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FIXED POINT THEOREMS FOR A CLASS OF CONTRACTIONS IN METRIC SPACES

(COMMUNICATED BY SIMEON RICH)

GBENGA AKINBO

Abstract. We prove some convergence and stability theoremss for Picard and Mann iteration for a general class of contractions. Our results generalize some of the results in the literature.

1. Introduction

The introduction of a general class of contractions, called A-contractions, by Akram *et al* [2] in 2008 represented an appreciable extension and generalization of the Banach's fixed point theorem. This class of contractions is defined as follows: Let A denote the set of all functions $\alpha : \mathbf{R}_+^3 \longrightarrow \mathbf{R}_+$ satisfying the following conditions.
(i) α is continuous on the set $\mathbf{R}_+^3$;
(ii) $a \leq kb$ for some $k \in [0,1)$ whenever $a \leq \alpha(a,b,b)$ or $a \leq \alpha(b,a,b)$ or $a \leq \alpha(b,b,a)$ for all $a, b \in \mathbf{R}_+$.
Then a self-mapping T of a metric space X is said to be an A-contraction if it satisfies $d(Tx,Ty) \leq \alpha(d(x,y), d(x,Tx), d(y,Ty))$ for all $x, y \in X$ and some $\alpha \in A$.

The authors of [2] proved existence of a unique fixed point of T in a complete metric space and showed that, given any initial approximation, the Picard iteration converges to the fixed point. Recently, Akinbo *et al* [1] obtained a more general result in which four maps $F, G, S, T : X \longrightarrow X$ satisfying $d(Sx, Ty) \leq \alpha(d(Gx, Fy), d(Gx, Sx), d(Fy, Ty))$ for all $x, y \in X$, and some $\alpha \in A$, were shown to have a unique common fixed point. Here, the metric space X does not have to be complete.

In both papers (interested readers may see [1] and [2]), the authors demonstrated how the class of A-contractions contains several other classes of contractions in the literature.

In this paper, we present some independent results which complement those found in [1] and [2]. We employ the following modified class of A-contractions and

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call it A'-contractions for the sake of identity.
We shall not impose the continuity condition (i) in obtaining our results.

Definition 1.1. A self-mapping T of a metric space X shall be called an A'-contraction if for some α satisfying condition (ii) above, the following condition holds:

$$d(Tx,Ty) \leq \alpha(d(x,y), d(x,Ty), d(y,Tx)) \quad for \;\; all \;\; x,y \in X \tag{1.1}$$

Remark. *It is clear that the fixed point of an A'-contraction is always unique if it exists. Indeed, suppose p and p' are fixed points of T, then* $d(p,p') = d(Tp,Tp') \leq \alpha(d(p,p'), d(p,Tp'), d(p',Tp))$ *so that* $d(p,p') \leq kd(p,p')$ *for some* $k \in [0,1)$. *This yields* $p = p'$.

Example The following functions $\alpha : \mathbf{R}_+^3 \longrightarrow \mathbf{R}_+$ satisfy condition (ii) above.
(a) $\alpha(u,v,w) = a \cdot max\{u,v,w\}$, where $a \in [0,1)$.
(b) $\alpha(u,v,w) = b(u+v+w)$, where $b \in [0,\frac{1}{3})$.

The following Lemma, which can be found in [3] and [4], shall be useful in **Section 3** of this paper.

Lemma 1.1. *Let δ be a real number satisfying $0 \leq \delta < 1$, and $\{\epsilon_n\}$ a positive sequence satisfying $\lim_{n\to\infty} \epsilon_n = 0$. Then for any positive sequence $\{u_n\}$ satisfying*

$$u_{n+1} \leq \delta u_n + \epsilon_n,$$

it follows that $\lim_{n\to\infty} u_n = 0$.

