\sigma}^{+\infty} \frac{g(s,x(p(s)))}{r_2(s)} ds - \int_{t+j\sigma}^{+\infty} \int_s^{+\infty} \frac{h(u,x(q(u)))}{r_2(s)r_1(u)} duds \\ \qquad - \int_{t+j\sigma}^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{l(v,x(\eta(v)))}{r_2(s)r_1(u)} dvduds \Big], \qquad t \geq T \\ x(T), \qquad t_0 \leq t < T. \end{cases}$$

It follows that for $t \geq T$,

$$x(t) - x(t-\sigma) = \int_t^{+\infty} \frac{g(s,x(p(s)))}{r_2(s)} ds - \int_t^{+\infty} \int_s^{+\infty} \frac{h(u,x(q(u)))}{r_2(s)r_1(u)} duds - \int_t^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{l(v,x(\eta(v)))}{r_2(s)r_1(u)} dvduds.$$

It's easy to verify that x is a bounded positive solution of Eq.(1.7).

Finally, we investigate that Eq.(1.7) possesses uncountably many bounded positive solutions. Let $L_1, L_2 \in (\overline{a}+c, \underline{b}-c)$ with $L_1 \neq L_2$. For each $j \in \{1,2\}$, choose a constant $T_j > t_0 + \sigma$ and a mapping Q_{L_j} to satisfy (2.11) and (2.12), where L and T are replaced by L_j and T_j, respectively, and

$$\sum_{j=1}^{+\infty} \Big[\int_{T_3+j\sigma}^{+\infty} \frac{\alpha(s)}{r_2(s)} ds + \int_{T_3+j\sigma}^{+\infty} \int_s^{+\infty} \frac{\beta(u)}{r_2(s)r_1(u)} duds + \int_{T_3+j\sigma}^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{\gamma(v)}{r_2(s)r_1(u)} dvduds \Big] < \frac{|L_1 - L_2|}{2}. \tag{2.15}$$

for some $T_3 > \max\{T_1, T_2\}$. Obviously, the mappings Q_{L_1} and Q_{L_2} have the fixed points $x, y \in \Omega(a,b)$, respectively. That is, x and y are bounded positive solutions of Eq.(1.7). Next we need only to prove that $x \neq y$. As a matter of fact, by (2.12) we get that for $t \geq T_3$,

$$x(t) = L_1 - \sum_{j=1}^{+\infty} \Big[\int_{t+j\sigma}^{+\infty} \frac{g(s,x(p(s)))}{r_2(s)} ds - \int_{t+j\sigma}^{+\infty} \int_s^{+\infty} \frac{h(u,x(q(u)))}{r_2(s)r_1(u)} duds - \int_{t+j\sigma}^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{l(v,x(\eta(v)))}{r_2(s)r_1(u)} dvduds \Big],$$

$$y(t) = L_2 - \sum_{j=1}^{+\infty} \Big[\int_{t+j\sigma}^{+\infty} \frac{g(s,y(p(s)))}{r_2(s)} ds - \int_{t+j\sigma}^{+\infty} \int_s^{+\infty} \frac{h(u,y(q(u)))}{r_2(s)r_1(u)} duds - \int_{t+j\sigma}^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{l(v,y(\eta(v)))}{r_2(s)r_1(u)} dvduds \Big],$$

which together with (iv) and (2.15) yield that

$$\begin{aligned} \big|x(t) - y(t)\big| &\geq |L_1 - L_2| - 2\sum_{j=1}^{+\infty} \Big[\int_{T_3+j\sigma}^{+\infty} \frac{\alpha(s)}{r_2(s)} ds + \int_{T_3+j\sigma}^{+\infty} \int_s^{+\infty} \frac{\beta(u)}{r_2(s)r_1(u)} duds \\ &\qquad + \int_{T_3+j\sigma}^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{\gamma(v)}{r_2(s)r_1(u)} dvduds \Big] \\ &> 0, \quad \forall t \geq T_3, \end{aligned}$$

that is, $x \neq y$. This completes the proof. □

Theorem 2.3. *Let $a, b \in C(I, \mathbb{R}^+)$ with $\overline{a} < \underline{b}$ and* (i), (ii), (iv) *and* (vi) *hold. If $c < \frac{\underline{b}-\overline{a}}{2}$ and φ is nondecreasing with $\varphi(t+) < t$ for each $t > 0$, then Eq.*(1.7) *possesses uncountably many bounded positive solutions in $\Omega(a,b)$.*

