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q-ANALOGUES OF SAIGO'S FRACTIONAL CALCULUS OPERATORS

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Abstract. M. Saigo [Math. Rep. Coll. Gen. Educ., Kyushu Univ., 11 (1978) 135-143] has defined a pair of fractional integral operators and fractional derivatives involving generalizd hypergeometric function. The aim of present paper is to define their q-analogues. First, we define a pair of q-analogues of Saigo's fractional integral operators and establish some results for it. Next, we define a pair of q-analogues of Saigo's fractional derivatives and prove that these are left inverses of the corresponding fractional integral operators. We also obtain q-Mellin transforms of all these operators.

1. Introduction

The concept of fractional calculus is believed to have stemmed from a question raised by L'Hospital to Leibniz in 1695 [12]. It has gained considerable popularity and importance during last three decades due to its distinguished applications in numerous diverse fields of science and engineering ([17], [13], [11]). The q-calculus was also initiated in twenties of the last century. A detail account of which can be seen in the books by Slater [20], Exton [6], Gasper [9] and a thesis [5].

The fractional q-calculus is the q-extension of the ordinary fractional calculus. The theory of q-calculus operators in recent past have been applied in the areas like ordinary fractional calculus, optimal control problems, solutions of the q-difference (differential) and q-integral equations, q-transform analysis and many more.

Al-Salam introduced the concept of fractional q-calculus, starting from the q-analogue of Cauchy's formula ([3],[4],[2]). Agarwal [1] studied certain fractional q-integral operators and q-derivatives, where he proved the semigroup properties for left and right Riemann-Liouville type fractional integral operators. Further, Isogawa et al. [10] studied some basic properties of fractional q-derivatives. Rajkovic et al. [15] generalized the notion of the left fractional q-integral operators and fractional q-derivatives by introducing variable lower limit and proved the semigroup

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properties. Garg et al. [8] introduced q-analogues of hyper-Bessel type Kober fractional derivatives. Further, Saxena et al. [19] and Yadav et al. ([22],[21]) have obtained images of various q-special functions under fractional q-calculus operators. Recently, Purohit and Yadav [14] have defined q-extensions of the Saigo's fractional integral operators [16].

In the theory of q-calculus [9] , $0 < |q| < 1$ the **q-shifted factorial** (q-analogue of the Pochhammer Symbol) is defined by

$$(a;q)_k = \begin{cases} \prod_{j=0}^{k-1}\left(1-aq^j\right) & \text{if } k>0 \\ 1 & \text{if } k=0 \\ \prod_{j=0}^{\infty}\left(1-aq^j\right) & \text{if } k\to\infty \end{cases} \tag{1.1}$$

or equivalently

$$(a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty}, k \in N$$

and for any complex number α,

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, \tag{1.2}$$

where the principal value of q^α is taken.
The **q-analogue of the power function** is defined and denoted as

$$\begin{aligned}(a-b)_\alpha &= a^\alpha \left({}^b\!/\!{}_a;q\right)_\alpha \\ &= a^\alpha \prod_{j=0}^{\infty}\left[\frac{1-\left({}^b\!/\!{}_a\right)q^j}{1-\left({}^b\!/\!{}_a\right)q^{j+\alpha}}\right] = a^\alpha \frac{\left({}^b\!/\!{}_a;q\right)_\infty}{\left(q^\alpha\, {}^b\!/\!{}_a;q\right)_\infty}, (a \neq 0)\,.\end{aligned} \tag{1.3}$$

The **q-gamma function** is defined by

$$\Gamma_q(\alpha) = \frac{G(q^\alpha)}{G(q)}(1-q)^{1-\alpha} = (1-q)_{\alpha-1}(1-q)^{1-\alpha}, \alpha \in R/\{0,-1,-2,...\}, \tag{1.4}$$

where

$$G(q^\alpha) = \frac{1}{(q^\alpha;q)_\infty}.$$

The **q-derivative** of a function $f(x)$ is given by [9]:

$$D_q f(x) = \frac{f(x)-f(qx)}{(1-q)x}, (x \neq 0)\, and\, (D_q f)(0) = \lim_{x\to 0}(D_q f)(x), \tag{1.5}$$

where $D_q \to {}^d\!/\!{}_{dx}$, as $q \to 1$.
We have

$$D_q^n x^\mu = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\mu-n+1)} x^{\mu-n}, \Re(\mu)+1>0. \tag{1.6}$$

