Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 3 Issue 4(2011), Pages 223-231.

FRACTIONAL CALCULUS ON UNIFIED CLASS OF SPIRAL-LIKE FUNCTIONS INVOLVING SRIVASTAVA-ATTIYA OPERATOR

(COMMUNICATED BY H.M.SRIVASTAVA)

M.K.AOUF, G.MURUGUSUNDARAMOORTHY AND K.THILAGAVATHI

Abstract. Making use of convolution product, we introduce a unified class of spiral-like functions and obtain the coefficient bounds, extreme points and radius of starlikeness for functions belonging to the generalized class $\mathcal{P}_{\mu}^{\lambda}(\alpha,\beta,\gamma)$. Furthermore, Distortion theorems for the fractional derivative and fractional integration are obtained. Also we get result about coefficient inequality.

1. Introduction

Let A denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1.1}$$

which are analytic and univalent in the open disc $U=\{z:z\in\mathcal{C},\ |z|<1\}$. For functions $f\in A$ given by (1.1) and $g\in A$ given by $g(z)=z+\sum\limits_{n=2}^{\infty}b_nz^n$, we define the Hadamard product (or convolution) of f and g by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n,\quad z\in U. \tag{1.2}$$

We recall here a general Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ defined by [18].

$$\Phi(z,s,a):=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s} \tag{1.3}$$

$(a\in\mathbb{C}\setminus\mathbb{Z}_0^-;s\in\mathbb{C},\text{when}|z|<1;\Re(s)>1\text{ when }|z|=1)$ where, as usual, $\mathbb{Z}_0^-:=\mathbb{Z}\setminus\{\mathbb{N}\}$, $(\mathbb{Z}:=\{0,\pm1,\pm2,\pm3,...\});\mathbb{N}:=\{1,2,3,...\}$. Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ can be found in the recent investigations by Choi and Srivastava [3], Ferreira and Lopez [4], Garg et

---

2000 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Analytic, univalent, Starlike function, Spiral-like function, Fractional derivative, Fractional integration, convexity, Hadamard product (or convolution).

Submitted July 7, 2011. Published November 28, 2011.

al. [6], Lin and Srivastava [8], Lin et al. [9], and others. Srivastava and Attiya [17] (see also Raducanu and Srivastava [15], and Prajapat and Goyal [14]) introduced and investigated the linear operator:

$$\mathcal{J}_b^{\mu} : \mathcal{A} \to \mathcal{A}$$

defined in terms of the Hadamard product by

$$\mathcal{J}_b^{\mu} f(z) = \mathcal{G}_b^{\mu} * f(z) \tag{1.4}$$

($z \in U; b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; \mu \in \mathbb{C}; f \in \mathcal{A}$), where, for convenience,

$$\mathcal{G}_b^{\mu} := (1+b)^{\mu}[\Phi(z,\mu,b) - b^{-\mu}] \quad (z \in U). \tag{1.5}$$

We recall here the following relationships (given earlier by [14], [15]) which follow easily by using (1.1), (3.3) and (1.5)

$$\mathcal{J}_b^{\mu} f(z) = z + \sum_{n=2}^{\infty} C_n(b,\mu) a_n z^n \tag{1.6}$$

where

$$C_n(b,\mu) = \left| \left(\frac{1+b}{n+b} \right)^{\mu} \right| \tag{1.7}$$

and (throughout this paper unless otherwise mentioned) the parameters μ, b are constrained as $b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; \mu \in \mathbb{C}$.

(1) For $\mu = 0$ and $b = 0$

$$\mathcal{J}_0^0 f(z) := f(z). \tag{1.8}$$

(2) For $\mu = 1$and $b = 0$

$$\mathcal{J}_0^1 f(z) := \int_0^z \frac{f(t)}{t} dt := \mathcal{L} f(z) := z + \sum_{n=2}^{\infty} \left(\frac{1}{n} \right) a_n z^n. \tag{1.9}$$

(3) For $\mu = 1$ and $b = \nu (\nu > -1)$

$$\mathcal{J}_\nu^1 f(z) := \frac{1+\nu}{z^\nu} \int_0^z t^{1-\nu} f(t) dt := \mathcal{B}_\nu f(z) := z + \sum_{n=2}^{\infty} \left(\frac{1+\nu}{n+\nu} \right) a_n z^n. \tag{1.10}$$

