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THE GROWTH OF ITERATED ENTIRE FUNCTIONS

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Abstract. In this paper we study growth of iterated entire functions to improve some earlier results.

1. Introduction, Definitions and Notations

For any two transcendental entire functions $f(z)$ and $g(z)$ defined in the open complex plane C, it is well known [1] that $\lim_{r\to\infty} \frac{\log T(r,f_o g)}{T(r,f)} = \infty$ and $\lim_{r\to\infty} \frac{\log T(r,f_o g)}{T(r,g)} = 0$. Later on Singh [10] investigated some comparative growth of $logT(r, f_o g)$ and $T(r, f)$. Further in [10] he raised the problem of investing the comparative growth of $logT(r, f_o g)$ and $T(r, g)$. However some results on the comparative growth of $logT(r, f_o g)$ and $T(r, g)$ are proved in [5].

Recently Lahiri and Datta [6] made close investigation on comparative growth properties of $logT(r, f_o g)$ and $T(r, g)$ together with that of $\log\log T(r, f_o g)$ and $T(r, f^{(k)})$.

In this paper, we first consider two entire functions $f(z)$ and $g(z)$ and following Lahiri and Banerjee [3] form the relative iterations of $f(z)$ with respect to $g(z)$ (defined below) and using this concept of relative iteration study growth of iterated entire functions to generalise some results of Lahiri and Datta [6].

Let

$$\begin{aligned}
f_1(z) &= f(z) \\
f_2(z) &= f(g(z)) = f(g_1(z)) \\
f_3(z) &= f(g(f(z))) = f(g_2(z)) = f(g(f_1(z))) \\
&\dots \qquad \dots \qquad \dots \\
&\dots \qquad \dots \qquad \dots \\
f_n(z) &= f(g(f........(f(z) \text{ or } g(z))........)), \\
&\quad \text{according as } n \text{ is odd or even,} \\
&= f(g_{n-1}(z)) = f(g(f_{n-2}(z))),
\end{aligned}$$

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and so are

$$\begin{aligned} g_1(z) &= g(z) \\ g_2(z) &= g(f(z)) = g(f_1(z)) \\ &\dots\dots \\ &\dots\dots \\ g_n(z) &= g(f_{n-1}(z)) = g(f(g_{n-2}(z))). \end{aligned}$$

Clearly all $f_n(z)$ and $g_n(z)$ are entire functions.

For two non-constant entire functions $f(z)$ and $g(z)$, we have the well known inequality

$$\log M(r, f(g)) \le \log M(M(r, g), f). \tag{1.1}$$

Definition 1.1. *The order ρ_f and lower order λ_f of a meromorphic function is defined as*

$$\rho_f = \lim\sup_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

and

$$\lambda_f = \lim\inf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

If f is entire then

$$\rho_f = \lim\sup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}$$

and

$$\lambda_f = \lim\inf_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

Definition 1.2. *The hyper order $\overline{\rho_f}$ and hyper lower order $\overline{\lambda}_f$ of a meromorphic function is defined as*

$$\overline{\rho_f} = \lim\sup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}$$

and

$$\overline{\lambda}_f = \lim\inf_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

If f is entire then

$$\overline{\rho_f} = \lim\sup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}$$

and

$$\overline{\lambda}_f = \lim\inf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}.$$

Definition 1.3. *A function $\lambda_f(r)$ is called a lower proximate order of a meromorphic function f if*

(i) $\lambda_f(r)$ is nonnegative and continuous for $r \ge r_0$, say;

(ii) $\lambda_f(r)$ is differentiable for $r \ge r_0$ except possibly at isolated points at which $\lambda_f'(r-0)$ and $\lambda_f'(r+0)$ exist;

(iii) $\lim_{r\to\infty} \lambda_f(r) = \lambda_f < \infty$;

(iv) $\lim_{r\to\infty} r\lambda_f'(r)\log r = 0$; *and*

(v) $\liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f(r)}} = 1$.

Notation 1.4. [9] *Let* $log^{[0]}x = x$*,* $exp^{[0]}x = x$ *and for positive integer* m*,* $log^{[m]}x = log(log^{[m-1]}x)$*,* $exp^{[m]}x = exp(exp^{[m-1]}x)$*.*

Throughout we assume f, g etc. are non constant entire functions having respective orders ρ_f, ρ_g and respective lower orders λ_f, λ_g. Also we do not explain the standard notations and definitions of the theory of entire and meromorphic functions because those are available in [2].

2. Lemmas

The following lemmas will be needed in the sequel.

Lemma 2.1. [2] *Let* $f(z)$ *be an entire function. For* $0 \leq r < R < \infty$*, we have*

$$T(r, f) \leq \log^{+} M(r, f) \leq \frac{R+r}{R-r} T(R, f).$$

Lemma 2.2. [8] *Let* $f(z)$ *and* $g(z)$ *be two entire functions. Then we have*

$$T(r, f(g)) \geq \frac{1}{3} \log M \left(\frac{1}{8} M \left(\frac{r}{4}, g \right) + O(1), f \right).$$

Lemma 2.3. [4] *Let* f *be an entire function. Then for* $k > 2$*,*

$$\liminf_{r\to\infty} \frac{\log^{[k-1]} M(r, f)}{\log^{[k-2]} T(r, f)} = 1.$$

Lemma 2.4. [6] *Let* f *be a meromorphic function. Then for* $\delta(> 0)$ *the function* $r^{\lambda_f\ +\delta-\lambda_f(r)}$ *is an increasing function of* r*.*

Lemma 2.5. [7] *Let* f *be an entire function of finite lower order. If there exist entire functions* a_i $(i = 1, 2, 3..........n;\ n \leq \infty)$ *satisfying* $T(r, a_i) = o\{T(r, f)\}$ *and*

$$\sum_{i=1}^{n} \delta(a_i, f) = 1 \quad \text{then} \quad \lim_{r\to\infty} \frac{T(r, f)}{\log M(r, f)} = \frac{1}{\pi}.$$