The **q-integral** of a function is defined as [9]:

$$\int_0^x f(t)\,d_q t = x(1-q)\sum_{k=0}^{\infty} q^k f\left(xq^k\right), \tag{1.7}$$

$$\int_x^\infty f(t)\,d_q t = x(1-q)\sum_{k=1}^{\infty} q^{-k} f\left(xq^{-k}\right), \tag{1.8}$$

$$\int_0^\infty f(t)\,d_qt=(1-q)\sum_{k=-\infty}^{\infty}q^kf\left(q^k\right), \tag{1.9}$$

The **q-binomial series** [9] is given by

$$_1\Phi_0\left[\begin{array}{c}\alpha\\ -\end{array};q,x\right]=\sum_{n=0}^{\infty}\frac{(\alpha;q)_n}{(q;q)_n}x^n=\frac{(\alpha x;q)_\infty}{(x;q)_\infty}. \tag{1.10}$$

Heine's **q-analogue of Gauss summation theorem** [9] is given by

$$\begin{aligned}{}_2\Phi_1\left[\begin{array}{c}q^a,q^b\\ q^c\end{array};q,q^{c-a-b}\right]&=\sum_{n=0}^{\infty}\frac{(q^a;q)_n\left(q^b;q\right)_n}{(q^c;q)_n(q;q)_n}\left(q^{c-a-b}\right)^n\\&=\frac{\Gamma_q(c)\,\Gamma_q(c-a-b)}{\Gamma_q(c-a)\,\Gamma_q(c-b)}.\end{aligned} \tag{1.11}$$

The **q-Mellin transform** of a suitable function $f(x)$ on $R_q^+=\{q^n;n\in Z\}$ is given by [7] :

$$M_q(f)(s)=\int_0^\infty x^{s-1}f(x)\,d_qx. \tag{1.12}$$

Also

$$\begin{aligned}M_q[D_q^nf(x)](s)&=[1-s]_q[2-s]_q...[n-s]_qM_q[f(x)](s-n)\\&=q^{n(n+1)/2-ns}(-1)^n\frac{\Gamma_q(s)}{\Gamma_q(s-n)}M_q[f(x)](s-n).\end{aligned} \tag{1.13}$$

where D_q is the q-derivative defined by (1.5).

A **q-analogue of Riemann-Liouville fractional integral operator** [1] is defined as:

$$I_q^\alpha(x)=\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x(tq/x;q)_{\alpha-1}f(t)\,d_qt;\Re(\alpha)>0, \tag{1.14}$$

and **q-analogue of Riemann-Liouville fractional derivative** [1] is defined as

$$D_{x,q}^\alpha f(x)=D_q^n\left(I_q^{n-\alpha}f\right)(x),\ n-1<\Re(\alpha)\le n,n\in N. \tag{1.15}$$

A **q-analogue of the Weyl fractional integral operator** [4], is defined as:

$$K_q^\alpha f(x)=\frac{q^{-\alpha(\alpha-1)/2}}{\Gamma_q(\alpha)}\int_x^\infty(x/t;q)_{\alpha-1}t^{\alpha-1}f\left(tq^{1-\alpha}\right)d_qt;\Re(\alpha)>0, \tag{1.16}$$

and **q-analogue of Weyl fractional derivative** [4] is defined as

$$_{-\infty}D_{x,q}^\alpha f(x)=(-1)^nD_q^nK_q^{n-\alpha}f(x),\ n-1<\Re(\alpha)\le n,n\in N. \tag{1.17}$$

Also, the **q-analogues of the Kober fractional integral operators** [4] are defined as

$$I_q^{\eta,\alpha}f(x)=\frac{x^{-\eta-1}}{\Gamma_q(\alpha)}\int_0^x(qt/x;q)_{\alpha-1}t^\eta f(t)\,d_qt;\Re(\alpha)>0, \tag{1.18}$$

$$K_q^{\eta,\alpha}f(x)=\frac{x^\eta q^{-\eta}}{\Gamma_q(\alpha)}\int_x^\infty(x/t;q)_{\alpha-1}t^{-\eta-1}f\left(tq^{1-\alpha}\right)d_qt;\Re(\alpha)>0. \tag{1.19}$$

The remaining paper is organized as follows. In Section 2, we define a pair of q-analogues of Saigo's fractional integral operators and establish some results for it. These involve the image of power function under these operators, composition formulas and fractional q-integration by parts. In Section 3, we first establish results which give extensions of q-analogues of Saigo's fractional integral operators for $\Re(\alpha) < 0$ and define a pair of q-analogues of Saigo's fractional derivatives. Next, we show that these operators are left inverse operators to the q-analogues of Saigo's fractional integral operators. We also obtain images of the power function under these operators. In Section 4, we obtain q-Mellin transforms of all these operators.