(4) For $\mu = \sigma (\sigma > 0)$ and $b = 1$

$$\mathcal{J}_1^{\sigma} f(z) := z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1} \right)^{\sigma} a_n z^n = \mathcal{I}^{\sigma} f(z), \tag{1.11}$$

where $\mathcal{L}$ and $\mathcal{B}_\nu$ are the integral operators introduced by Alexandor [1] and Bernardi [2], respectively, and $\mathcal{I}^{\sigma}(f)$ is the Jung-Kim-Srivastava integral operator [7] closely related to some multiplier transformation studied by Fleet [5].

For β real, $|\beta| < \frac{\pi}{2}$ a function f in the form (1.1) is said in S_p^{β} the class of β-spiral-like functions satisfying the analytic criteria

$$Re\left(e^{i\beta} \frac{zf'(z)}{f(z)} \right) > 0, z \in U$$

was introduced and shown to be a subfamily of A by Spacek [16]. Later, Zamorski[20] obtained sharp coefficient bounds for the function class class.Motivated by the earlier works on analytic functions involving Hurwitz-Lerch Zeta functions (see[10, 11, 12, 13, 14, 17] and making use of the operator $\mathcal{J}_b^{\mu}$, we introduce a new subclass of spiral-like functions and discuss some some usual properties of the geometric function theory of this generalized function class.

For $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$, $0 \leq \gamma < 1$, and $|\beta| < \frac{\pi}{2}$, we let $\mathcal{P}_{\mu}^{\lambda}(\alpha, \beta, \gamma)$ be the subclass of A consisting of functions of the form (1.1) and satisfying the inequality

$$\left| \frac{F(z) - 1}{(2\alpha - 1)[F(z) - 1] + 2\alpha(1 - \gamma)e^{-i\beta} cos\beta} \right| < 1 \tag{1.12}$$

where

$$F(z) = \frac{z(\mathcal{J}_b^{\mu} f)'(z) + \lambda z^2 (\mathcal{J}_b^{\mu} f)''(z)}{(1 - \lambda)\mathcal{J}_b^{\mu} f(z) + \lambda z(\mathcal{J}_b^{\mu} f)'(z)} \tag{1.13}$$

where $z \in U$, $\mathcal{J}_b^{\mu} f(z)$ is given by (1.6) . In particular, for $0 \leq \lambda \leq 1$, the class $\mathcal{P}_{\mu}^{\lambda}(\alpha, \beta, \gamma)$ provides a transition from starlike functions to convex functions.By suitably specializing the values of μ, α, β and λ the class $\mathcal{P}_{\mu}^{\lambda}(\alpha, \beta, \gamma)$ reduces to the various new subclasses.

As illustrations, we present few following examples:

Example 1: If $\mu = 0$ and $b = 0$, then

$$\mathcal{P}(\alpha, \beta, \gamma) := \left\{ f \in A : \left| \frac{F(z) - 1}{(2\alpha - 1)[F(z) - 1] + 2\alpha(1 - \gamma)e^{-i\beta} cos\beta} \right| < 1 \right\}, \tag{1.14}$$

where $F(z) = \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)}$.

Example 2: If $\mu = 1$ and $f(z)$ is as defined in (1.9), then

$$\mathcal{P}_1^0(\alpha, \beta, \gamma) \equiv \mathcal{L}_{\lambda}(\alpha, \beta, \gamma) := \left\{ f \in A : \left| \frac{F(z) - 1}{(2\alpha - 1)[F(z) - 1] + 2\alpha(1 - \gamma)e^{-i\beta} cos\beta} \right| < 1 \right\},$$

where $F(z) = \frac{z(\mathcal{L}f)'(z) + \lambda z^2 (\mathcal{L}f)''(z)}{(1-\lambda)\mathcal{L}f(z) + \lambda z(\mathcal{L}f)'(z)}$.

