Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 4 Issue 1 (2012.), Pages 197-207

TRIPLE FIXED POINTS IN ORDERED METRIC SPACES

(COMMUNICATED BY SIMEON REICH)

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Abstract. In this paper, we prove triple fixed point theorems in partially ordered metric spaces depended on another function. The presented results generalize the theorem of Berinde and Borcut [*Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, Nonlinear Anal. 74(15) (2011) 4889–4897*]. Also, we state some examples showing that our results are effective.

1. Introduction

Banach fixed point theorem and its applications are well known. Many authors have extended this theorem, introducing more general contractive conditions, which imply the existence of a fixed point. Existence of fixed points in ordered metric spaces was investigated in 2004 by Ran and Reurings [29], and then by Nieto and López [28]. For some other results in ordered metric spaces, see e.g. [3, 4, 5, 23, 24, 25, 26, 27].

Bhashkar and Lakshmikantham in [11] introduced the concept of a coupled fixed point of a mapping $F : X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered complete metric spaces. Later, various results on coupled fixed point have been obtained, see e.g. [1, 6, 7, 8, 12, 17, 18, 19, 20, 30].

On the other hand, Berinde and Borcut [10] introduced the concept of triple fixed point (see also the papers [2, 9, 31]). The following two definitions are from [10].

Definition 1.1. *Let $(X, \leq)$ be a partially ordered set and $F : X \times X \times X \to X$. The mapping F is said to has the mixed monotone property if for any $x, y, z \in X$*

$$x_1,\ x_2 \in X, \quad x_1 \leq x_2 \Longrightarrow F(x_1, y, z) \leq F(x_2, y, z),$$

$$y_1,\ y_2 \in X, \quad y_1 \leq y_2 \Longrightarrow F(x, y_1, z) \geq F(x, y_2, z),$$

$$z_1,\ z_2 \in X, \quad z_1 \leq z_2 \Longrightarrow F(x, y, z_1) \leq F(x, y, z_2),$$

2000 *Mathematics Subject Classification.* 46T99, 54H25, 47H10, 54E50.
Key words and phrases. Triple fixed point, ordered sets, metric spaces, nonlinear contractions.

Submitted January 29, 2012. Accepted March 7, 2012.

Definition 1.2. *Let $F : X \times X \times X \to X$. An element (x, y, z) is called a triple fixed point of F if*

$$F(x, y, z) = x, \quad F(y, x, y) = y \quad \text{and} \quad F(z, y, x) = z.$$

Also, Berinde and Borcut [10] proved the following result.

Theorem 1.1. *Let $(X, \leq, d)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F : X \times X \times X \to X$ such that F has the mixed monotone property and there exist $j, r, l \geq 0$ with $j + r + l < 1$ such that*

$$d(F(x, y, z), F(u, v, w)) \leq jd(x, u) + rd(y, v) + ld(z, w), \tag{1}$$

for any $x, y, z \in X$ for which $x \leq u$, $v \leq y$ and $z \leq w$. Suppose either F is continuous or X has the following property:

1. *if a non-decreasing sequence $x_n \to x$, then $x_n \leq x$ for all n,*
2. *if a non-increasing sequence $y_n \to y$, then $y \leq y_n$ for all n.*

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, z_0)$ and $z_0 \leq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that

$$F(x, y, z) = x, \quad F(y, x, y) = y \quad \text{and} \quad F(z, y, x) = z,$$

that is, F has a triple fixed point.

In this paper we give some triple fixed point theorems for mappings having the mixed monotone property in partially ordered metric spaces depended on another function which are generalization of the main results of Berinde and Borcut [10].

2. MAIN RESULTS

We start with the following definition (see e.g. [13, 14, 22, 21]).