Lemma 2.6. *Let* $f(z)$ *and* $g(z)$ *be two non-constant entire functions such that* $0 < \lambda_f \leq \rho_f < \infty$ *and* $0 < \lambda_g \leq \rho_g < \infty$*. Then for any* ε $(0 < \varepsilon < min\{\lambda_f, \lambda_g\})$

$$\log^{[n-1]} T(r, f_n) \leq \begin{cases} (\rho_f + \varepsilon) \log M(r, g) + O(1) & \text{when } n \text{ is even} \\ (\rho_g + \varepsilon) \log M(r, f) + O(1) & \text{when } n \text{ is odd} \end{cases}$$

and

$$\log^{[n-1]} T(r, f_n) \geq \begin{cases} (\lambda_f - \varepsilon) \log M \left(\frac{r}{4^{n-1}}, g \right) + O(1) & \text{when } n \text{ is even} \\ (\lambda_g - \varepsilon) \log M \left(\frac{r}{4^{n-1}}, f \right) + O(1) & \text{when } n \text{ is odd.} \end{cases}$$

Proof. For $\varepsilon(>0)$ we get from Lemma 2.1 and (1.1) for all large values of r

$$\begin{aligned} T(r,f_n) &\le \log M(r,f_n)\\ &\le \log M(M(r,g_{n-1}),f)\\ &\le [M(r,g_{n-1})]^{\rho_f+\varepsilon},\\ \text{that is, }\log\ T(r,f_n) &\le (\rho_f+\varepsilon)\log M(r,g_{n-1})\\ &\le (\rho_f+\varepsilon)\log M(M(r,f_{n-2}),g)\\ &\le (\rho_f+\varepsilon)[M(r,f_{n-2})]^{\rho_g+\varepsilon}.\\ \text{So, }\log^{[2]}\ T(r,f_n) &\le (\rho_g+\varepsilon)\log M(M(r,g_{n-3}),f)+O(1)\\ &\le (\rho_g+\varepsilon)[M(r,g_{n-3})]^{\rho_f+\varepsilon}+O(1).\\ \text{Therefore, }\log^{[n-1]}\ T(r,f_n) &\le (\rho_f+\varepsilon)\log M(r,g)+O(1)\quad\text{when } n \text{ is even.}\end{aligned}$$

Similarly

$$\log^{[n-1]}\ T(r,f_n)\le(\rho_g+\varepsilon)\log M(r,f)+O(1)\quad\text{when } n \text{ is odd.}$$

Again for ε $(0<\varepsilon<\min\{\lambda_f,\lambda_g\})$ we get from Lemma 2.1 and Lemma 2.2 for all large values of r

$$\begin{aligned} T(r,f_n) &= T(r,f(g_{n-1}))\\ &\ge \frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4},g_{n-1}\right)+O(1),f\right)\\ &\ge \frac{1}{3}\left[\frac{1}{8}M\left(\frac{r}{4},g_{n-1}\right)+O(1)\right]^{\lambda_f-\varepsilon}\\ &\ge \frac{1}{3}\left[\frac{1}{9}M\left(\frac{r}{4},g_{n-1}\right)\right]^{\lambda_f-\varepsilon},\\ \text{that is, }\log T(r,f_n) &\ge (\lambda_f-\varepsilon)\log M\left(\frac{r}{4},g_{n-1}\right)+O(1)\\ &\ge (\lambda_f-\varepsilon)T\left(\frac{r}{4},g_{n-1}\right)+O(1)\\ &\ge (\lambda_f-\varepsilon)\frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4^2},f_{n-2}\right)+O(1),g\right)+O(1)\\ &\ge (\lambda_f-\varepsilon)\frac{1}{3}\left[\frac{1}{8}M\left(\frac{r}{4^2},f_{n-2}\right)+O(1)\right]^{\lambda_g-\varepsilon}+O(1)\\ &\ge (\lambda_f-\varepsilon)\frac{1}{3}\left[\frac{1}{9}M\left(\frac{r}{4^2},f_{n-2}\right)\right]^{\lambda_g-\varepsilon}+O(1),\\ \text{that is, }\log^{[2]}T(r,f_n) &\ge (\lambda_g-\varepsilon)\log M\left(\frac{r}{4^2},f_{n-2}\right)+O(1).\\ \text{Therefore, }\log^{[n-2]}\ T(r,f_n) &\ge (\lambda_g-\varepsilon)\log M\left(\frac{r}{4^{n-2}},f_og\right)+O(1). \quad (2.1)\\ \text{So, }\log^{[n-1]}\ T(r,f_n) &\ge (\lambda_f-\varepsilon)\log M\left(\frac{r}{4^{n-1}},g\right)+O(1)\quad\text{when } n \text{ is even.}\end{aligned}$$

Similarly

$$\log^{[n-1]}\ T(r,f_n)\ge(\lambda_g-\varepsilon)\log M\left(\frac{r}{4^{n-1}},f\right)+O(1)\quad\text{when } n \text{ is odd.}$$

This proves the lemma. □

3. Theorems

Theorem 3.1. *Let f and g be two non-constant entire functions having finite lower orders. Then*

$$(i)\quad \liminf_{r\to\infty} \frac{\log^{[n-1]} T(r,f_n)}{T(r,g)} \leq 3\rho_f 2^{\lambda_g},$$

$$(ii)\quad \limsup_{r\to\infty} \frac{\log^{[n-1]} T(r,f_n)}{T(r,g)} \geq \frac{\lambda_f}{(4^{n-1})^{\lambda_g}}$$

when n is even
and

$$(iii)\quad \liminf_{r\to\infty} \frac{\log^{[n-1]} T(r,f_n)}{T(r,\ f)} \leq 3\rho_g 2^{\lambda_f},$$

$$(iv)\quad \limsup_{r\to\infty} \frac{\log^{[n-1]} T(r,f_n)}{T(r,f)} \geq \frac{\lambda_g}{(4^{n-1})^{\lambda_f}}$$

when n is odd.