Example 3: If $\mu = 1, b = \nu(\nu > -1)$ and $f(z)$is as defined in (1.10), then

$$\mathcal{P}_1^0(\alpha, \beta, \gamma) \equiv \mathcal{F}_{\nu}^{\lambda}(\alpha, \beta, \gamma) := \left\{ f \in A : \left| \frac{F(z) - 1}{(2\alpha - 1)[F(z) - 1] + 2\alpha(1 - \gamma)e^{-i\beta} cos\beta} \right| < 1 \right\},$$

where $F(z) = \frac{z(\mathcal{B}f)'(z) + \lambda z^2 (\mathcal{B}f)''(z)}{(1-\lambda)\mathcal{B}f(z) + \lambda z(\mathcal{B}f)'(z)}$.

Example 4: If $\mu = \sigma(\sigma > 0)$, $b = 1$ and $f(z)$ is defined in (1.11), then

$$\mathcal{P}_{\sigma}^1(\alpha, \beta, \gamma) \quad \equiv \quad \mathcal{I}_{\lambda}^{\sigma}(\alpha, \beta, \gamma) := \left\{ f \in A : \left| \frac{F(z) - 1}{(2\alpha - 1)[F(z) - 1] + 2\alpha(1 - \gamma)e^{-i\beta} cos\beta} \right| < 1 \right\}.$$

where $F(z) = \frac{z(\mathcal{I}^{\sigma} f)'(z) + \lambda z^2 (\mathcal{I}^{\sigma} f)''(z)}{(1-\lambda)\mathcal{I}^{\sigma} f(z) + \lambda z(\mathcal{I}^{\sigma} f)'(z)}$.

Similarly when $\lambda = 0$ and $\lambda = 1$ one can state various subclasses of starlike and convex functions respectively.

The main object of this paper is to study the coefficient bounds,for functions belong to the generalized class $\mathcal{P}_{\mu}^{\lambda}(\alpha, \beta, \gamma)$. Furthermore, Distortion theorems for the fractional derivative and fractional integration are obtained. Also we get result about coefficient inequality.

2. Coefficient Bounds

In this section we obtain a necessary and sufficient condition for functions $f(z)$ in the classes $P_{\mu}^{\lambda}(\alpha, \beta, \gamma)$

Theorem 2.1. *A function $f(z)$ of the form (1.1) is in $P_\mu^\lambda(\alpha,\beta,\gamma)$ if and only if*

$$\sum_{n=2}^{\infty}[(n-1)(1-\alpha)+\alpha(1-\gamma)|e^{-i\beta}cos\beta|](1-\lambda+\lambda n)|C_n(b,\mu)||a_n| \leq \alpha(1-\gamma)|e^{-i\beta}cos\beta|, \tag{2.1}$$

$0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$, $0 \leq \gamma < 1$, *and* $|\beta| < \frac{\pi}{2}$ *where* $C_n(b,\mu)$ *is given by (1.7).*

Proof. For $|z| = 1$, we have

$$|F(z)-1| - |(2\alpha-1)[F(z)-1] + 2\alpha(1-\gamma)e^{-i\beta}cos\beta| = |\sum_{n=2}^{\infty}[1-\lambda+n\lambda](n-1)C_n(b,\mu)a_nz^n|$$

$$-|2\alpha(1-\gamma)e^{-i\beta}cos\beta z - \sum_{n=2}^{\infty}[1-\lambda+n\lambda][(1-n)(1-2\alpha)-2\alpha(1-\gamma)e^{-i\beta}cos\beta]C_n(b,\mu)a_nz^n|$$

$$\leq \sum_{n=2}^{\infty}[1-\lambda+n\lambda](n-1)C_n(b,\mu)a_n - 2\alpha(1-\gamma)e^{-i\beta}cos\beta$$

$$-\sum_{n=2}^{\infty}[1-\lambda+n\lambda][(n-1)(1-2\alpha)-2\alpha(1-\gamma)e^{-i\beta}cos\beta]C_n(b,\mu)a_n$$

$$= \sum_{n=2}^{\infty}[(n-1)(1-\alpha)+\alpha(1-\gamma)|e^{-i\beta}cos\beta|][1-\lambda+n\lambda]C_n(b,\mu)a_n - \alpha(1-\gamma)|e^{-i\beta}cos\beta| \leq 0,$$