2. q-analogues of Saigo's fractional integral operators

Recently, Purohit and Yadav [14] have given definition of q-analogues of Saigo's fractional integral operators under the restriction that one of the parameters η is a non negative integer. It was not possible to give the definition of fractional derivatives under that restriction. To overcome these difficulties, we give the following definitions of q-analogues of the Saigo's fractional integral operators. For $\Re(\alpha) > 0$, β and η being real or complex.

$$\begin{aligned} I_q^{\alpha,\beta,\eta} f(x) &= \frac{x^{-\beta-1}}{\Gamma_q(\alpha)} \int_0^x (tq/x;q)_{\alpha-1} \\ &\times \sum_{m=0}^{\infty} \frac{\left(q^{\alpha+\beta};q\right)_m \left(q^{-\eta};q\right)_m}{\left(q^{\alpha};q\right)_m (q;q)_m} q^{(\eta-\beta)m} (-1)^m q^{-\binom{m}{2}} \left(\frac{t}{x}-1\right)_m f(t)\, d_q t, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} K_q^{\alpha,\beta,\eta} f(x) &= \frac{q^{-\alpha(\alpha+1)/2-\beta}}{\Gamma_q(\alpha)} \int_x^{\infty} (x/t;q)_{\alpha-1} t^{-\beta-1} \\ &\times \sum_{m=0}^{\infty} \frac{\left(q^{\alpha+\beta};q\right)_m \left(q^{-\eta};q\right)_m}{\left(q^{\alpha};q\right)_m (q;q)_m} q^{(\eta-\beta)m} (-1)^m q^{-\binom{m}{2}} \left(\frac{x}{qt}-1\right)_m f\left(tq^{1-\alpha}\right) d_q t. \end{aligned} \tag{2.2}$$

For $q \to 1$ the operators (2.1) and (2.2) reduce to Saigo's fractional integral operators $I^{\alpha,\beta,\eta}$ and $K^{\alpha,\beta,\eta}$ respectively which are defined as follows . For $\Re(\alpha) > 0$, β and η being real or complex [16].

$$\begin{aligned} I^{\alpha,\beta,\eta} f(x) &= \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \\ &\times {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right) f(t)\, dt \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} K^{\alpha,\beta,\eta} f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} t^{-\alpha-\beta} \\ &\times {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right) f(t)\, dt \end{aligned} \tag{2.4}$$

Definitions given by (2.1) and (2.2), in view of (1.7) and (1.8) can be written as

$$
\begin{aligned}
&I_q^{\alpha,\beta,\eta} f(x) = x^{-\beta}(1-q)^{\alpha} \\
&\times \sum_{m=0}^{\infty} \frac{\left(q^{\alpha+\beta};q\right)_m (q^{-\eta};q)_m}{(q;q)_m} q^{(\eta-\beta+1)m} \sum_{k=0}^{\infty} q^k \frac{(q^{\alpha+m};q)_k}{(q;q)_k} f\left(xq^{k+m}\right),
\end{aligned} \tag{2.5}
$$

and

$$
\begin{aligned}
&K_q^{\alpha,\beta,\eta} f(x) = x^{-\beta} q^{-\alpha(\alpha+1)/2}(1-q)^{\alpha} \\
&\times \sum_{m=0}^{\infty} \frac{\left(q^{\alpha+\beta};q\right)_m (q^{-\eta};q)_m}{(q;q)_m} q^{\eta m} \sum_{k=0}^{\infty} q^{\beta k} \frac{(q^{\alpha+m};q)_k}{(q;q)_k} f\left(xq^{-\alpha-k-m}\right).
\end{aligned} \tag{2.6}
$$

It is easy to observe that $I_q^{0,0,\eta}$ and $K_q^{0,0,\eta}$are identity operators.
Further, q-analogues of Riemann-Liouville, Weyl and Kober fractional integral operators are recovered as special cases of our operators $I_q^{\alpha,\beta,\eta}$ and $K_q^{\alpha,\beta,\eta}$ as follows

$$I_q^{\alpha,-\alpha,\eta} f(x) = I_q^{\alpha} f(x), \tag{2.7}$$

$$K_q^{\alpha,-\alpha,\eta} f(x) = K_q^{\alpha} f(x), \tag{2.8}$$

$$I_q^{\alpha,0,\eta} f(x) = I_q^{\eta,\alpha} f(x), \tag{2.9}$$

$$K_q^{\alpha,0,\eta} f(x) = q^{-\alpha(\alpha+1)/2} K_q^{\eta,\alpha} f(x). \tag{2.10}$$

Now, we obtain image of the power function under fractional q-integral operators $I_q^{\alpha,\beta,\eta}$ and $K_q^{\alpha,\beta,\eta}$.