Definition 2.1. *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be ICS if T is injective, continuous and has the property: for every sequence $\{x_n\}$ in X, if $\{Tx_n\}$ is convergent then $\{x_n\}$ is also convergent.*

Let Φ be the set of all functions $\phi : [0, \infty) \to [0, \infty)$ such that

1. ϕ is non-decreasing,
2. $\phi(t) < t$ for all $t > 0$,
3. $\lim_{r \to t^+} \phi(r) < t$ for all $t > 0$.

From now on, we denote $X^3 = X \times X \times X$. Our first result is given by the following:

Theorem 2.1. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $T : X \to X$ is an ICS mapping and $F : X^3 \to X$ is such that F has the mixed monotone property. Assume that there exists $\phi \in \Phi$ such that*

$$d(TF(x, y, z), TF(u, v, w)) \leq \phi\Bigg(\max\{d(Tx, Tu), d(Ty, Tv), d(Tz, Tw)\}\Bigg) \tag{2}$$

for any $x, y, z \in X$ for which $x \leq u$, $v \leq y$ and $z \leq w$. Suppose either

(a) *F is continuous, or*
(b) *X has the following property:*

(*i*) *if non-decreasing sequence* $x_n \to x$ *(respectively,* $z_n \to z$*), then* $x_n \leq x$ *(respectively,* $z_n \leq z$*) for all* n*,*

(*ii*) *if non-increasing sequence* $y_n \to y$*, then* $y_n \geq y$ *for all* n*.*

If there exist $x_0, y_0, z_0 \in X$ *such that* $x_0 \leq F(x_0, y_0, z_0)$*,* $y_0 \geq F(y_0, x_0, y_0)$ *and* $z_0 \leq F(z_0, y_0, x_0)$*, then there exist* $x, y, z \in X$ *such that*

$$F(x,y,z) = x, \quad F(y,x,y) = y \quad \text{and} \quad F(z,y,x) = z,$$

that is, F *has a triple fixed point.*

Proof. Let $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$ and $z_0 \leq F(z_0, y_0, x_0)$. Set

$$x_1 = F(x_0, y_0, z_0), \quad y_1 = F(y_0, x_0, y_0) \quad \text{and} \quad z_1 = F(z_0, y_0, x_0). \tag{3}$$

Continuing this process, we can construct sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X such that

$$x_{n+1} = F(x_n, y_n, z_n), \quad y_{n+1} = F(y_n, x_n, y_n) \quad \text{and} \quad z_{n+1} = F(z_n, y_n, x_n). \tag{4}$$

Since F has the mixed monotone property, then using a mathematical induction it is easy that

$$x_n \leq x_{n+1}, \quad y_{n+1} \leq y_n, \quad z_n \leq z_{n+1}, \text{ for } n = 0, 1, 2, ... \tag{5}$$

Assume for some $n \in \mathbb{N}$,

$$x_n = x_{n+1}, \quad y_n = y_{n+1} \quad \text{and} \quad z_n = z_{n+1},$$

then, by (4), (x_n, y_n, z_n) is a triple fixed point of F. From now on, assume for any $n \in \mathbb{N}$ that at least

$$x_n \neq x_{n+1} \quad \text{or} \quad y_n \neq y_{n+1} \quad \text{or} \quad z_n \neq z_{n+1}. \tag{6}$$

Since T is injective, then by (6), for any $n \in \mathbb{N}$

$$0 < \max\{d(Tx_n, Tx_{n+1}), d(Ty_n, Ty_{n+1}, d(Tz_n, Tz_{n+1})\}.$$

Due to (2) and (4), we have

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= d(TF(x_{n-1}, y_{n-1}, z_{n-1}), TF(x_n, y_n, z_n)) \\ &\leq \phi\Bigg(\max\{d(Tx_{n-1}, Tx_n), d(Ty_{n-1}, Ty_n), d(Tz_{n-1}, Tz_n)\}\Bigg) \end{aligned} \tag{7}$$