Proof. We may clearly assume $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \lambda_g \leq \rho_g < \infty$. Now from Lemma 2.6 for arbitrary $\varepsilon > 0$

$$\log^{[n-1]}\ T(r,f_n) \leq (\rho_f + \varepsilon)\log M(r,g) + O(1) \tag{3.1}$$

when n is even.

Let $0 < \varepsilon <\min\{1, \lambda_f, \lambda_g\}$. Since

$$\liminf_{r\to\infty} \frac{T(r,g)}{r^{\lambda_g(r)}} = 1,$$

there is a sequence of values of r tending to infinity for which

$$T(r,g) < (1+\varepsilon) r^{\lambda_g(r)} \tag{3.2}$$

and for all large value of r

$$T(r,g) > (1-\varepsilon) r^{\lambda_g(r)}. \tag{3.3}$$

Thus for a sequence of values of r tending to infinity we get for any $\delta(>0)$

$$\begin{aligned}\frac{\log M(r,g)}{T(r,g)} &\leq \frac{3T(2r,g)}{T(r,g)} \leq \frac{3(1+\varepsilon)}{1-\varepsilon}\frac{(2r)^{\lambda_g+\delta}}{(2r)^{\lambda_g+\delta-\lambda_g(2r)}}\frac{1}{r^{\lambda_g(r)}}\\ &\leq \frac{3(1+\varepsilon)}{1-\varepsilon}2^{\lambda_g+\delta}\end{aligned}$$

because $r^{\lambda_g+\delta-\lambda_g(r)}$ is an increasing function of r.
Since ε, $\delta > 0$ be arbitrary, we have

$$\liminf_{r\to\infty} \frac{\log M(r,g)}{T(r,g)} \leq 3.2^{\lambda_g}. \tag{3.4}$$

Therefore from (3.1) and (3.4) we get

$$\liminf_{r\to\infty} \frac{\log^{[n-1]} T(r,f_n)}{T(r,g)} \leq 3\rho_f 2^{\lambda_g}.$$

when n is even.
Again for even n we have from Lemma 2.6

$$\begin{aligned}\log^{[n-1]} T(r,f_n) &\geq (\lambda_f-\varepsilon)\log M\left(\frac{r}{4^{n-1}},g\right)+O(1)\\ &\geq (\lambda_f-\varepsilon)T\left(\frac{r}{4^{n-1}},g\right)+O(1)\\ &\geq (\lambda_f-\varepsilon)(1-\varepsilon)(1+O(1))\frac{\left(\frac{r}{4^{n-1}}\right)^{\lambda_g+\delta}}{\left(\frac{r}{4^{n-1}}\right)^{\lambda_g+\delta-\lambda_g\left(\frac{r}{4^{n-1}}\right)}},\ \text{by (3.3)}.\end{aligned}$$

Since $r^{\lambda_g+\delta-\lambda_g(r)}$ is an increasing function of r, we have

$$\log^{[n-1]} T(r,f_n)\geq(\lambda_f-\varepsilon)(1-\varepsilon)(1+O(1))\frac{r^{\lambda_g(r)}}{(4^{n-1})^{\lambda_g+\delta}}$$

for all large values of r.

So by (3.2) for a sequence of values of r tending to infinity

$$\log^{[n-1]} T(r,f_n)\geq(\lambda_f-\varepsilon)\frac{1-\varepsilon}{1+\varepsilon}(1+O(1))\frac{T(r,g)}{(4^{n-1})^{\lambda_g+\delta}}.$$

Since ε and δ are arbitrary, it follows from the above that

$$\limsup_{r\to\infty}\frac{\log^{[n-1]} T(r,f_n)}{T(r,g)}\geq\frac{\lambda_f}{(4^{n-1})^{\lambda_g}}.$$

Similarly for odd n we get the second part of the theorem.
This proves the theorem. □

Theorem 3.2. *Let f and g be two non-constant entire functions such that λ_f and $\lambda_g(>0)$ are finite . Also there exist entire functions a_i $(i=1,2,3..........n;\ n\leq\infty)$ satisfying $T(r,a_i)=o\{T(r,g)\}$ as $r\to\infty$ and*

$$\sum_{i=1}^{n}\delta(a_i,g)=1.$$

Then

$$\frac{\pi\lambda_f}{(4^{n-1})^{\lambda_g}}\leq\limsup_{r\to\infty}\frac{\log^{[n-1]} T(r,f_n)}{T(r,g)}\leq\pi\rho_f$$

when n is even.

Proof. If $\lambda_f=0$ then the first inequality is obvious. Now we suppose that $\lambda_f>0$. For $0<\varepsilon<\min\{1,\lambda_f,\lambda_g\}$ we have from Lemma 2.6. for all large values of r

$$\begin{aligned}\frac{\log^{[n-1]}\ T(r,f_n)}{T(r,g)} &\geq (\lambda_f-\varepsilon)\frac{\log M\left(\frac{r}{4^{n-1}},g\right)}{T(r,g)}+O(1)\quad\text{when } n \text{ is even}\\ &\geq (\lambda_f-\varepsilon)\frac{\log M\left(\frac{r}{4^{n-1}},g\right)}{T\left(\frac{r}{4^{n-1}},g\right)}\frac{T\left(\frac{r}{4^{n-1}},g\right)}{T(r,g)}+O(1). \qquad (3.5)\end{aligned}$$

Also from (3.2) and (3.3) we get for a sequence of values of $r\to\infty$ and for $\delta>0$

$$\begin{aligned}\frac{T\left(\frac{r}{4^{n-1}},g\right)}{T(r,g)} &> \frac{1-\varepsilon}{1+\varepsilon}\frac{\left(\frac{r}{4^{n-1}}\right)^{\lambda_g+\delta}}{\left(\frac{r}{4^{n-1}}\right)^{\lambda_g+\delta-\lambda_g\left(\frac{r}{4^{n-1}}\right)}}\frac{1}{r^{\lambda_g(r)}}\\ &\geq \frac{1-\varepsilon}{1+\varepsilon}\frac{1}{(4^{n-1})^{\lambda_g+\delta}}\end{aligned}$$

because $r^{\lambda_g+\delta-\lambda_g(r)}$ is an increasing function of r.
Since $\varepsilon,\ \delta>0$ be arbitrary, so using Lemma 2.5, we have from (3.5)

$$\limsup_{r\to\infty}\frac{\log^{[n-1]}T(r,f_n)}{T(r,g)}\geq\frac{\pi\lambda_f}{(4^{n-1})^{\lambda_g}}.$$