Proof. Put $L \in (\overline{a} + c, \underline{b} - c)$. In view of (i), there exists $T \geq t_0 + \sigma$ sufficiently large satisfying (2.1). Define a mapping $Q_L : \Omega(a,b) \to C(I,\mathbb{R})$ by

$$(Q_L x)(t) = \begin{cases} L + f(t, x(t-\sigma)) + \int_t^{+\infty} \frac{g(s,x(p(s)))}{r_2(s)} ds - \int_t^{+\infty} \int_s^{+\infty} \frac{h(u,x(q(u)))}{r_2(s)r_1(u)} duds \\ \qquad - \int_t^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{l(v,x(\eta(v)))}{r_2(s)r_1(u)} dvduds, \quad t \geq T \\ (Q_L x)(T), \qquad t_0 \leq t < T. \end{cases} \tag{2.16}$$

Firstly, we assure that $Q_L x \in \Omega(a,b)$ for all $x \in \Omega(a,b)$. In terms of (ii), (iv), (2.1) and (2.16), we infer that for each $x \in \Omega(a,b)$,

$$\begin{aligned} &(Q_L x)(t) \\ &\leq L + c(t) + \Bigg(\int_T^{+\infty} \frac{\alpha(s)}{r_2(s)} ds + \int_T^{+\infty} \int_s^{+\infty} \frac{\beta(u)}{r_2(s)r_1(u)} duds \\ &\qquad + \int_T^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{\gamma(v)}{r_2(s)r_1(u)} dvduds \Bigg) \\ &\leq L + c + (\underline{b} - c - L) \\ &\leq b(t), \quad t \geq T, \end{aligned} \tag{2.17}$$

$$\begin{aligned} &(Q_L x)(t) \\ &\geq L - c(t) - \Bigg(\int_T^{+\infty} \frac{\alpha(s)}{r_2(s)} ds + \int_T^{+\infty} \int_s^{+\infty} \frac{\beta(u)}{r_2(s)r_1(u)} duds \\ &\qquad + \int_T^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{\gamma(v)}{r_2(s)r_1(u)} dvduds \Bigg) \\ &\geq L - c - (L - c - \overline{a}) \\ &\geq a(t), \quad t \geq T. \end{aligned} \tag{2.18}$$

Thus $Q_L\Omega(a,b) \subseteq \Omega(a,b)$.

Secondly, we claim that

$$\lim_{t \to 0^+} \varphi(t) = 0 = \varphi(0). \tag{2.19}$$

Because $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing and nonnegative, we deduce that

$$0 \leq \varphi(0) \leq \varphi(t) \leq \varphi(s), \quad \forall s > t > 0,$$

which together with $\varphi(t+) < t$ for each $t > 0$ ensures that

$$0 \leq \varphi(0) \leq \varphi(t) \leq \lim_{s \to t^+} \varphi(s) = \varphi(t+) < t, \quad \forall t > 0.$$

Letting $t \to 0^+$ in the above inequalities, we get that (2.19) holds.

Thirdly, we prove that Q_L is continuous. Let $x_0 \in \Omega(a,b)$ and $\{x_k\}_{k \geq 0} \subset \Omega(a,b)$ with $x_k \to x_0$ as $k \to +\infty$. Let $D_k = \{x_k, x_0\}$ for $k \geq 1$. It follows from (vi), (2.16) and (2.19) that

$$
\begin{aligned}
\|Q_L x_k - Q_L x_0\| &= \sup_{t\in I} |(Q_L x_k)(t) - (Q_L x_0)(t)| \\
&\le \sup_{t\ge T} \bigg[|f(t, x_k(t-\sigma)) - f(t, x_0(t-\sigma))| \\
&\qquad + \int_t^{+\infty} \frac{|g(s, x_k(p(s))) - g(s, x_0(p(s)))|}{r_2(s)} ds \\
&\qquad + \int_t^{+\infty} \int_s^{+\infty} \frac{|h(u, x_k(q(u))) - h(u, x_0(q(u)))|}{r_2(s) r_1(u)} duds \\
&\qquad + \int_t^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{|l(v, x_k(\eta(v))) - l(v, x_0(\eta(v)))|}{r_2(s) r_1(u)} dvduds \bigg] \\
&\le \sup_{t\ge T} \varphi(D_k(t)) \\
&= \sup_{t\ge T} \varphi\big(|x_k(t) - x_0(t)|\big) \\
&\le \varphi(\|x_k - x_0\|) \\
&\to 0 \text{ as } k \to +\infty.
\end{aligned}
$$