$$\begin{aligned} d(Ty_{n+1}, Ty_n) &= d(TF(y_n, x_n, y_n), TF(y_{n-1}, x_{n-1}, y_{n-1})) \\ &\leq \phi(\{d(Ty_{n-1}, Ty_n), d(Tx_{n-1}, Tx_n), d(Ty_{n-1}, Ty_n)\}) \\ &= \phi(\max\{d(Ty_{n-1}, Ty_n), d(Tx_{n-1}, Tx_n)\}) \\ &\leq \phi(\max\{d(Tz_{n-1}, Tz_n), d(Ty_{n-1}, Ty_n), d(Tx_{n-1}, Tx_n)\}), \end{aligned} \tag{8}$$

and

$$\begin{aligned} d(Tz_n, Tz_{n+1}) &= d(TF(z_{n-1}, y_{n-1}, x_{n-1}), TF(z_n, y_n, x_n)) \\ &\leq \phi(\max\{d(Tz_{n-1}, Tz_n), d(Ty_{n-1}, Ty_n), d(Tx_{n-1}, Tx_n)\}). \end{aligned} \tag{9}$$

Having in mind that $\phi(t) < t$ for all $t > 0$, so from (7)-(9) we obtain that

$$\begin{aligned} 0 <& \max\{d(Tx_n, Tx_{n+1}), d(Ty_n, Ty_{n+1}), d(Tz_n, Tz_{n+1})\} \\ \leq& \phi(\max\{d(Tz_{n-1}, Tz_n), d(Ty_{n-1}, Ty_n), d(Tx_{n-1}, Tx_n)\}) \\ <& \max\{d(Tz_{n-1}, Tz_n), d(Ty_{n-1}, Ty_n), d(Tx_{n-1}, Tx_n)\}. \end{aligned} \tag{10}$$

It follows that

$$\max\{d(Tx_n, Tx_{n+1}), d(Ty_n, Ty_{n+1}, d(Tz_n, Tz_{n+1})\}$$
$$< \max\{d(Tz_{n-1}, Tz_n), d(Ty_{n-1}, Ty_n), d(Tx_{n-1}, Tx_n)\}.$$

Thus, $\{\max\{d(Tx_n, Tx_{n+1}), d(Ty_n, Ty_{n+1}), d(Tz_n, Tz_{n+1})\}\}$ is a positive decreasing sequence. Hence, there exists $r \geq 0$ such that

$$\lim_{n\to+\infty} \max\{d(Tx_n, Tx_{n+1}), d(Ty_n, Ty_{n+1}), d(Tz_n, Tz_{n+1})\} = r.$$

Suppose that $r > 0$. Letting $n \to +\infty$ in (10), we obtain that

$$0 < r \leq \lim_{n\to+\infty} \phi(\max\{d(Tz_{n-1}, Tz_n), d(Ty_{n-1}, Ty_n), d(Tx_{n-1}, Tx_n)\}) = \lim_{t\to r^+} \phi(t) < r, \tag{11}$$

it is a contradiction. We deduce that

$$\lim_{n\to+\infty} \max\{d(Tx_n, Tx_{n+1}), d(Ty_n, Ty_{n+1}), d(Tz_n, Tz_{n+1})\} = 0. \tag{12}$$

We shall show that $\{Tx_n\}$, $\{Ty_n\}$ and $\{Tz_n\}$ are Cauchy sequences. Assume the contrary, that is, $\{Tx_n\}$, $\{Ty_n\}$ or $\{Tz_n\}$ is not a Cauchy sequence, that is,

$$\lim_{n,m\to+\infty} d(Tx_m, Tx_n) \neq 0, \quad \text{or} \quad \lim_{n,m\to+\infty} d(Ty_m, Ty_n) \neq 0,$$

or $\lim_{n,m\to+\infty} d(Tz_m, Tz_n) \neq 0$. This means that there exists $\varepsilon > 0$ for which we can find subsequences of integers (m_k) and (n_k) with $n_k > m_k > k$ such that

$$\max\{d(Tx_{m_k}, Tx_{n_k}), d(Ty_{m_k}, Ty_{n_k}), d(Tz_{m_k}, Tz_{n_k})\} \geq \varepsilon. \tag{13}$$