by hypothesis.Thus by Maximum Modulus theorem$f \in P_\mu^\lambda(\alpha,\beta,\gamma)$. Conversely, assume

$$\left|\frac{F(z)-1}{(2\alpha-1)[F(z)-1]+2\alpha(1-\gamma)e^{-i\beta}cos\beta}\right|$$

$$= \left|\frac{\frac{z(\mathcal{J}_b^\mu f)'(z)+\lambda z^2(\mathcal{J}_b^\mu f)''(z)}{(1-\lambda)(\mathcal{J}_b^\mu f)(z)+\lambda z(\mathcal{J}_b^\mu f)'(z)} - 1}{(2\alpha-1)[\frac{z(\mathcal{J}_b^\mu f)'(z)+\lambda z^2(\mathcal{J}_b^\mu f)''(z)}{(1-\lambda)\mathcal{J}_b^\mu f(z)+\lambda z(\mathcal{J}_b^\mu f)'(z)} - 1] + 2\alpha(1-\gamma)e^{-i\beta}cos\beta}\right| < 1.$$

Substituting the values of $\mathcal{J}_b^\mu f(z), (\mathcal{J}_b^\mu f(z))'$ $(\mathcal{J}_b^\mu f(z))''$ and simple computation yields,

$$\left|\frac{\sum_{n=2}^{\infty}[1-\lambda+n\lambda](n-1)C_n(b,\mu)a_nz^n}{2\alpha(1-\gamma)e^{-i\beta}cos\beta z - \sum_{n=2}^{\infty}[1-\lambda+n\lambda][(n-1)(1-2\alpha)-2\alpha(1-\gamma)e^{-i\beta}cos\beta]C_n(b,\mu)a_nz^n}\right| < 1.$$

$$Re\left(\frac{\sum_{n=2}^{\infty}[1-\lambda+n\lambda](n-1)C_n(b,\mu)a_nz^n}{2\alpha(1-\gamma)e^{-i\beta}cos\beta z - \sum_{n=2}^{\infty}[1-\lambda+n\lambda][(n-1)(1-2\alpha)-2\alpha(1-\gamma)e^{-i\beta}cos\beta]C_n(b,\mu)a_nz^n}\right) < 1.$$

We can choose value of z on the real axis so that $\mathcal{J}_b^\mu f(z)$ is real and letting $z \to 1^-$ through real values, so we can write the above inequality as

$$\sum_{n=2}^{\infty}[(n-1)(1-\alpha)+\alpha(1-\gamma)|e^{-i\beta}cos\beta|](1-\lambda+\lambda n)|C_n(b,\mu)||a_n| \leq \alpha(1-\gamma)|e^{-i\beta}cos\beta|$$

and hence the proof is complete. □

Corollary 2.2. *Let $f(z)$ to be in the class $P_\mu^\lambda(\alpha,\beta,\gamma)$ then*

$$a_n \leq \frac{\alpha(1-\gamma)|e^{-i\beta}cos\beta|}{[(n-1)(1-\alpha)+\alpha(1-\gamma)|e^{-i\beta}cos\beta|](1-\lambda+n\lambda)C_n(b,\mu)}, \tag{2.2}$$

and it is sharp for the function

$$f(z) = z + \frac{\alpha(1-\gamma)|e^{-i\beta}cos\beta|}{[(n-1)(1-\alpha)+\alpha(1-\gamma)|e^{-i\beta}cos\beta|](1-\lambda+n\lambda)C_n(b,\mu)} z^2, z \in U. \tag{2.3}$$

For $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$, $0 \leq \gamma < 1$, and $|\beta| < \frac{\pi}{2}$, in view of the Examples 1 to 4 in Section 1 and Theorem 2.2, we have following corollaries for the classes defined in these examples.