Thereupon, Q_L is continuous in $\Omega(a, b)$.

Lastly, we demonstrate that Q_L is a condensing mapping. Let $\epsilon > 0$. For any nonempty subset D of $\Omega(a, b)$ with $\alpha(D) > 0$, where α denotes the Kuratowski measure of noncompactness, there exist finitely many subsets $D_1, D_2, \dots, D_n$ of $\Omega(a, b)$ such that

$$
D \subseteq \bigcup_{m=1}^{n} D_m, \ \mathrm{diam} D_m \le \alpha(D) + \epsilon, \quad \forall m \in \{1, 2, \dots, n\}. \tag{2.20}
$$

It follows from (vi) and (2.16) that for any $x, y \in D_m,\ m \in \{1, 2, \dots, n\}$,

$$
\begin{aligned}
\|Q_L x - Q_L y\| &= \sup_{t\in I} |(Q_L x)(t) - (Q_L y)(t)| \\
&\le \sup_{t\ge T} \bigg[|f(t, x(t-\sigma)) - f(t, y(t-\sigma))| \\
&\qquad + \int_t^{+\infty} \frac{|g(s, x(p(s))) - g(s, y(p(s)))|}{r_2(s)} ds \\
&\qquad + \int_t^{+\infty} \int_s^{+\infty} \frac{|h(u, x(q(u))) - h(u, y(q(u)))|}{r_2(s) r_1(u)} duds \\
&\qquad + \int_t^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{|l(v, x(\eta(v))) - l(v, y(\eta(v)))|}{r_2(s) r_1(u)} dvduds \bigg] \\
&\le \sup_{t\ge T} \varphi(D_m(t)) \\
&\le \varphi(\mathrm{diam} D_m),
\end{aligned}
$$

which means that

$$
\mathrm{diam}(Q_L D_m) \le \varphi(\mathrm{diam} D_m), \quad \forall m \in \{1, 2, \dots, n\}. \tag{2.21}
$$

According to (2.20) and (2.21), we derive that

$$\begin{aligned}\alpha(Q_LD) &\le \alpha(\bigcup_{m=1}^{n} Q_LD_m) = \max_{1\le m\le n}\{\alpha(Q_LD_m)\}\\ &\le \max_{1\le m\le n}\operatorname{diam}(Q_LD_m) \le \max_{1\le m\le n}\varphi(\operatorname{diam}D_m)\\ &\le \varphi(\alpha(D)+\epsilon).\end{aligned}$$

Setting $\epsilon \to 0$ in the above inequality, we gain that

$$\alpha(Q_LD) \le \varphi(\alpha(D)+0) < \alpha(D),$$

which implies that Q_L is condensing. Lemma 1.3 ensures that there exists $x \in \Omega(a,b)$ with $Q_Lx = x$, which is also a solution of Eq.(1.7). The rest of the proof is similar to that of Theorem 2.1. This completes the proof. □

Theorem 2.4. *Let $a, b \in C(I, \mathbb{R}^+)$ with $\overline{a} < \underline{b}$ and* (i)-(iv), (vii) *and* (viii) *hold. If $c < \frac{\underline{b}-\overline{a}}{2}$ and $d \in (0,1)$, then Eq.*(1.7) *possesses uncountably many bounded positive solutions in $\Omega(a,b)$.*