Further, corresponding to m_k we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (13). Then

$$\max\{d(Tx_{m_k}, Tx_{n_k-1}), d(Ty_{m_k}, Ty_{n_k-1}), d(Tz_{m_k}, Tz_{n_k-1})\} < \varepsilon. \tag{14}$$

By triangular inequality and (14), we have

$$\begin{aligned} d(Tx_{m_k}, Tx_{n_k}) &\leq d(Tx_{m_k}, Tx_{n_k-1}) + d(Tx_{n_k-1}, Tx_{n_k}) \\ &< \epsilon + d(Tx_{n_k-1}, Tx_{n_k}). \end{aligned}$$

Thus, by (12) we obtain

$$\lim_{k\to+\infty} d(Tx_{m_k}, Tx_{n_k}) \leq \lim_{k\to+\infty} d(Tx_{m_k}, Tx_{n_k-1}) \leq \varepsilon. \tag{15}$$

Similarly, we have

$$\lim_{k\to+\infty} d(Ty_{m_k}, Ty_{n_k}) \leq \lim_{k\to+\infty} d(Ty_{m_k}, Ty_{n_k-1}) \leq \varepsilon. \tag{16}$$

$$\lim_{k\to+\infty} d(Tz_{m_k}, Tz_{n_k}) \leq \lim_{k\to+\infty} d(Tz_{m_k}, Tz_{n_k-1}) \leq \varepsilon. \tag{17}$$

Again by (14), we have

$$\begin{aligned} d(Tx_{m_k}, Tx_{n_k}) &\leq d(Tx_{m_k}, Tx_{m_k-1}) + d(Tx_{m_k-1}, Tx_{n_k-1}) + d(Tx_{n_k-1}, Tx_{n_k}) \\ &\leq d(Tx_{m_k}, Tx_{m_k-1}) + d(Tx_{m_k-1}, Tx_{m_k}) \\ &\quad + d(Tx_{m_k}, Tx_{n_k-1}) + d(Tx_{n_k-1}, Tx_{n_k}) \\ &< d(Tx_{m_k}, Tx_{m_k-1}) + d(Tx_{m_k-1}, Tx_{m_k}) + \varepsilon + d(Tx_{n_k-1}, Tx_{n_k}). \end{aligned}$$

Letting $k \to +\infty$ and using (12), we get

$$\lim_{k\to+\infty} d(Tx_{m_k}, Tx_{n_k}) \leq \lim_{k\to+\infty} d(Tx_{m_k-1}, Tx_{n_k-1}) \leq \varepsilon. \tag{18}$$

$$\lim_{k\to+\infty} d(Ty_{m_k}, Ty_{n_k}) \leq \lim_{k\to+\infty} d(Ty_{m_k-1}, Ty_{n_k-1}) \leq \varepsilon. \tag{19}$$

$$\lim_{k\to+\infty} d(Tz_{m_k}, Tz_{n_k}) \leq \lim_{k\to+\infty} d(Tz_{m_k-1}, Tz_{n_k-1}) \leq \varepsilon. \tag{20}$$

Using (13) and (18)-(20), we have

$$\begin{aligned} &\lim_{k\to+\infty} \max\{d(Tx_{m_k}, Tx_{n_k}), d(Ty_{m_k}, Ty_{n_k}), d(Tz_{m_k}, Tz_{n_k})\} \\ &= \lim_{k\to+\infty} \max\{d(Tx_{m_k-1}, Tx_{n_k-1}), d(Ty_{m_k-1}, Ty_{n_k-1}), d(Tz_{m_k-1}, Tz_{n_k-1})\} \\ &= \varepsilon. \end{aligned} \tag{21}$$