Theorem 2.1. *For* $0 < |q| < 1$, $\Re(\alpha) > 0$, β *and* η *being real or complex*

(a). *If* $\Re(\mu+1) > 0$ *and* $\Re(\mu-\beta+\eta+1) > 0$, *then*

$$I_q^{\alpha,\beta,\eta}\left(x^{\mu}\right) = \frac{\Gamma_q(\mu+1)\,\Gamma_q(\mu-\beta+\eta+1)}{\Gamma_q(\mu-\beta+1)\,\Gamma_q(\mu+\alpha+\eta+1)} x^{\mu-\beta}. \tag{2.11}$$

(b). *If* $\Re(\beta-\mu) > 0$ *and* $\Re(\eta-\mu) > 0$, *then*

$$K_q^{\alpha,\beta,\eta}\left(x^{\mu}\right) = \frac{\Gamma_q(\beta-\mu)\,\Gamma_q(\eta-\mu)}{\Gamma_q(-\mu)\,\Gamma_q(\beta+\alpha+\eta-\mu)} x^{\mu-\beta} q^{-\alpha\mu-\alpha(\alpha+1)/2}. \tag{2.12}$$

Proof. (a). Taking $f(x) = x^{\mu}$ in (2.5), we get

$$
\begin{aligned}
&I_q^{\alpha,\beta,\eta}\left(x^{\mu}\right) = x^{\mu-\beta}(1-q)^{\alpha} \\
&\times \sum_{m=0}^{\infty} \frac{\left(q^{\alpha+\beta};q\right)_m (q^{-\eta};q)_m}{(q;q)_m} q^{(\eta+\mu-\beta+1)m} \sum_{k=0}^{\infty} q^{k(1+\mu)} \frac{(q^{\alpha+m};q)_k}{(q;q)_k},
\end{aligned} \tag{2.13}
$$

Using (1.10) and (1.2) it is further simplified as

$$I_q^{\alpha,\beta,\eta}\left(x^{\mu}\right) = x^{\mu-\beta}(1-q)^{\alpha} \sum_{m=0}^{\infty} \frac{\left(q^{\alpha+\beta};q\right)_m (q^{-\eta};q)_m}{(q;q)_m \left(q^{1+\mu};q\right)_{\alpha+m}} q^{(\eta+\mu-\beta+1)m}, \tag{2.14}$$

On doing some simplifications and using q-analogue of Gauss summation theorem (1.11), we get the desired result (2.11) .
The part (b) can be established on similar lines, by using (2.6) and known theorems given by (1.10) and (1.11). □

Theorem 2.2. *For the function* $f(x)$ *represented by a power series* $\sum_{n=0}^{\infty} a_n x^n$, *with radius of convergence R, the following three composition formulas hold. For* $0 < |q| < 1$, $\Re(\alpha) > 0$, β *and* η *being real or complex* (a). *If* $\Re(\alpha+\eta-\delta+1)>0$ *and* $\Re(\eta-\delta-\beta+1)>0$*, then*

$$I_q^{\alpha,\beta,\eta} I_q^{\gamma,\delta,\alpha+\eta} f(x) = I_q^{\alpha+\gamma,\beta+\delta,\eta} f(x). \tag{2.15}$$

(b). *If* $\Re(\eta-\beta-\gamma-2\delta+1)>0$ *and* $\Re(\eta-\beta-\delta+1)>0$*, then*

$$I_q^{\alpha,\beta,\eta} I_q^{\gamma,\delta,\eta-\beta-\gamma-\delta} f(x) = I_q^{\alpha+\gamma,\beta+\delta,\eta-\gamma-\delta} f(x). \tag{2.16}$$

(c). *If* $\Re(\alpha+\beta+\eta+1)>0$*,* $\Re(\beta+1)>0$ *and* $\Re(\eta+1)>0$*, then*

$$I_q^{\alpha,\beta,\eta} x^{\beta+\delta} I_q^{\gamma,\delta,\alpha+\beta+\eta+\delta} f(x) = x^{\delta} I_q^{\alpha+\gamma,\beta+\delta,\eta+\delta} x^{\beta} f(x). \tag{2.17}$$

Proof. (a). We have

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad |x|<R, \tag{2.18}$$

The left hand side of (2.15) on using (2.18), changing the orders of summations and applying Thorem 2.1(a), can be written as

$$I_q^{\alpha,\beta,\eta} \sum_{n=0}^{\infty} a_n \frac{\Gamma_q(n+1)\,\Gamma_q(n-\delta+\alpha+\eta+1)}{\Gamma_q(n-\delta+1)\,\Gamma_q(n+\gamma+\alpha+\eta+1)} x^{n-\delta}. \tag{2.19}$$