Proof. Put $L \in (\overline{a}+c, \underline{b}-c)$. Due to (i) and (viii), we derive that there exists $T \ge t_0 + \sigma$ large enough satisfying (2.1) and

$$\begin{aligned}&\int_T^{+\infty}\frac{\lambda(s)}{r_2(s)}ds + \int_T^{+\infty}\int_s^{+\infty}\frac{\tau(u)}{r_2(s)r_1(u)}duds + \int_T^{+\infty}\int_s^{+\infty}\int_u^{+\infty}\frac{\zeta(v)}{r_2(s)r_1(u)}dvduds\\ &< \frac{1-d}{2}.\end{aligned}\tag{2.22}$$

Define a mapping $Q_L : \Omega(a,b) \to C(I,\mathbb{R})$ by (2.16). Just as (2.17) and (2.18), we can demonstrate that Q_L is a self-mapping on $\Omega(a,b)$ by (ii), (iv) and (2.1).

We now investigate that Q_L is a contraction mapping. According to (iii), (vii) and (2.22), we get that

$$\begin{aligned}
&|(Q_Lx)(t) - (Q_Ly)(t)| \\
&\le |f(t, x(t-\sigma)) - f(t, y(t-\sigma))| + \int_t^{+\infty} \frac{|g(s, x(p(s))) - g(s, y(p(s)))|}{r_2(s)} ds \\
&\quad + \int_t^{+\infty} \int_s^{+\infty} \frac{|h(u, x(q(u))) - h(u, y(q(u)))|}{r_2(s)r_1(u)} duds \\
&\quad + \int_t^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{|l(v, x(\eta(v))) - l(v, y(\eta(v)))|}{r_2(s)r_1(u)} dvduds \\
&\le d(t)|x(t-\sigma) - y(t-\sigma)| + \int_t^{+\infty} \frac{\lambda(s)|x(p(s)) - y(p(s))|}{r_2(s)} ds \\
&\quad + \int_t^{+\infty} \int_s^{+\infty} \frac{\tau(u)|x(q(u)) - y(q(u))|}{r_2(s)r_1(u)} duds \\
&\quad + \int_t^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{\zeta(v)|x(\eta(v)) - y(\eta(v))|}{r_2(s)r_1(u)} dvduds \\
&\le \bigg(d + \int_T^{+\infty} \frac{\lambda(s)}{r_2(s)} ds + \int_T^{+\infty} \int_s^{+\infty} \frac{\tau(u)}{r_2(s)r_1(u)} duds \\
&\qquad + \int_T^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{\zeta(v)}{r_2(s)r_1(u)} dvduds \bigg) \|x - y\| \\
&< \frac{1+d}{2} \|x - y\|, \qquad t \ge T,
\end{aligned}$$

which infers that $\|Q_Lx - Q_Ly\| < \frac{1+d}{2}\|x - y\|$ for any $x, y \in \Omega(a, b)$. Clearly, Q_L is a contraction mapping by $d \in (0, 1)$. Consequently, Q_L has a unique fixed point $x \in \Omega(a, b)$, which is a bounded positive solution of Eq.(1.7). The rest of the proof is similar to that of Theorem 2.1 and is omitted. This completes the proof. □

3. Algorithm and convergence

In this section, a perturbed Mann iteration method with error is constructed for approximating the solution of the third order nonlinear neutral delay differential equation (1.7), and the convergence and stability of the iterative sequence generated by the algorithm are discussed.

Lemma 3.1. *([7]) Let $\{a_n\}_{n\ge0}, \{b_n\}_{n\ge0}, \{c_n\}_{n\ge0}$ be nonnegative sequences satisfying*

$$a_{n+1} \le (1 - \lambda_n)a_n + \lambda_n b_n + c_n, \quad \forall n \ge 0,$$

where

$$\{\lambda_n\}_{n=0}^{\infty} \subset [0, 1], \sum_{n=0}^{\infty} \lambda_n = +\infty, \sum_{n=0}^{\infty} c_n < +\infty, \lim_{n\to\infty} b_n = 0.$$

Then $\lim_{n\to\infty} a_n = 0$.