If $\rho_f=\infty$, the second inequality is obvious. So we may assume $\rho_f<\infty$. Then the second inequality follows from Lemma 2.5 and Lemma 2.6.
This proves the theorem. □

Theorem 3.3. *Let f and g be two non-constant entire functions such that $\lambda_f(>0)$ and λ_g are finite. Also there exist entire functions a_i $(i=1,2,3..........n;\ n\leq\infty)$ satisfying $T(r,a_i)=o\{T(r,f)\}$ as $r\to\infty$ and*

$$\sum_{i=1}^{n}\delta(a_i,f)=1.$$

Then

$$\frac{\pi\lambda_g}{(4^{n-1})^{\lambda_f}}\leq\limsup_{r\to\infty}\frac{\log^{[n-1]}T(r,f_n)}{T(r,f)}\leq\pi\rho_g$$

when n is odd.

Theorem 3.4. *Let f and g be two non-constant entire functions such that $0<\lambda_f\leq\rho_f<\infty$ and $0<\lambda_g\leq\rho_g<\infty$. Then for $k=0,1,2,3,.......$*

$$\frac{\overline{\lambda}_g}{\rho_g}\leq\liminf_{r\to\infty}\frac{\log^{[n+1]}T(r,f_n)}{\log\ T(r,g^{(k)})}\leq\limsup_{r\to\infty}\frac{\log^{[n+1]}T(r,f_n)}{\log\ T(r,g^{(k)})}\leq\frac{\overline{\rho_g}}{\lambda_g}$$

when n is even and

$$\frac{\overline{\lambda}_f}{\rho_f}\leq\liminf_{r\to\infty}\frac{\log^{[n+1]}T(r,f_n)}{\log\ T(r,f^{(k)})}\leq\limsup_{r\to\infty}\frac{\log^{[n+1]}T(r,f_n)}{\log\ T(r,f^{(k)})}\leq\frac{\overline{\rho_f}}{\lambda_f}$$

when n is odd, where $f^{(k)}$ denote the k-th derivative of f.

Proof. First suppose that n is even. Then for given $\varepsilon(0<\varepsilon<min\{\lambda_f,\lambda_g\})$ we get from Lemma 2.6 for all large values of r

$$\begin{aligned}\log^{[n-1]}\ T(r,f_n)&\geq(\lambda_f-\varepsilon)\log M\left(\frac{r}{4^{n-1}},g\right)+O(1)\\&\geq(\lambda_f-\varepsilon)T\left(\frac{r}{4^{n-1}},g\right)+O(1)\\\text{that is,}\quad\log^{[n]}\ T(r,f_n)&\geq\log T\left(\frac{r}{4^{n-1}},g\right)+O(1).\\\text{So,}\quad\log^{[n+1]}\ T(r,f_n)&\geq\log^{[2]}T\left(\frac{r}{4^{n-1}},g\right)+O(1).\end{aligned}$$

So for all large values of r

$$\frac{\log^{[n+1]}T(r,f_n)}{\log\ T(r,g^{(k)})}\geq\frac{\log^{[2]}T\left(\frac{r}{4^{n-1}},g\right)}{\log\ \frac{r}{4^{n-1}}}.\frac{\log\frac{r}{4^{n-1}}}{\log\ T(r,g^{(k)})}+o(1).\tag{3.6}$$

Since

$$\limsup_{r\to\infty}\frac{\log T(r,g^{(k)})}{\log r}=\rho_g,$$

so for all large values of r and arbitrary $\varepsilon > 0$ we have

$$\log T(r, g^{(k)}) < (\rho_g + \varepsilon) \log r. \tag{3.7}$$

Since $\varepsilon > 0$ is arbitrary, so from (3.6) and (3.7) we have

$$\begin{aligned} \liminf_{r\to\infty} \frac{\log^{[n+1]} T(r, f_n)}{\log\ T(r, g^{(k)})} &\geq \liminf_{r\to\infty} \frac{\log^{[2]} T\left(\frac{r}{4^{n-1}}, g\right)}{\log\ \frac{r}{4^{n-1}}} . \left(\frac{\log r - \log 4^{n-1}}{\rho_g \log\ r}\right) \\ &\geq \frac{\overline{\lambda}_g}{\rho_g}. \end{aligned} \tag{3.8}$$

Again from Lemma 2.6 for all large values of r

$$\begin{aligned} \log^{[n-1]}\ T(r, f_n) &\leq (\rho_f + \varepsilon) \log M(r, g) + O(1) \\ \text{i.e. } \frac{\log^{[n+1]} T(r, f_n)}{\log\ T(r, g^{(k)})} &\leq \frac{\log^{[3]} M(r, g)}{\log\ T(r, g^{(k)})} + o(1). \end{aligned} \tag{3.9}$$

Since

$$\liminf_{r\to\infty} \frac{\log T(r, g^{(k)})}{\log r} = \lambda_g,$$

so for all large values of r and arbitrary $\varepsilon (0 < \varepsilon < \lambda_g)$ we have

$$\log T(r, g^{(k)}) > (\lambda_g - \varepsilon) \log r. \tag{3.10}$$

Since $\varepsilon > 0$ is arbitrary, so from (3.9) and (3.10) we have

$$\limsup_{r\to\infty} \frac{\log^{[n+1]} T(r, f_n)}{\log\ T(r, f^{(k)})} \leq \frac{\overline{\rho_g}}{\lambda_g}. \tag{3.11}$$

Combining (3.8) and (3.11) we obtain the first part of the theorem.