Again on using the result (2.11) and doing some simplifications, we get

$$\sum_{n=0}^{\infty} a_n \frac{\Gamma_q(n+1)\,\Gamma_q(n-\delta-\beta+\eta+1)}{\Gamma_q(n-\delta-\beta+1)\,\Gamma_q(n+\gamma+\alpha+\eta+1)} x^{n-\delta-\beta}, \tag{2.20}$$

which in view of Theorem2.1(a) is equivalent to the right side of (2.15).
Parts (b) and (c) can be established on similar lines, by using the result (2.11) . □

Theorem 2.3. *For the function* $f(x)$ *represented by a power series* $\sum_{n=0}^{\infty} a_n x^n$*, with radius of convergence R, the following three composition formulas hold. For* $0<|q|<1$*,* $\Re(\alpha)>0$*,* β *and* η *being real or complex*

(a). *If* $\beta,\delta \notin Z_0 = Z-\{0\}$ *and* $\eta \notin Z$*, then*

$$K_q^{\gamma,\delta,\alpha+\eta} K_q^{\alpha,\beta,\eta} f(x) = q^{\gamma(\alpha+\beta)} K_q^{\alpha+\gamma,\beta+\delta,\eta} f(x). \tag{2.21}$$

(b). *If* $\beta,\delta \notin Z_0 = Z-\{0\}$ *and* $\eta,\gamma \notin Z$*, then*

$$K_q^{\gamma,\delta,\eta-\beta-\gamma-\delta} K_q^{\alpha,\beta,\eta} f(x) = q^{\gamma(\alpha+\beta)} K_q^{\alpha+\gamma,\beta+\delta,\eta-\gamma-\delta} f(x). \tag{2.22}$$

(c). *If* $\beta \notin Z_0 = Z-\{0\}$ *and* $\eta \notin Z$*, then*

$$K_q^{\gamma,\delta,\alpha+\beta+\eta+\delta} x^{\beta+\delta} K_q^{\alpha,\beta,\eta} f(x) = q^{\alpha(\gamma+\delta)} x^{\beta} K_q^{\alpha+\gamma,\beta+\delta,\eta+\delta} x^{\delta} f(x). \tag{2.23}$$

Theorem2.3, can be proved on the lines of Theorem2.2 by using Theorem2.1(b).

Theorem 2.4. *If* $f(x)$ *and* $g(x)$ *are functions expressible in power series with radii of convergnce R and S respectively, then for* $0<|q|<1, \Re(\alpha)>0$ *,* β *and* η *being real or complex, we have the following result for* ***fractional q-integration by parts***

$$\int_0^{\infty} f(x)\, K_q^{\alpha,\beta,\eta} g(x)\, d_q x = q^{-\alpha(\alpha+1)/2} \int_0^{\infty} g\left(xq^{-\alpha}\right) I_q^{\alpha,\beta,\eta} f(x)\, d_q x, \tag{2.24}$$

provided the q-integrals exist.

Proof. The left hand side of (2.24) on using (2.6) , can be written as

$$\begin{aligned} &q^{-\alpha(\alpha+1)/2}(1-q)^{\alpha}\int_0^{\infty} f(x)x^{-\beta} \\ &\times\sum_{m=0}^{\infty}\frac{\left(q^{\alpha+\beta};q\right)_m(q^{-\eta};q)_m}{(q;q)_m}q^{\eta m}\sum_{k=0}^{\infty}q^{\beta k}\frac{(q^{\alpha+m};q)_k}{(q;q)_k}g\left(xq^{-\alpha-k-m}\right)d_qx \end{aligned} \tag{2.25}$$

On interchanging the orders of integration and summations and using the definition (1.9), we get

$$\begin{aligned} &q^{-\alpha(\alpha+1)/2}(1-q)^{\alpha+1}\sum_{m=0}^{\infty}\frac{\left(q^{\alpha+\beta};q\right)_m(q^{-\eta};q)_m}{(q;q)_m}q^{\eta m} \\ &\times\sum_{k=0}^{\infty}q^{\beta k}\frac{(q^{\alpha+m};q)_k}{(q;q)_k}\sum_{r=-\infty}^{\infty}q^{r(1-\beta)}g\left(q^{r-\alpha-k-m}\right)f\left(q^r\right) \\ &= q^{-\alpha(\alpha+1)/2}(1-q)^{\alpha+1} \\ &\times\sum_{r=-\infty}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(q^{\alpha+\beta};q\right)_m(q^{-\eta};q)_m}{(q;q)_m}q^{(\eta+1)m}q^k\frac{(q^{\alpha+m};q)_k}{(q;q)_k}g\left(q^{r-\alpha}\right)q^{r(1-\beta)}f\left(q^{k+m+r}\right) \end{aligned} \tag{2.26}$$