Corollary 2.3. *A necessary and sufficient condition for $f(z)$ of the form (1.1) to be in the class $\mathcal{P}(\alpha,\beta,\gamma)$, is that*

$$\sum_{n=2}^{\infty}[(n-1)(1-\alpha)+\alpha(1-\gamma)|e^{-i\beta}cos\beta|](1-\lambda+\lambda n)|a_n| \leq \alpha(1-\gamma)|e^{-i\beta}cos\beta|$$

Corollary 2.4. *A necessary and sufficient condition for $f(z)$ of the form (1.1) to be in the class $\mathcal{P}_\lambda(\alpha,\beta,\gamma)$, is that*

$$\sum_{n=2}^{\infty}[(n-1)(1-\alpha)+\alpha(1-\gamma)|e^{-i\beta}cos\beta|](1-\lambda+\lambda n)|a_n| \leq \alpha(1-\gamma)|e^{-i\beta}cos\beta|,$$

Corollary 2.5. *A necessary and sufficient condition for $f(z)$ of the form (1.1) to be in the class $\mathcal{L}_\lambda(\alpha,\beta,\gamma)$, is that*

$$\sum_{n=2}^{\infty}[(n-1)(1-\alpha)+\alpha(1-\gamma)|e^{-i\beta}cos\beta|](1-\lambda+\lambda n)\frac{|a_n|}{n} \leq \alpha(1-\gamma)|e^{-i\beta}cos\beta|$$

Corollary 2.6. *A necessary and sufficient condition for $f(z)$ of the form (1.1) to be in the class $\mathcal{F}_\nu^\lambda(\alpha,\beta,\gamma)$, is that*

$$\sum_{n=2}^{\infty}[(n-1)(1-\alpha)+\alpha(1-\gamma)|e^{-i\beta}cos\beta|](1-\lambda+\lambda n)\left(\frac{1+\nu}{n+\nu}\right)|a_n| \leq \alpha(1-\gamma)|e^{-i\beta}cos\beta|$$

Corollary 2.7. *A necessary and sufficient condition for $f(z)$ of the form (1.1) to be in the class $\mathcal{I}_\lambda^\sigma(\alpha,\beta,\gamma)$, is that*

$$\sum_{n=2}^{\infty}[(n-1)(1-\alpha)+\alpha(1-\gamma)|e^{-i\beta}cos\beta|](1-\lambda+\lambda n)\left(\frac{2}{n+1}\right)^\sigma a_n \leq \alpha(1-\gamma)|e^{-i\beta}cos\beta|.$$

Similarly by taking $\lambda = 0$ and $\lambda = 1$ many known results can be obtained as particular cases so we omit stating the particular cases for the above corollaries.

3. Fractional Calculus and Distortion Bounds

We recall the following definitions due to Srivastava and Owa [19] .

Definition 1. *Let the function $f(z)$ be analytic in a simply - connected region of the $z-$ plane containing the origin. The fractional integral of f of order δ is defined by*

$$D_z^{-\delta}f(z) = \frac{1}{\Gamma(\delta)}\int_0^z \frac{f(\xi)}{(z-\xi)^{1-\delta}}d\xi, \quad \delta > 0, \tag{3.1}$$

where the multiplicity of $(z-\xi)^{1-\delta}$ *is removed by requiring* $log(z-\xi)$ *to be real when* $z-\xi>0$.

Definition 2. *The fractional derivatives of order* δ*, is defined for a function* $f(z)$*, by*

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\delta}} d\xi, \quad 0 \le \delta < 1, \tag{3.2}$$

where the function $f(z)$ *is constrained, and the multiplicity of the function* $(z-\xi)^{-\delta}$ *is removed as in Definition 3.10.*

Definition 3. *Under the hypothesis of Definition 3.11, the fractional derivative of order* $n+\delta$ *is defined by*

$$D_z^{n+\delta} f(z) = \frac{d^n}{dz^n} D_z^{\delta} f(z), \quad (0 \le \delta < 1 \ ; \ n \in N_0). \tag{3.3}$$

Theorem 3.1. *Let* $f(z)$ *to be in the class* $P_{\mu}^{\lambda}(\alpha,\beta,\gamma)$ *then*

$$|D_z^{-\delta} f(z)| \le \frac{1}{\Gamma(2+\delta)} |z|^{1+\delta} \left[1 + \frac{2\alpha(1-\gamma)|e^{-i\beta}cos\beta||z|}{(2+\delta)(1+\alpha((1-\gamma)|e^{-i\beta}cos\beta|-1)(1+\lambda)C_n(b,\mu))} \right] \tag{3.4}$$

$$|D_z^{-\delta} f(z)| \ge \frac{1}{\Gamma(2+\delta)} |z|^{1+\delta} \left[1 - \frac{2\alpha(1-\gamma)|e^{-i\beta}cos\beta||z|}{(2+\delta)(1+\alpha((1-\gamma)|e^{-i\beta}cos\beta|-1)(1+\lambda)C_n(b,\mu))} \right] \tag{3.5}$$