Similarly when n is odd then we have the second part of the theorem. This proves the theorem. □

Theorem 3.5. *Let f and g be two non-constant entire functions such that $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \lambda_g \leq \rho_g < \infty$. Then*

$$(i)\quad \frac{\lambda_g}{\rho_g} \leq \liminf_{r\to\infty} \frac{\log^{[n]} T(r, f_n)}{\log\ T(r, g)} \leq 1 \leq \limsup_{r\to\infty} \frac{\log^{[n]} T(r, f_n)}{\log\ T(r, g)} \leq \frac{\rho_g}{\lambda_g}$$

when n is even and

$$(ii)\quad \frac{\lambda_f}{\rho_f} \leq \liminf_{r\to\infty} \frac{\log^{[n]} T(r, f_n)}{\log\ T(r, f)} \leq 1 \leq \limsup_{r\to\infty} \frac{\log^{[n]} T(r, f_n)}{\log\ T(r, f)} \leq \frac{\rho_f}{\lambda_f}$$

when n is odd.

Proof. First suppose that n is even. Then for given $\varepsilon (0 < \varepsilon < min\{\lambda_f, \lambda_g\})$ we get from Lemma 2.6 for all large values of r

$$\begin{aligned} \log^{[n-1]}\ T(r, f_n) &\leq (\rho_f + \varepsilon) \log M(r, g) + O(1) \\ \text{i.e. } \log^{[n]}\ T(r, f_n) &\leq \log^{[2]} M(r, g) + O(1) \\ \text{i.e. } \frac{\log^{[n]} T(r, f_n)}{\log\ T(r, g)} &\leq \frac{\log^{[2]} M(r, g)}{\log\ T(r, g)} + o(1) \end{aligned} \tag{3.12}$$

$$\text{i.e. } \liminf_{r\to\infty} \frac{\log^{[n]} T(r, f_n)}{\log\ T(r, g)} \leq 1 \quad \text{[by Lemma 2.3]}. \tag{3.13}$$

Also,

$$\begin{aligned} \log^{[n-1]}\ T(r,f_n) &\geq (\lambda_f-\varepsilon)\log M\left(\frac{r}{4^{n-1}},g\right)+O(1)\\ \text{i.e.}\quad \log^{[n]}\ T(r,f_n) &\geq \log^{[2]} M\left(\frac{r}{4^{n-1}},g\right)+O(1). \end{aligned}$$

So

$$\begin{aligned} \frac{\log^{[n]} T(r,f_n)}{\log\ T(r,g)} &\geq \frac{\log T\left(\frac{r}{4^{n-1}},g\right)}{\log\ \frac{r}{4^{n-1}}}.\left(\frac{\log r-\log 4^{n-1}}{\rho_g \log\ r}\right)+o(1)\\ \text{i.e.}\quad \liminf_{r\to\infty}\frac{\log^{[n]} T(r,f_n)}{\log\ T(r,g)} &\geq \frac{\lambda_g}{\rho_g}. \end{aligned} \tag{3.14}$$

Also from (3.12), we get for all large values of r,

$$\begin{aligned} \frac{\log^{[n]} T(r,f_n)}{\log\ T(r,g)} &\leq \frac{\log^{[2]} M(r,g)}{\log r}\frac{\log r}{\log T(r,g)}+o(1)\\ \therefore \limsup_{r\to\infty}\frac{\log^{[n]} T(r,f_n)}{\log T(r,g)} &\leq \frac{\rho_g}{\lambda_g}. \end{aligned} \tag{3.15}$$

Again from Lemma 2.6,

$$\begin{aligned} \log^{[n-1]}\ T(r,f_n) &\geq (\lambda_f-\varepsilon)\log M\left(\frac{r}{4^{n-1}},g\right)+O(1)\\ \text{i.e.}\quad \log^{[n]}\ T(r,f_n) &\geq \log^{[2]} M\left(\frac{r}{4^{n-1}},g\right)+O(1). \end{aligned} \tag{3.16}$$

From (3.3) we obtain for all large values of r and for $\delta>0$ and $\varepsilon(0<\varepsilon<1)$

$$\begin{aligned} \log M\left(\frac{r}{4^{n-1}},g\right) &> (1-\varepsilon)\frac{\left(\frac{r}{4^{n-1}}\right)^{\lambda_g+\delta}}{\left(\frac{r}{4^{n-1}}\right)^{\lambda_g+\delta-\lambda_g\left(\frac{r}{4^{n-1}}\right)}}\\ &\geq \frac{1-\varepsilon}{(4^{n-1})^{\lambda_g+\delta}}r^{\lambda_g(r)} \end{aligned}$$

because $r^{\lambda_g+\delta-\lambda_g(r)}$ is an increasing function of r.

So by (3.2) we get for a sequence of value of r tending to infinity

$$\begin{aligned} \log M\left(\frac{r}{4^{n-1}},g\right) &\geq \frac{1-\varepsilon}{1+\varepsilon}\frac{1}{(4^{n-1})^{\lambda_g+\delta}}T(r,g)\\ \text{i.e.}\ \log^{[2]} M\left(\frac{r}{4^{n-1}},g\right) &\geq \log T(r,g)+O(1). \end{aligned} \tag{3.17}$$

Now from (3.16) and (3.17)

$$\limsup_{r\to\infty}\frac{\log^{[n]} T(r,f_n)}{\log\ T(r,f)}\geq 1. \tag{3.18}$$

So the theorem follows from (3.13), (3.14), (3.15) and (3.18) when n is even. Similarly when n is odd we get (ii). □

Corollary 3.6. *Using the hypothesis of Theorem 3.5 if f and g are of regular growth then*

$$\lim_{r\to\infty}\frac{\log^{[n]} T(r,f_n)}{\log\ T(r,g)}=\lim_{r\to\infty}\frac{\log^{[n]} T(r,f_n)}{\log\ T(r,f)}=1.$$

Remark 3.7. *The conditions $\lambda_f,\lambda_g>0$ and $\rho_f,\rho_g<\infty$ are necessary for Theorem 3.5 and Corollary 3.6, which are shown by the following examples.*