Using (1.9) to replace the basic bilateral series by the integral, we get

$$\begin{aligned} &q^{-\alpha(\alpha+1)/2}(1-q)^{\alpha}\int_0^{\infty} g\left(xq^{-\alpha}\right)x^{-\beta} \\ &\times\sum_{m=0}^{\infty}\frac{\left(q^{\alpha+\beta};q\right)_m(q^{-\eta};q)_m}{(q;q)_m}q^{(\eta-\beta+1)m}\sum_{k=0}^{\infty}q^{k}\frac{(q^{\alpha+m};q)_k}{(q;q)_k}f\left(xq^{k+m}\right)d_qx \end{aligned} \tag{2.27}$$

Finally, on using (2.5) it gives the right hand side of (2.24). □

On reducing $I_q^{\alpha,\beta,\eta}$ and $K_q^{\alpha,\beta,\eta}$ to q-analogues of Riemann-Liouville, Weyl and Kober operators, we shall get fractional q-integration by parts for these operators as given by Agrawal [1].

3. *q*-analogues of Saigo's fractional derivatives

First, we shall establish following result, which gives the extensions of operators $I_q^{\alpha,\beta,\eta}$ and $K_q^{\alpha,\beta,\eta}$ for $\Re(\alpha) < 0$. This will be required to define the fractional q-derivatives.

Theorem 3.1. *For the function $f(x)$ represented by a power series $\sum_{n=0}^{\infty} a_n x^n$, with radius of convergence R, the following relations hold. For $0 < |q| < 1$, $\Re(\alpha + n) > 0$,$n \in N$,β and η being real or complex*

(a). *If $\Re(\eta-\beta+1) > 0$, then*

$$I_q^{\alpha,\beta,\eta}f(x) = D_q^n I_q^{\alpha+n,\beta-n,\eta-n}f(x). \tag{3.1}$$

(b). *If $\beta \notin Z_0 = Z - \{0\}$ and $\eta \notin Z$, then*

$$K_q^{\alpha,\beta,\eta}f(x) = \left(-q^{(\alpha+\beta)}D_q\right)^n K_q^{\alpha+n,\beta-n,\eta}f(x). \tag{3.2}$$

Proof. (a). We first establish the following

$$I_q^{\alpha,\beta,\eta} f(x) = D_q I_q^{\alpha+1,\beta-1,\eta-1} f(x). \tag{3.3}$$

We have

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad |x| < R, \tag{3.4}$$

By using (3.4) in right side of (3.3), changing the orders of summations and using Theorem2.1(a), it can be written as

$$D_q \sum_{n=0}^{\infty} a_n \frac{\Gamma_q(n+1)\,\Gamma_q(n-\beta+\eta+1)}{\Gamma_q(n-\beta+2)\,\Gamma_q(n+\alpha+\eta+1)} x^{n-\beta+1}. \tag{3.5}$$

On using (1.6) and doing some simplifications, we get

$$\sum_{n=0}^{\infty} a_n \frac{\Gamma_q(n+1)\,\Gamma_q(n-\beta+\eta+1)}{\Gamma_q(n-\beta+1)\,\Gamma_q(n+\alpha+\eta+1)} x^{n-\beta}, \tag{3.6}$$

which in view of Theorem2.1(a) is equivalent to the left side of (3.3).
Repeated application of this result will give (3.2).
The part (b) can be established on similar lines, by using Theorem2.1(b) and (1.6). □

Now, we define **q-analogues of Saigo's fractional derivatives** for $n-1 < \Re(\alpha) \leqslant n, n \in N$,$\beta$ and η being real or complex as follows

$$D_q^{\alpha,\beta,\eta} f(x) = D_q^n I_q^{-\alpha+n,-\beta-n,\alpha+\eta-n} f(x) \tag{3.7}$$

and

$$P_q^{\alpha,\beta,\eta} f(x) = q^{\alpha(\alpha+\beta)} \left(-q^{-(\alpha+\beta)} D_q\right)^n K_q^{-\alpha+n,-\beta-n,\alpha+\eta} f(x), \tag{3.8}$$

where $I_q^{\alpha,\beta,\eta}$ and $K_q^{\alpha,\beta,\eta}$ are given by (2.1) and (2.2) respectively.
For $q \to 1$, the operators (3.7) and (3.8) reduce to the Saigo's fractional derivatives as given in [16] .
Also, if we take $\beta = -\alpha$ in (3.7) and (3.8) they reduce to q-analogues of Riemann-Liouville and Weyl fractional derivatives given by (1.15) and (1.17) respectively.
In the following theorem, we shall prove that, the q-analogues of Saigo's fractional derivatives (3.7) and (3.8) act as left inverse to the fractional integral operators (2.1) and (2.2) respectively.