2. Convergence of the Picard iteration to the fixed point of A'-contractions

Theorem 2.1. *Let X be a metric space, and let $T : X \longrightarrow X$ belong to the class of A'-contractions. Suppose that the set $F_T = \{x : Tx = x\}$ of fixed points of T is not empty, then the Picard iterative sequence $\{x_n\}$ defined by*

$$x_{n+1} = Tx_n, \qquad n = 0,1,2,\ldots$$

converges to the unique fixed of T.

Proof. Let $F_T = \{u\}$. For any $x_0 \in X$, since T belongs to the class of A'-contractions, there exists $\alpha : \mathbf{R}_+^3 \longrightarrow \mathbf{R}_+$ satisfying *condition (ii)* such that,

$$\begin{aligned} d(u,x_{n+1}) &= d(Tu,Tx_n) \\ &\leq \alpha(d(u,x_n), d(u,Tx_n), d(Tu,x_n)) \\ &= \alpha(d(u,x_n), d(u,x_{n+1}), d(u,x_n)) \end{aligned} \tag{2.1}$$

This implies that there exists some $k \in [0,1)$ such that

$$d(u,x_{n+1}) \leq kd(u,x_n) \quad n = 0,1,2,... \tag{2.2}$$

Consequently, we have

$$d(u,x_{n+1}) \leq k^{n+1} d(u,x_0), \quad n = 0,1,2,....$$

That is, $\lim_{n\to\infty} d(u, x_n)$ exists. Suppose $\lim_{n\to\infty} d(u, x_n) = L > 0$. Then by (2.2), we have

$$L = \lim_{n\to\infty} d(u, x_{n+1}) \leq k \lim_{n\to\infty} d(u, x_n) = kL$$

That is, $(1-k)L \leq 0$. Contradiction.
Therefore, $\lim_{n\to\infty} d(u, x_n) = L = 0$. So that $\{x_n\} \to u$ as $n \to \infty$. □

Corollary 2.2. *Let T be a self-mapping of a metric space X satisfying*

$$d(Tx, Ty) \leq a(d(x, y)d(x, Ty)d(y, Tx))^{\frac{1}{3}} \tag{2.3}$$

for all $x, y \in X$, and some constant $a \in [0, 1)$. If the set $F_T = \{x : Tx = x\}$ of fixed points of T is not empty, then the Picard iterative sequence $\{x_n\}$ defined by

$$x_{n+1} = Tx_n, \qquad n = 0, 1, 2, \ldots$$

converges to the unique fixed point of T.

Proof. We observe that the inequality $3uvw \leq u^3 + v^3 + w^3$ holds for nonnegative numbers u, v and w. Let $u^3 = d(x, y)$, $v^3 = d(x, Ty)$ and $w^3 = d(y, Tx)$. Then for some $b \in [0, \frac{1}{3})$, we have

$$3b(d(x, y)d(x, Ty)d(y, Tx))^{\frac{1}{3}} \leq b(d(x, y) + d(x, Ty) + d(y, Tx)).$$

Consequently, from inequality (2.3) and the (b) part of our Example, there exists $\alpha : \mathbf{R}^{\mathbf{3}}_{+} \longrightarrow \mathbf{R}_{+}$ satisfying condition (ii) such that

$$\begin{aligned} d(Tx, Ty) &\leq a(d(x, y)d(x, Ty)d(y, Tx))^{\frac{1}{3}} \\ &\leq b(d(x, y) + d(x, Ty) + d(y, Tx)) \\ &= \alpha(d(x, y), d(x, Ty), d(y, Tx)), \end{aligned} \tag{2.4}$$

where $0 \leq 3b = a < 1$. Hence, by virtue of *Theorem 1*, the sequence defined above converges to the unique fixed point of T. □

Since the geometric mean of any pair of nonnegative numbers is not greater than their arithmetic mean, that is, $2\sqrt{uv} \leq u + v$, letting $u = d(x, y)$ and $v = d(x, Ty) + d(y, Tx)$ and following a similar procedure as in the proof of Corollary 2.2, we obtain

Corollary 2.3. *Let T be a self-mapping of a metric space X satisfying*

$$d(Tx, Ty) \leq h(d(x, y)d(x, Ty) + d(x, y)d(y, Tx))^{\frac{1}{2}} \tag{2.5}$$

for all $x, y \in X$, and some constant $h \in [0, \frac{2}{3})$. Suppose $F_T = \{x : Tx = x\}$ is a nonempty set, then the Picard iteration converges to the unique fixed point of T.