Example 3.8. *Let $f = z, g = \exp z$. Then $\lambda_f = \rho_f = 0$ and $0 < \lambda_g = \rho_g < \infty$. Now when n is even then*

$$f_n = \exp^{[\frac{n}{2}]} z.$$

Therefore,

$$T(r, f_n) \leq \log M(r, f_n) = \exp^{[\frac{n}{2}-1]} r.$$

So,

$$\begin{aligned}\log^{[n]} T(r, f_n) &\leq \log^{[n]}(\exp^{[\frac{n}{2}-1]} r) \\ &= \log^{[n-\frac{n}{2}+1]} r \\ &= \log^{[\frac{n}{2}+1]} r.\end{aligned}$$

Also when n is odd

$$f_n = \exp^{[\frac{n-1}{2}]} z.$$

Therefore,

$$T(r, f_n) \leq \log M(r, f_n) = \exp^{[\frac{n-1}{2}-1]} r.$$

So,

$$\begin{aligned}\log^{[n]} T(r, f_n) &\leq \log^{[n]}(\exp^{[\frac{n-1}{2}-1]} r) \\ &= \log^{[n-\frac{n-1}{2}+1]} r \\ &= \log^{[\frac{n+1}{2}+1]} r.\end{aligned}$$

Now

$$\log T(r, f) = \log r \quad \textit{and} \ \log T(r, g) = \log r - \log \pi.$$

Therefore when n is even

$$\frac{\log^{[n]} T(r, f_n)}{\log T(r, g)} \leq \frac{\log^{[\frac{n}{2}+1]} r}{\log r - \log \pi} \to 0 \quad as \quad r \to \infty,$$

and when n is odd

$$\frac{\log^{[n]} T(r, f_n)}{\log T(r, f)} \leq \frac{\log^{[\frac{n+1}{2}+1]} r}{\log r} \to 0 \quad as \quad r \to \infty.$$

Example 3.9. *Let $f = \exp^{[2]} z, g = \exp z$. Then $\lambda_f = \rho_f = \infty, \quad \lambda_g = \rho_g = 1$.*
Now when n is even

$$f_n = \exp^{[\frac{3n}{2}]} z.$$

Therefore

$$\begin{aligned}3T(2r, f_n) &\geq \log M(r, f_n) = \exp^{[\frac{3n}{2}-1]} r \\ i.e. \quad T(r, f_n) &\geq \frac{1}{3} \exp^{[\frac{3n}{2}-1]} \frac{r}{2} \\ \therefore \quad \log^{[n]} T(r, f_n) &\geq \log^{[n]}(\exp^{[\frac{3n}{2}-1]} \frac{r}{2}) + o(1) \\ &= \exp^{[\frac{n}{2}-1]} \frac{r}{2} + o(1).\end{aligned}$$

Also when n is odd

$$f_n = \exp^{[\frac{3n+1}{2}]} z.$$

Therefore

$$\begin{aligned} 3T(2r,f_n) &\geq \log M(r,f_n)=\exp^{[\frac{3n+1}{2}-1]} r \\ i.e. \quad T(r,f_n) &\geq \frac{1}{3}\exp^{[\frac{3n-1}{2}]}\frac{r}{2} \\ \therefore \quad \log^{[n]} T(r,f_n) &\geq \log^{[n]}(\exp^{[\frac{3n-1}{2}]}\frac{r}{2})+o(1) \\ &= \exp^{[\frac{n-1}{2}]}\frac{r}{2}+o(1). \end{aligned}$$

Also

$$T(r,f)\leq e^r \text{ and } T(r,g)=\frac{r}{\pi}.$$

Therefore when n is even

$$\frac{\log^{[n]} T(r,f_n)}{\log T(r,g)}\geq \frac{\exp^{[\frac{n}{2}-1]}\frac{r}{2}+o(1)}{\log r-\log\pi}\to\infty \quad as \quad r\to\infty,$$

and when n is odd

$$\frac{\log^{[n]} T(r,f_n)}{\log T(r,f)}\geq \frac{\exp^{[\frac{n-1}{2}]}\frac{r}{2}+o(1)}{r}\to\infty \quad as \quad r\to\infty.$$

Theorem 3.10. *Let f and g be two entire functions such that $0<\lambda_f\leq\rho_f<\infty$ and $0<\lambda_g\leq\rho_g<\infty$. Then for $k=0,1,2,3,......$*

$$(i) \quad \frac{\lambda_g}{\rho_f}\leq \liminf_{r\to\infty}\frac{\log^{[n]} T(r,f_n)}{\log\ T(r,f^{(k)})}\leq \limsup_{r\to\infty}\frac{\log^{[n]} T(r,f_n)}{\log\ T(r,f^{(k)})}\leq\frac{\rho_g}{\lambda_f}$$

when n is even.

$$(ii) \quad \frac{\lambda_f}{\rho_g}\leq \liminf_{r\to\infty}\frac{\log^{[n]} T(r,f_n)}{\log\ T(r,g^{(k)})}\leq \limsup_{r\to\infty}\frac{\log^{[n]} T(r,f_n)}{\log\ T(r,g^{(k)})}\leq\frac{\rho_f}{\lambda_g}$$

when n is odd.