Proof. By using Theorem 2.2,we have

$$\sum_{n=2}^{\infty} a_n \le \frac{\alpha(1-\gamma)|e^{-i\beta}cos\beta|}{[(n-1)(1-\alpha)+\alpha(1-\gamma)|e^{-i\beta}cos\beta|](1-\lambda+n\lambda)C_n(b,\mu)} \tag{3.6}$$

By Definition 1 we have

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(2+\delta)} z^{1+\delta} + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1+\delta)} a_n z^{1+\delta} \tag{3.7}$$

and

$$\Gamma(2+\delta) z^{-\delta} D_z^{-\delta} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)} a_n z^n = z + \sum_{n=2}^{\infty} \theta(n) a_z z^n \tag{3.8}$$

where

$$\theta(n) = \frac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)}. \tag{3.9}$$

We know that $\theta(n)$ is a decreasing function of n and $0 < \theta(n) \le \theta(2) = \frac{2}{2+\delta}$.

Using (3.6)and (3.8) we have

$$\Gamma(2+\delta) z^{-\delta} D_z^{-\delta} f(z)| \le |z| + \theta(2)|z|^2 \sum_{n=2}^{\infty} a_n \le |z| + \frac{2\alpha(1-\gamma)|e^{-i\beta}cos\beta||z|^2}{(2+\delta)(1+\alpha((1-\gamma)|e^{-i\beta}cos\beta|-1)(1+\lambda)C_n(b,\mu)}$$

which gives(3.4);we also have

$$|\Gamma(2+\delta) z^{-\delta} D_z^{-\delta} f(z)| \ge |z| + \theta(2)|z|^2 \sum_{n=2}^{\infty} a_n \ge |z| - \frac{2\alpha(1-\gamma)|e^{-i\beta}cos\beta||z|^2}{(2+\delta)(1+\alpha((1-\gamma)|e^{-i\beta}cos\beta|-1)(1+\lambda)C_n(b,\mu)}$$

which gives(3.5). □

Theorem 3.2. *Let $f(z)$ to be in the class $P_{\mu}^{\lambda}(\alpha,\beta,\gamma)$ then*

$$|D_z^{\delta}f(z)| \leq \frac{1}{\Gamma(2-\delta)}|z|^{1-\delta}\left[1+\frac{2\alpha(1-\gamma)|e^{-i\beta}cos\beta||z|}{(2-\delta)(1+\alpha((1-\gamma)|e^{-i\beta}cos\beta|-1)(1+\lambda)C_n(b,\mu))}\right] \tag{3.10}$$

$$|D_z^{\delta}f(z)| \geq \frac{1}{\Gamma(2-\delta)}|z|^{1-\delta}\left[1-\frac{2\alpha(1-\gamma)|e^{-i\beta}cos\beta||z|}{(2-\delta)(1+\alpha((1-\gamma)|e^{-i\beta}cos\beta|-1)(1+\lambda)C_n(b,\mu))}\right] \tag{3.11}$$

The inequalities in(3.10),(3.11) are attained for the function $f(z)$ given by(2.3)

Proof. From Definition 2, we have

$$D_z^{\delta}f(z) = \frac{z^{1-\delta}}{\Gamma(2-\delta)} + \sum_{n=2}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1-\delta)}a_n z^{n-\delta}$$

$$\Gamma(2-\delta)z^{\delta}D_z^{\delta} = z + \sum_{n=2}^{\infty}\frac{\Gamma(n+1)(2-\delta)}{\Gamma(n+1-\delta)}a_n z^z = z + \sum_{n=2}^{\infty}\phi(n)a_n z^n$$

where $\phi(n) = \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}$ for $n \geq 2$, $\phi(n)$ is a decreasing function of n, then

$$\phi(n) \leq \phi(2) = \frac{\Gamma(3)\Gamma(2-\delta)}{\Gamma(3-\delta)} = \frac{2}{2-\delta}. \tag{3.12}$$