Theorem 3.2. *For the function $f(x)$ represented by a power series $\sum_{n=0}^{\infty} a_n x^n$, with radius of convergence R, the following results hold. For $0 < |q| < 1$, $\Re(\alpha) > 0$, β and η being real or complex*

(a).

$$D_q^{\alpha,\beta,\eta} I_q^{\alpha,\beta,\eta} f(x) = f(x). \tag{3.9}$$

(b).

$$P_q^{\alpha,\beta,\eta} K_q^{\alpha,\beta,\eta} f(x) = f(x). \tag{3.10}$$

Proof. By using the definition (3.7) and Theorem2.2(a), we obtain

$$D_q^{\alpha,\beta,\eta} I_q^{\alpha,\beta,\eta} f(x) = D_q^n I_q^{n,-n,\alpha+\eta-n} f(x), \tag{3.11}$$

on using Theorem3.1(a) in the right hand side of (3.11), we get

$$D_q^{\alpha,\beta,\eta} I_q^{\alpha,\beta,\eta} f(x) = I_q^{0,0,\alpha+\eta} f(x). \tag{3.12}$$

Since $I_q^{0,0,\alpha+\eta}$is an identity operator for any η, we get the result (3.9).
The part (b) can be established on similar lines, on using definition (3.8), Theorem2.3(a) and Theorem3.1(b). □

In the next theorem, we obtain images of the power function under fractional derivatives (3.7) and (3.8).

Theorem 3.3. *For $0 < |q| < 1$, $n-1 < \Re(\alpha) \leqslant n, n \in N$, β and η being real or complex*

(a). *If $\Re(\mu+1) > 0$ and $\Re(\mu+\alpha+\beta 1+\eta+1) > 0$, then*

$$D_q^{\alpha,\beta,\eta} x^\mu = \frac{\Gamma_q(\mu+1)\,\Gamma_q(\mu+\alpha+\beta+\eta+1)}{\Gamma_q(\mu+\beta+1)\,\Gamma_q(\mu+\eta+1)} x^{\mu+\beta}. \tag{3.13}$$

(b). *If $\Re(-\beta-\mu) > 0$ and $\Re(\alpha+\eta-\mu) > 0$, then*

$$P_q^{\alpha,\beta,\eta} x^\mu = \frac{\Gamma_q(-\beta-\mu)\,\Gamma_q(\alpha+\eta-\mu)}{\Gamma_q(-\mu)\,\Gamma_q(-\beta+\eta-\mu)} q^{\alpha(\beta+\mu)+\alpha(\alpha-1)/2} x^{\mu+\beta}. \tag{3.14}$$

Proof. (a). To obtain $D_q^{\alpha,\beta,\eta} x^\mu$, we use the definition (3.7) and the result (2.11), to get

$$D_q^{\alpha,\beta,\eta} x^\mu = D_q^n \frac{\Gamma_q(\mu+1)\,\Gamma_q(\mu+\alpha+\beta+\eta+1)}{\Gamma_q(\mu+\beta+n+1)\,\Gamma_q(\mu+\eta+1)} x^{\mu+\beta+n}. \tag{3.15}$$

On using (1.6), we get the result (3.13).
The part (b), can be established on similar lines, by using the definition (3.8) and the results (2.12) and (1.6). □

4. q-Mellin transform of q-analogues of Saigo's fractional integral operators and derivatives

Theorem 4.1. *For $0 < |q| < 1$, $\Re(\alpha) > 0$, β and η being real or complex*

(a). *If $\Re(1+\beta-s) > 0$ and $\Re(1+\eta-s) > 0$, then*

$$M_q\left(I_q^{\alpha,\beta,\eta} f(x)\right)(s) = \frac{\Gamma_q(1+\beta-s)\,\Gamma_q(1+\eta-s)}{\Gamma_q(1-s)\,\Gamma_q(1+\beta+\alpha+\eta-s)} M_q(f(x))(s-\beta). \tag{4.1}$$