Definition 3.2. *([8])??Let $n \ge 0$, T be a self-mapping of $H, x_0 \in H$, $x_{n+1} = f(T, x_n)$ be an iteration procedure which yields a sequence of points $\{x_n\}_{n\ge0} \subset H$, where f is a continuous mapping. Suppose that $\{x \in H : Tx = x\} \ne \varnothing$ and $\{x_n\}_{n\ge0}$ converges to a fixed point x^* of T. Let $\{u_n\}_{n\ge0} \subset H, E_n = \|u_{n+1} - f(T, u_n)\|$. If*

$\lim_{n\to\infty} E_n = 0$ *implies that* $\lim_{n\to\infty} u_n = x^*$, *then the iteration procedure defined by* $x_{n+1} = f(T, x_n)$ *is said to be* $T-$*stable or stable with respect to* T.

Algorithm 3.3. *Let* $\sigma, r_1, r_2, f, g, h, l, p, q, \eta$ *be same as those in Theorem 2.4. For any* $x_0(t) \in C(I, \mathbb{R})$, *define an iterative sequence* $\{x_n(t)\}_{n\geq 0}$ *on* I *by*

$$x_{n+1}(t) =(1 - a_n)x_n(t) + a_n(Q_L x_n)(t) + a_n e_n(t), \ \forall n \geq 0, \tag{3.1}$$

where Q_L *is the same as in (2.16),* $\{e_n(t)\}_{n\geq 0} \subset C(I, \mathbb{R})$ *is a sequence introduced to take into account possible inexact computation which satisfies*

$$\lim_{n\to\infty} \|e_n\| = 0,$$

and the sequence $\{a_n\}_{n\geq 0}$ *satisfies the following condition*

$$0 < a \leq a_n \leq 1, \ \forall n \geq 0,$$

where a *is a constant. Let* $\{z_n(t)\}_{n\geq 0} \subset C(I, \mathbb{R})$ *be any sequence and define* ε_n *for* $n \geq 0$ *by*

$$\varepsilon_n = \big\|z_{n+1} - \big[(1 - a_n)z_n + a_n(Q_L z_n) + a_n e_n\big]\big\|. \tag{3.2}$$

Theorem 3.4. *Let all conditions of Theorem 2.4 hold. Then*

(1) the iterative sequence $\{x_n(t)\}_{n\geq 0}$ *generated by Algorithm 3.3 converges to a solution* $x(t)$ *relative to* L *of Eq.*(1.7),

(2) for any sequence $\{z_n(t)\}_{n\geq 0} \subset C(I, \mathbb{R})$, $\lim_{n\to\infty} z_n(t) = x(t)$ *if and only if* $\lim_{n\to\infty} \varepsilon_n = 0$, *where* ε_n *is defined by Algorithm 3.3.*

Proof. First to prove (1). It follows from Theorem 2.4 that Eq.(1.7) has a solution $x(t) \subset C(I, \mathbb{R})$ relative to L. Consequently,

$$x(t) =(1 - a_n)x(t) + a_n(Q_L x)(t). \tag{3.3}$$

By (2.22) and for $t \in I$,

$$\begin{aligned}
&\|x_{n+1} - x\| \\
&\leq (1 - a_n)\|x_n - x\| + a_n\Big\{d(t)\|x_n - x\| \\
&\quad + \int_T^{+\infty} \frac{\lambda(s)\|x_n - x\|}{r_2(s)} ds + \int_T^{+\infty}\int_s^{+\infty} \frac{\tau(u)\|x_n - x\|}{r_2(s)r_1(u)} duds \\
&\quad + \int_T^{+\infty}\int_s^{+\infty}\int_u^{+\infty} \frac{\zeta(v)\|x_n - x\|}{r_2(s)r_1(u)} dvduds\Big\} + a_n\|e_n\| \\
&\leq \Big\{1 - a_n\Big[1 - \Big(d + \int_T^{+\infty} \frac{\lambda(s)}{r_2(s)} ds + \int_T^{+\infty}\int_s^{+\infty} \frac{\tau(u)}{r_2(s)r_1(u)} duds \\
&\quad + \int_T^{+\infty}\int_s^{+\infty}\int_u^{+\infty} \frac{\zeta(v)}{r_2(s)r_1(u)} dvduds\Big)\Big]\Big\}\|x_n - x\| + a_n\|e_n\| \\
&\leq (1 - \frac{1-d}{2} a_n)\|x_n - x\| + \frac{1-d}{2} a_n \frac{\|e_n\|}{\frac{1-d}{2}}, \quad \forall n \geq 0,
\end{aligned} \tag{3.4}$$

which means that $x_n(t) \to x(t)$ as $n \to \infty$ by Algorithm 3.3 and Lemma 3.1.