By using Theorem 2.2 , we have

$$\sum_{n=2}^{\infty}a_n \leq \frac{\alpha(1-\gamma)|e^{-i\beta}cos\beta|}{[(n-1)(1-\alpha)+\alpha(1-\gamma)|e^{-i\beta}cos\beta|](1-\lambda+n\lambda)C_n(b,\mu)} \tag{3.13}$$

Thus

$$|\Gamma(2-\delta)z^{\delta}D_z^{\delta}f(z)| \leq |z|+\phi(2)|z|^2\sum_{n=2}^{\infty}a_n \leq |z|+\frac{2\alpha(1-\gamma)|e^{-i\beta}cos\beta||z|^2}{(2-\delta)(1+\alpha((1-\gamma)|e^{-i\beta}cos\beta|-1)(1+\lambda)C_2(b,\mu)}$$

which yields (3.10) and by same way we obtain

$$|\Gamma(2-\delta)z^{\delta}D_z^{-\delta}f(z)| \geq |z|-\phi(2)|z|^2\sum_{n=2}^{\infty}a_n \geq |z|-\frac{2\alpha(1-\gamma)|e^{-i\beta}cos\beta||z|^2}{(2-\delta)(1+\alpha((1-\gamma)|e^{-i\beta}cos\beta|-1)(1+\lambda)C_2(b,\mu)}$$

which gives(3.11). □

Corollary 3.3. *For every $f(z)$ to be in the class $P_{\mu}^{\lambda}(\alpha,\beta,\gamma)$ we have*

$$\frac{|z|^2}{2}\left[1-\frac{2\alpha(1-\gamma)|e^{-i\beta}cos\beta||z|}{3(1+\alpha((1-\gamma)|e^{-i\beta}cos\beta|-1)(1+\lambda)C_n(b,\mu)}\right] \leq |\int_0^2 f(t)dt| \tag{3.14}$$

$$\leq \frac{|z|^2}{2}\left[1+\frac{2\alpha(1-\gamma)|e^{-i\beta}cos\beta||z|}{3(1+\alpha((1-\gamma)|e^{-i\beta}cos\beta|-1)(1+\lambda)C_n(b,\mu)}\right]$$

and

$$|z|\left[1-\frac{\alpha(1-\gamma)|e^{-i\beta}cos\beta||z|}{(1+\alpha((1-\gamma)|e^{-i\beta}cos\beta|-1)(1+\lambda)C_n(b,\mu)}\right] \leq |f(z)|$$

$$\leq |z|\left[1+\frac{\alpha(1-\gamma)|e^{-i\beta}cos\beta||z|}{(1+\alpha((1-\gamma)|e^{-i\beta}cos\beta|-1)(1+\lambda)C_n(b,\mu)}\right] \tag{3.15}$$

Corollary 3.4. *$D_z^{-\delta}f(z)$ and $D_z^{\delta}f(z)$ are included in the disk with center at the origin and radii*

$$\frac{1}{\Gamma(2+\delta)}\left[1+\frac{2\alpha(1-\gamma)|e^{-i\beta}cos\beta||z|}{(2+\delta)(1+\alpha((1-\gamma)|e^{-i\beta}cos\beta|-1)(1+\lambda)C_2(b,\mu)}\right], \tag{3.16}$$

$$\frac{1}{\Gamma(2-\delta)}\left[1+\frac{2\alpha(1-\gamma)|e^{-i\beta}cos\beta||z|}{(2-\delta)(1+\alpha((1-\gamma)|e^{-i\beta}cos\beta|-1)(1+\lambda)C_2(b,\mu)}\right]. \tag{3.17}$$

The proof of the corollaries 3.1 and 3.2 follows immediately by Definitions 1 and 2 and Theorems 3.1 and 3.2for $\delta = 1$ and $\delta = 0$ respectively, hence we omit the details.

Concluding Remarks: By suitably specializing the various parameters involved in Theorem 2.2 to Theorem 3.2, one can state the corresponding results for many relatively more familiar function classes.

Acknowledgments. The authors would like to thank the referee(s) for his suggestions.