(b). *If $\Re(s) > 0$ and $\Re(s-\beta+\eta) > 0$, then*

$$M_q\left(K_q^{\alpha,\beta,\eta} f(x)\right)(s) = \frac{\Gamma_q(s)\,\Gamma_q(s-\beta+\eta)}{\Gamma_q(s-\beta)\,\Gamma_q(s+\alpha+\eta)} q^{-\alpha(\alpha+1)/2} M_q\left(f\left(xq^{-\alpha}\right)\right)(s-\beta). \tag{4.2}$$

(c). *If $\Re(1-\beta-s) > 0$ and $\Re(1+\alpha+\eta-s) > 0$, then*

$$M_q\left(D_q^{\alpha,\beta,\eta} f(x)\right)(s) = \frac{\Gamma_q(1-\beta-s)\,\Gamma_q(1+\alpha+\eta-s)}{\Gamma_q(1-s)\,\Gamma_q(1-\beta+\eta-s)} M_q(f(x))(s+\beta). \tag{4.3}$$

(d). *If $\Re(s) > 0$ and $\Re(s+\beta+\alpha+\eta) > 0$, then*

$$\begin{aligned} &M_q\left(P_q^{\alpha,\beta,\eta} f(x)\right)(s) \\ &= q^{\alpha\beta+\alpha(\alpha+1)/2-n(s+\beta)} \tfrac{\Gamma_q(s)\Gamma_q(s+\beta+\eta+\alpha)}{\Gamma_q(s+\beta)\Gamma_q(s+\eta)} M_q\left(f\left(xq^{\alpha-n}\right)\right)(s+\beta), \end{aligned} \tag{4.4}$$

where $n = [R(\alpha)] + 1, n \in N$.

Proof. (a). The left hand side of (4.1) on using definition (1.12) and (2.5), can be written as

$$(1-q)^{\alpha}\int_0^{\infty} x^{s-\beta-1}\sum_{m=0}^{\infty}\frac{\left(q^{\alpha+\beta};q\right)_m\left(q^{-\eta};q\right)_m}{(q;q)_m}q^{(\eta-\beta+1)m} \times\sum_{k=0}^{\infty}q^k\frac{\left(q^{\alpha+m};q\right)_k}{(q;q)_k}f\left(xq^{k+m}\right)d_qx \tag{4.5}$$

On interchanging the orders of integration and summations and using the definition (1.9), we get

$$(1-q)^{\alpha+1}\sum_{m=0}^{\infty}\frac{\left(q^{\alpha+\beta};q\right)_m\left(q^{-\eta};q\right)_m}{(q;q)_m}q^{(\eta-\beta+1)m} \times\sum_{k=0}^{\infty}q^k\frac{\left(q^{\alpha+m};q\right)_k}{(q;q)_k}\sum_{r=-\infty}^{\infty}q^{r(s-\beta)}f\left(q^{k+m+r}\right) \tag{4.6}$$

$$=(1-q)^{\alpha+1}\sum_{m=0}^{\infty}\frac{\left(q^{\alpha+\beta};q\right)_m\left(q^{-\eta};q\right)_m}{(q;q)_m}q^{(\eta+s+1)m} \times\sum_{k=0}^{\infty}q^{k(1-s+\beta)}\frac{\left(q^{\alpha+m};q\right)_k}{(q;q)_k}\sum_{r=-\infty}^{\infty}q^{r(s-\beta)}f\left(q^{r}\right) \tag{4.7}$$

Using (1.10) and (1.2) it is further simplified as

$$(1-q)^{\alpha+1}\sum_{m=0}^{\infty}\frac{\left(q^{\alpha+\beta};q\right)_m(q^{-\eta};q)_m}{(q;q)_m(q^{1-s+\beta};q)_{\alpha+m}}q^{(\eta+s+1)m}\sum_{r=-\infty}^{\infty}q^{r(s-\beta)}f\left(q^{r}\right) \tag{4.8}$$

On doing some simplifications, using q-analogue of Gauss summation theorem (1.11) and the result (1.9), we get the desired result (4.1).
The part (b) can be established on similar lines, by using (2.6) and (1.9).

To establish the part (c), we use the definition (3.7) and the result (1.13) in left side of (4.3) to get

$$q^{n(n+1)/2-ns}(-1)^n\frac{\Gamma_q(s)}{\Gamma_q(s-n)}M_q[I_q^{-\alpha+n,-\beta-n,\alpha+\eta-n}f(x)](s-n) \tag{4.9}$$

On using part (a) and doing some simplifications, we arrive at right side of (4.3).

The part (d) can be established on similar lines, by using definition (3.8) with the results (1.13) and (4.2). □

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