Now to prove (2). Using (3.2) and (3.3), similar to the proof of (3.4), we deduce that

$$\begin{aligned}\|z_{n+1}-x\| \le & \|z_{n+1}-[(1-a_n)z_n+a_n(Q_Lz_n)+a_ne_n]\| \\ & + \|(1-a_n)z_n+a_n(Q_Lz_n)+a_ne_n-x\| \\ \le & (1-\frac{1-d}{2}a_n)\|z_n-x\|+\frac{1-d}{2}a_n\frac{\|e_n\|+\frac{\epsilon_n}{a}}{\frac{1-d}{2}}, \quad \forall n\ge 0.\end{aligned} \tag{3.5}$$

Suppose that $\lim_{n\to\infty}\varepsilon_n=0$. By (3.5), Algorithm 3.3 and Lemma 3.1, we get that $z_n(t)\to x(t)$ as $n\to\infty$. Conversely, suppose that $z_n(t)\to x(t)$ as $n\to\infty$. In view of (3.2), we infer that

$$\begin{aligned}\varepsilon_n &\le \|z_{n+1}-x\|+\|(1-a_n)z_n+a_n(Q_Lz_n)+a_ne_n-x\| \\ &\le \|z_{n+1}-x\|+(1-\frac{1-d}{2}a_n)\|z_n-x\|+a_n\|e_n\| \\ &\le \|z_{n+1}-x\|+\|z_n-x\|+\|e_n\|, \quad \forall n\ge 0.\end{aligned}$$

Hence, $\lim_{n\to\infty}\varepsilon_n=0$. This completes the proof. □

4. Examples

In this section, two examples are given to illustrate how to apply the above results.

Example 4.1. *Consider the following third order nonlinear neutral delay differential equation:*

$$\begin{aligned}&\frac{d}{dt}\Big\{t^3\frac{d}{dt}\Big[t^2\frac{d}{dt}\Big(x(t)-\frac{\sin^2 t}{x(t-\sigma)+1}\Big)\Big]\Big\} \\ &\quad+\frac{d}{dt}\Big[t^3\frac{d}{dt}\frac{x(e^t)}{t}\Big]+\frac{d}{dt}[tx(t^2)]=\frac{1}{t^2+x(\sqrt{t})}, \quad t\ge t_0=1,\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}&\sigma>0,\ r_1(t)=t^3,\ r_2(t)=t^2,\ f(t,u)=\frac{\sin^2 t}{u+1},\ g(t,u)=\frac{u}{t}, \\ &h(t,u)=tu,\ l(t,u)=\frac{1}{t^2+u},\ p(t)=e^t,\ q(t)=t^2,\ \eta(t)=\sqrt{t}.\end{aligned} \tag{4.2}$$

Choose $a(t)=2+\sin t, b(t)=6+\cos t$. *Then,* $\underline{a}=1,\overline{a}=3,\underline{b}=5,\overline{b}=7$. *We can take*

$$c(t)=\frac{\sin^2 t}{2},\ d(t)=\frac{\sin^2 t}{4},\ \alpha(t)=\frac{7}{t},\ \beta(t)=7t,\ \gamma(t)=\frac{1}{t^2+1}. \tag{4.3}$$

It is easy to verify that the conditions of Theorem 2.1 are satisfied. Therefore Theorem 2.1 ensures that (4.1) has uncountably many bounded positive solutions in $\Omega(2+\sin t,6+\cos t)$.