References

[1] J. W. Alexander, Functions which map the interior of the unit circle upon simple regions, Ann. of Math., **17**(1915), 1222.

[2] S. D. Bernardi, *Convex and starlike univalent functions,* Trans. Amer. Math. Soc., **135**(1969), 429–446.

[3] J. Choi and H. M. Srivastava, Certain families of series associated with the Hurwitz-Lerch Zeta function, Appl. Math. Comput., **170** (2005), 399-409.

[4] C. Ferreira and J. L. Lopez, Asymptotic expansions of the Hurwitz-Lerch Zeta function, J. Math. Anal. Appl., **298**(2004), 210-224.

[5] T. M. Flett, The dual of an inequality of Hardy and Littlewood and some related inequalities, J. Math. Anal. Appl., **38**(1972), 746765

[6] M. Garg, K. Jain and H. M. Srivastava, Some relationships between the generalized Apostol-Bernoulli polynomials and Hurwitz-Lerch Zeta functions, Integral Transform. Spec. Funct., **17**(2006), 803-815.

[7] I. B. Jung, Y. C. Kim AND H. M. Srivastava, The Hardy space of analytic functions associated with certain one-parameter families of integral operators, J. Math. Anal. Appl., **176**(1993), 138147.

[8] S.-D. Lin and H. M. Srivastava, Some families of the Hurwitz-Lerch Zeta functions and associated fractional derivative and other integral representations, Appl. Math. Comput., **154**(2004), 725-733.

[9] S.-D. Lin, H. M. Srivastava and P.-Y. Wang, Some espansion formulas for a class of generalized Hurwitz-Lerch Zeta functions, Integral Transform. Spec. Funct., **17**(2006), 817-827.

[10] G.Murugusundaramoorthy , Unified class of analytic functions with negative coefficients involving the Hurwitz-Lerch Zeta function ,Bulletin of Mathematical Analysis and Applications, Volume 1 Issue 3(2009), 71-84.

[11] G.Murugusundaramoorthy, Subordination results and integral means inequalities for k- uniformly starlike functions defined by convolution involving the Hurwitz-Lerch Zeta function ,Studia univ. "babes-bolyai", mathematica Volume LV, Number 4, December 2010

[12] G.Murugusundaramoorthy,A subclass of analytic functions associated with the Hurwitz-Lerch Zeta function , Hacettepe J. Maths .,(2010) , Vol.39, No.2, 265–272

[13] G. Murugusundaramoorthy, Subordination results for certain subclasses of starlike functions associated with Hurwitz-Lerch Zeta function ,Advan. Studies in Contemp. Math. Vol ., 21,No.1,95-105(2011)

[14] J. K. Prajapat and S. P. Goyal, Applications of Srivastava-Attiya operator to the classes of strongly starlike and strongly convex functions, J. Math. Inequal., **3**(2009), 129-137.

[15] D. Raducanu and H. M. Srivastava, A new class of analytic functions defined by means of a convolution operator involving the Hurwitz-Lerch Zeta function, Integral Transform. Spec. Funct., **18**(2007), 933-943.
[16] L. Spacek, Contribution a la theorie des fonctions univalentes, Casopis pest. Mat. 62 (1932), 12–19.
[17] H. M. Srivastava and A. A. Attiya, An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination, Integral Transform. Spec. Funct., **18**(2007), 207–216.
[18] H. M. Srivastava and J. Choi, Series associated with the Zeta and related functions, Dordrecht, Boston, London: Kluwer Academic Publishers, 2001.
[19] H.M.Srivastava and S. Owa, An application of the fractional derivative, Math. Japonica , **29**, (1984), 383 - -389.
[20] J. Zamorski, About the extremal sprial schlicht functions, Ann. Polon. Math. 9 (1962), 265–273.

M.K.Aouf
Mathematics Department , Faculty of Science,,, Mansoura 35516, Egypt
E-mail address: mkaouf127@yahoo.com

G.Murugusundaramoorthy
School of Advanced Sciences , VIT University,, Vellore - 632014, India.
E-mail address: gmsmoorthy@yahoo.com

K.Thilagavathi
School of Advanced Sciences , VIT University,, Vellore - 632014, India.