Proof. First suppose that n is even. Then for given $\varepsilon(0<\varepsilon<min\{\lambda_f,\lambda_g\})$ we have from Lemma 2.6 for all large values of r,

$$\begin{aligned} \log^{[n-1]}\ T(r,f_n) &\leq (\rho_f+\varepsilon)\log M(r,g)+O(1) \\ \text{i.e.} \quad \log^{[n]}\ T(r,f_n) &\leq \log^{[2]} M(r,g)+O(1). \end{aligned}$$

Also we know that

$$\liminf_{r\to\infty}\frac{\log T(r,g^{(k)})}{\log r}=\lambda_g.$$

Now

$$\begin{aligned} \limsup_{r\to\infty}\frac{\log^{[n]} T(r,f_n)}{\log\ T(r,f^{(k)})} &\leq \limsup_{r\to\infty}\frac{\log^{[2]} M(r,g)}{\log\ T(r,f^{(k)})} \\ &\leq \limsup_{r\to\infty}\left[\frac{\log^{[2]} M(r,g)}{\log r}.\frac{\log r}{\log T(r,f^{(k)})}\right] \\ &= \frac{\rho_g}{\lambda_f} \qquad (3.19) \end{aligned}$$

Again from lemma 2.6 we have for all large values of r,

$$\begin{aligned}\log^{[n-1]}\ T(r,f_n) &\geq (\lambda_f-\varepsilon)\log M\left(\frac{r}{4^{n-1}},g\right)+O(1)\\ &\geq (\lambda_f-\varepsilon)\left(\frac{r}{4^{n-1}}\right)^{\lambda_g-\varepsilon}+O(1)\\ \text{i.e.,}\quad \log^{[n]}\ T(r,f_n) &\geq (\lambda_g-\varepsilon)\log\ r+O(1).\end{aligned}$$

Also

$$\log\ T(r,f^{(k)})<(\rho_f+\varepsilon)\log\ r.$$

Therefore,

$$\frac{\log^{[n]}T(r,f_n)}{\log\ T(r,f^{(k)})}\geq\frac{(\lambda_g-\varepsilon)\log\ r+O(1)}{(\rho_f+\varepsilon)\log\ r}.$$

Since $\varepsilon>0$ is arbitrary we get

$$\liminf_{r\to\infty}\frac{\log^{[n]}T(r,f_n)}{\log\ T(r,f^{(k)})}\geq\frac{\lambda_g}{\rho_f}. \tag{3.20}$$

Therefore from (3.19) and (3.20) we have the result for even n.

Similarly for odd n we have (ii).

This proves the Theorem. □

Theorem 3.11. *Let f and g be two entire functions such that $0<\lambda_f\leq\rho_f<\infty$ and $\rho_g<\infty$. Then for $k=0,1,2,3,......$*

$$\lim_{r\to\infty}\frac{\log^{[n-1]}T(r,f_n)}{T(\exp(r),f^{(k)})}=0\quad \textit{for all natural number } n(\geq 2).$$

Proof. First suppose n is even. Then by Lemma 2.6 for all sufficiently large values of r and $\varepsilon(0<\varepsilon<\lambda_f)$

$$\begin{aligned}\log^{[n-1]}\ T(r,f_n) &\leq (\rho_f+\varepsilon)\log M(r,g)+O(1)\\ \log M(r,g) &< r^{\rho_g+\varepsilon}\\ \text{and}\quad T(\exp(r),f^{(k)}) &> e^{r^{(\lambda_f-\varepsilon)}}.\end{aligned}$$

So

$$\begin{aligned}\frac{\log^{[n-1]}T(r,f_n)}{T(\exp(r),f^{(k)})} &\leq \frac{(\rho_f+\varepsilon)r^{\rho_g+\varepsilon}}{e^{r^{(\lambda_f-\varepsilon)}}}+o(1)\\ \therefore\lim_{r\to\infty}\frac{\log^{[n-1]}T(r,f_n)}{T(\exp(r),f^{(k)})} &= 0.\end{aligned}$$

Similarly for odd n we have

$$\begin{aligned}\log^{[n-1]}\ T(r,f_n) &\leq (\rho_g+\varepsilon)\log M(r,f)+O(1),\\ \text{and}\quad \log M(r,f) &< r^{\rho_f+\varepsilon}\end{aligned}$$

So

$$\begin{aligned}\frac{\log^{[n-1]}T(r,f_n)}{T(\exp(r),f^{(k)})} &\leq \frac{(\rho_g+\varepsilon)r^{\rho_f+\varepsilon}}{e^{r^{(\lambda_f-\varepsilon)}}}+o(1)\\ \therefore\lim_{r\to\infty}\frac{\log^{[n-1]}T(r,f_n)}{T(\exp(r),f^{(k)})} &= 0.\end{aligned}$$

This proves the theorem. □

Remark 3.12. *The condition $\rho_g < \infty$ is the necessary for Theorem 3.11, which is shown by the following example.*

Example 3.13. *Let $f = \exp z$ and $g = \exp^{[2]} z$ then $\lambda_f = \rho_f = 1$ and $\rho_g = \infty$.*
Now when n is even

$$f_n = \exp^{[\frac{3n}{2}]} z$$

Therefore,

$$\begin{aligned} 3T(2r, f_n) &\geq \log M(r, f_n) = \exp^{[\frac{3n}{2}-1]} r \\ i.e. \quad T(r, f_n) &\geq \frac{1}{3} \exp^{[\frac{3n}{2}-1]} \frac{r}{2} \\ \therefore \quad \log^{[n-1]} T(r, f_n) &\geq \exp^{[\frac{3n}{2}-1-n+1]} \frac{r}{2} + o(1) \\ &= \exp^{[\frac{n}{2}]} \frac{r}{2} + o(1). \end{aligned}$$

Also when n is odd,

$$f_n = \exp^{[\frac{3n-1}{2}]} z$$

Therefore

$$\begin{aligned} 3T(2r, f_n) &\geq \log M(r, f_n) = \exp^{[\frac{3n-1}{2}-1]} r \\ i.e. \quad T(r, f_n) &\geq \frac{1}{3} \exp^{[\frac{3n-1}{2}-1]} \frac{r}{2} \\ \therefore \quad \log^{[n-1]} T(r, f_n) &\geq \exp^{[\frac{3n-1}{2}-1-n+1]} \frac{r}{2} + o(1) \\ &= \exp^{[\frac{n-1}{2}]} \frac{r}{2} + o(1). \end{aligned}$$