The proof of the following theorem is similar to that of Theorem 2.1.

Theorem 2.4. *Let X be a metric space, and let T be a selfmap of X such that the following inequality holds for some $\alpha : \mathbf{R}^{\mathbf{3}}_{+} \longrightarrow \mathbf{R}_{+}$ satisfying* **condition (ii)** *and constant $\lambda \geq 0$:*

$$d(Tx, Ty) \leq \alpha(d(x, y), d(x, Ty), d(y, Tx)) + \lambda d(x, Tx)d(y, Ty) \quad x, y \in X. \tag{2.6}$$

Suppose T has a fixed point in X, then the Picard iteration converges to the unique fixed of T.

Remark. *The results of Corollaries 2.2 and 2.3 still hold if $\lambda d(x, Tx)d(y, Ty)$ is added to the right hand side of inequalities (2.3) and (2.5).*

3. Stability of Picard iteration with respect to the A'-contractions

Let (X, d) be a metric space, $T : X \longrightarrow X$ and $x_0 \in X$. Suppose that the iteration procedure

$$x_{n+1} = f(T, x_n),\ n = 0, 1, 2, ... \tag{3.1}$$

converges to a fixed point u of T.
Let $\{y_n\}$ be an arbitrary sequence in X and set

$$\epsilon_n = d(y_{n+1}, f(T, y_n)),\ n = 0, 1, 2, ... \tag{3.2}$$

We say that the fixed point iteration process (3.1) is *T-stable* or *stable with respect to T* if and only if

$$\lim_{n\to\infty} \epsilon_n = 0 \Leftrightarrow \lim_{n\to\infty} y_n = u.$$

Theorem 3.1. *Let X be a metric space and T a selfmap of X satisfying the conditions of Theorem 2.4. If T has a fixed point, then the Picard iterative process is T-stable.*

Proof. We first observe that, for all $x \in X$, $d(u, Tx)$ is bounded above by $kd(u, x)$, where $k \in [0, 1)$. Indeed,

$$\begin{aligned} d(u, Tx) &= d(Tu, Tx) \\ &\le \alpha(d(u, x), d(u, Tx), d(x, Tu)) + \lambda d(u, Tu)d(x, Tx) \\ &= \alpha(d(u, x), d(u, Tx), d(u, x)), \end{aligned} \tag{3.3}$$

so that $d(u, Tx) \le kd(u, x)$.
Assume $\epsilon_n = d(y_{n+1}, Ty_n) = 0$. Then,

$$\begin{aligned} d(y_{n+1}, u) &\le d(y_{n+1}, Ty_n) + d(u, Ty_n) = \epsilon_n + d(u, Ty_n) \\ &\le \epsilon_n + kd(u, y_n). \end{aligned} \tag{3.4}$$

Now, using Lemma 1.1 with $\delta = k$, we have $\lim_{n\to\infty} y_n = 0$.
Conversely, assume $\lim_{n\to\infty} y_n = u$. Then,

$$\begin{aligned} \epsilon_n &= d(y_{n+1}, Ty_n) \\ &\le d(y_{n+1}, u) + d(u, Ty_n) \\ &\le d(y_{n+1}, u) + kd(u, y_n)). \end{aligned} \tag{3.5}$$

Consequently, $\lim_{n\to\infty} \epsilon_n = 0$. □

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Gbenga Akinbo
Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria
E-mail address: agnebg@yahoo.co.uk