Example 4.2. *Consider the following third order nonlinear neutral delay differential equation:*

$$\begin{aligned}&\frac{d}{dt}\Big\{t^4\frac{d}{dt}\Big[t^3\frac{d}{dt}\Big(x(t)-x(t-\sigma)\Big)\Big]\Big\}+\frac{d}{dt}\Big[t^4\frac{d}{dt}\frac{\sqrt{x(2^t)}}{t}\Big] \\ &\quad+\frac{d}{dt}[t^2\cos x(2+\ln t)]=\frac{\sin x(1+\arctan t)}{t^3}, \quad t\ge t_0=2,\end{aligned} \tag{4.4}$$

where

$$\begin{aligned} &\sigma > 0,\ r_1(t) = t^4,\ r_2(t) = t^3,\ g(t,u) = \frac{\sqrt{u}}{t},\ h(t,u) = t^2 \cos u, \\ &l(t,u) = \frac{\sin u}{t^3},\ p(t) = 2^t,\ q(t) = 2 + \ln t,\ \eta(t) = 1 + \arctan t. \end{aligned} \tag{4.5}$$

Choose $a(t) \equiv 1, b(t) \equiv 4$. *We can take*

$$\alpha(t) = \frac{2}{t},\ \beta(t) = t^2,\ \gamma(t) = \frac{1}{t^3}. \tag{4.6}$$

It can be verified that the assumptions of Theorem 2.2 are fulfilled. It follows from Theorem 2.2 that (4.3) has uncountably many bounded positive solutions in $\Omega(1,4)$.

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References

[1] R.P. Agarwal, D. O'Regan and S.H. Saker, *Oscillation criteria for second-order nonlinear neutral delay dynamic equations*, J. Math. Anal. Appl. **300** (2004), 203–217.

[2] L.H. Erbe, W.K. Kong and B.G. Zhang, *Oscillatory theory for functional differential equations*, Marcel Dekker, New York (1995).

[3] H. El-Metwally, M.R.S. Kulenovic and S. Hadziomerspahic *Nonoscillatory solutions for system of neutral delay equation*, Nonlinear Anal. **54** (2003), 63–81.

[4] L. Hanustiakova and R. Olach *Nonoscillatory bounded solutions of neutral differential systems*, Nonlinear Anal. **68**(2008), 1816–1824.

[5] Muhammand N. IsIam and Youssef N. Raffoul *Periodic solutions of neutral nonlinear system of differential equations with functional delay* , J. Math. Anal. Appl. **331** (2007), 1175–1186.

[6] B.M. Levitan *Some problems of the theory of almost periodic functions I* , Uspekhi Mat. Nauk. **2(5)** (1947), 133–192.

[7] L.S. Liu *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive mappings in Banach spaces* , Proc. Amer. Math. Soc. **194** (1995), 114-125.

[8] Z. Liu, R.P. Agarwal and S.M. Kang *On perturbed three-step iterative algorithm for completely generalized nonlinear mixed quasivariational inclusions* , Dyn. Contin. Discr. Impuls. Syst. Ser. A, Math. Anal. **15(2)**(2008), 229–241.

[9] Z. Liu, H.Y. Gao, S.M. Kang and S.H. Shim *Existence and Mann iterative approximations of nonoscillatory solutions of nth-order neutral delay differential equations* , J. Math. Anal. Appl. **329** (2007), 515–529.

[10] X. Y. Lin *Oscillatory of second-order nonlinear neutral differential equations* , J. Math. Anal. Appl. **309**(2005), 442–452.

[11] N. Parhi and R.N. Rath *Oscillation critiria for forced first order neutral differential equations with variable coefficients* , J. Math. Anal. Appl. **256** (2001), 525–541.

[12] B.N. Sadovskii *A fixed point principle* Funct. Anal. Appl. **1** (1967), 151–153.

[13] Y. Yu and H. Wang *Nonoscillatory solutions of second-order nonlinear neutral delay equations* , J. Math. Anal. Appl. **311**(2005), 445–456.

[14] Y. Zhou *Existence for nonoscillatory solutions of second-order nonlinear differential equations* , J. Math. Anal. Appl. **331**(2007), 91–96.

[15] W.P. Zhang, W. Feng, J.R. Yan and J.S. Song *Existence of nonoscillatory solutions of first-order linear neutral delay differential equations* , Compu. Math. Appl. **49** (2005), 1021–1027.

[16] Y. Zhou and B.G. Zhang *Existence of nonoscillatory solutions of higher-order neutral differential equations with positive and negative coefficients* , Appl. Math. Lett. **15**(2002), 867–874.

Min Liu, Zhenyu Guo
School of Sciences, Liaoning Shihua University, Postal 113001, Fushun, Liaoning, China
E-mail address: min_liu@yeah.net
E-mail address: guozy@163.com