Also

$$T(\exp(r), f^{(k)}) = \frac{e^r}{\pi}$$

Therefore when n is even

$$\frac{\log^{[n-1]} T(r, f_n)}{T(\exp(r), f^{(k)})} \geq \frac{\exp^{[\frac{n}{2}]} \frac{r}{2} + o(1)}{e^r/\pi} \to \infty \quad as \quad r \to \infty,$$

and when n is odd

$$\frac{\log^{[n-1]} T(r, f_n)}{T(\exp(r), f^{(k)})} \geq \frac{\exp^{[\frac{n-1}{2}]} \frac{r}{2} + o(1)}{e^r/\pi} \to \infty \quad as \quad r \to \infty.$$

Theorem 3.14. *Let f and g be two entire functions such that $0 < \lambda_g \leq \rho_g < \infty$ and $\rho_f < \infty$. Then for $k = 0, 1, 2, 3, \ldots\ldots$*

$$\lim_{r\to\infty} \frac{\log^{[n-1]} T(r, f_n)}{T(\exp(r), g^{(k)})} = 0 \quad \text{for all natural number } n(\geq 2).$$

Theorem 3.15. *Let f and g be two entire functions such that $\rho_g < \lambda_f \leq \rho_f < \infty$. Then for $k = 0, 1, 2, 3, \ldots\ldots$*

$$\lim_{r\to\infty} \frac{\log^{[n-1]} T(r, f_n)}{T(r, f^{(k)})} = 0 \quad \text{when } n \text{ is even.}$$

Proof. From Lemma 2.6 we have for arbitrary $\varepsilon(>0)$ such that $\rho_g+\varepsilon<\lambda_f-\varepsilon$ and for large values of r,

$$\begin{aligned}\log^{[n-1]}\ T(r,f_n) &\le (\rho_f+\varepsilon)\log M(r,g)+O(1)\\ \log M(r,g) &< r^{\rho_g+\varepsilon},\\ \text{and}\ \ T(r,f^{(k)}) &> r^{\lambda_f-\varepsilon}.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\log^{[n-1]}T(r,f_n)}{T(r,f^{(k)})} &\le \frac{(\rho_f+\varepsilon)r^{\rho_g+\varepsilon}}{r^{\lambda_f-\varepsilon}}+o(1)\\ \text{i.e.}\quad \lim_{r\to\infty}\frac{\log^{[n-1]}T(r,f_n)}{T(r,f^{(k)})} &= 0.\end{aligned}$$

This proves the theorem. □

Theorem 3.16. *Let f and g be two entire functions such that $\rho_f<\lambda_g\le\rho_g<\infty$. Then for $k=0,1,2,3,......$,*

$$\lim_{r\to\infty}\frac{\log^{[n-1]}T(r,f_n)}{T(r,g^{(k)})}=0\quad \text{when } n \text{ is odd.}$$

In [6] Lahiri and Datta proved the following theorem.
Theorem A. *Let f and g be two transcendental entire functions such that*

$$\text{(i) } 0<\lambda_g\le\rho_g<\infty,\quad \text{(ii) } \lambda_f>0,\quad \text{and}\quad \text{(iii)}\ \ \delta(0;f)<1.$$

Then for any real number A,

$$\limsup_{r\to\infty}\frac{\log T(r,f_og)}{\log\ T(r^A,g^{(k)})}=\infty$$

for $k=0,1,2,3,......$

Now we generalise the above as follows.

Theorem 3.17. *Let f and g be two transcendental entire functions such that*

$$\text{(i) } 0<\lambda_g\le\rho_g<\infty,\quad \text{(ii) } \lambda_f>0,\quad \text{and}\quad \text{(iii)}\ \ \delta(0;f)<1.$$

Then for any real number A,

$$\limsup_{r\to\infty}\frac{\log^{[n-1]}T(r,f_n)}{\log\ T(r^A,g^{(k)})}=\infty$$

for $k=0,1,2,3,......$ and n is even.

Proof. When n is even then from (2.1),

$$\begin{aligned}\log^{[n-2]}\ T(r,f_n) &\ge (\lambda_g-\varepsilon)\log M\left(\frac{r}{4^{n-2}},fg\right)+O(1)\\ &\ge (\lambda_g-\varepsilon)T\left(\frac{r}{4^{n-2}},fg\right)+O(1)\\ \text{that is,}\ \log^{[n-1]}\ T(r,f_n) &\ge \log T\left(\frac{r}{4^{n-2}},fg\right)+O(1)\ .\end{aligned}$$

Therefore

$$\begin{aligned}\limsup_{r\to\infty}\frac{\log^{[n-1]}T(r,f_n)}{\log\ T(r^A,g^{(k)})} &\geq \limsup_{r\to\infty}\frac{\log T\left(\frac{r}{4^{n-2}},fg\right)}{\log\ T(r^A,g^{(k)})}\\ &\geq \limsup_{r\to\infty}\left[\frac{\log T\left(\frac{r}{4^{n-2}},fg\right)}{\log\ T\left(\left(\frac{r}{4^{n-2}}\right)^A,g^{(k)}\right)}\frac{\log\ T\left(\left(\frac{r}{4^{n-2}}\right)^A,g^{(k)}\right)}{\log\ T(r^A,g^{(k)})}\right] \quad (3.21)\end{aligned}$$

But

$$\limsup_{r\to\infty}\frac{\log\ T\left(\left(\frac{r}{4^{n-2}}\right)^A,g^{(k)}\right)}{\log\ T(r^A,g^{(k)})}\geq\frac{\lambda_g}{\rho_g}. \quad (3.22)$$

So proceeding as in Theorem A we have the result by using (3.21) and (3.22).

This proves the theorem. □

Theorem 3.18. *Let f and g be two transcendental entire functions such that*

$$\text{(i) } 0<\lambda_f\leq\rho_f<\infty,\quad \text{(ii) } \lambda_g>0,\quad \text{and}\quad \text{(iii) } \delta(0;g)<1.$$

Then for any real number A,

$$\limsup_{r\to\infty}\frac{\log^{[n-1]}T(r,f_n)}{\log\ T(r^A,f^{(k)})}=\infty$$

for $k=0,1,2,3,......$ and n is odd.

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