Now, using inequality (2) we obtain

$$\begin{aligned} d(Tx_{m_k}, Tx_{n_k}) &= d(TF(x_{m_k-1}, y_{m_k-1}, z_{m_k-1}), TF(x_{n_k-1}, y_{n_k-1}, z_{n_k-1})) \\ &\leq \phi\Bigg(\max\{d(Tx_{m_k-1}, Tx_{n_k-1}), d(Ty_{m_k-1}, Ty_{n_k-1}), d(Tz_{m_k-1}, Tz_{n_k-1})\}\Bigg) \end{aligned} \tag{22}$$

$$\begin{aligned} d(Ty_{m_k}, Ty_{n_k}) &= d(TF(y_{m_k-1}, x_{m_k-1}, y_{m_k-1}), TF(y_{n_k-1}, x_{n_k-1}, y_{n_k-1})) \\ &\leq \phi\Bigg(\max\{d(Ty_{m_k-1}, Ty_{n_k-1}), d(Tx_{m_k-1}, Tx_{n_k-1})\}\Bigg) \end{aligned} \tag{23}$$

and

$$\begin{aligned} d(Tz_{m_k}, Tz_{n_k}) &= d(TF(z_{m_k-1}, y_{m_k-1}, x_{m_k-1}), TF(z_{n_k-1}, y_{n_k-1}, x_{n_k-1})) \\ &\leq \phi\Bigg(\max\{d(Tx_{m_k-1}, Tx_{n_k-1}), d(Ty_{m_k-1}, Ty_{n_k-1}), d(Tz_{m_k-1}, Tz_{n_k-1})\}\Bigg). \end{aligned} \tag{24}$$

We deduce from (22)-(24) that

$$\begin{aligned} &\max\{d(Tx_{m_k}, Tx_{n_k}), d(Ty_{m_k}, Ty_{n_k}), d(Tz_{m_k}, Tz_{n_k})\} \\ &\leq \phi(\max\{d(Tx_{m_k-1}, Tx_{n_k-1}), d(Ty_{m_k-1}, Ty_{n_k-1}), d(Tz_{m_k-1}, Tz_{n_k-1})\}). \end{aligned} \tag{25}$$

Letting $k \to +\infty$ in (25) and having in mind (21), we get that

$$0 < \varepsilon \leq \lim_{t\to\varepsilon^+} \phi(t) < \varepsilon,$$

it is a contradiction. Thus, $\{Tx_n\}$, $\{Ty_n\}$ and $\{Tz_n\}$ are Cauchy sequences in (X, d). Since X is a complete metric space, $\{Tx_n\}$, $\{Ty_n\}$ and $\{Tz_n\}$ are convergent sequences.

Since T is an ICS mapping, there exist $x, y, z \in X$ such that

$$\lim_{n\to+\infty} x_n = x, \quad \lim_{n\to+\infty} y_n = y, \quad \text{and} \quad \lim_{n\to+\infty} z_n = z. \tag{26}$$

Since T is continuous, we have

$$\lim_{n\to+\infty} Tx_n = Tx, \quad \lim_{n\to+\infty} Ty_n = Ty, \quad \text{and} \quad \lim_{n\to+\infty} Tz_n = Tz. \tag{27}$$

Suppose now the assumption (a) holds, that is, F is continuous. By (4), (26) and (27) we obtain

$$x = \lim_{n\to+\infty} x_{n+1} = \lim_{n\to+\infty} F(x_n, y_n, z_n) = F(\lim_{n\to+\infty} x_n, \lim_{n\to+\infty} y_n, \lim_{n\to+\infty} z_n) = F(x, y, z),$$

$$y = \lim_{n\to+\infty} y_{n+1} = \lim_{n\to+\infty} F(y_n, x_n, y_n) = F(\lim_{n\to+\infty} y_n, \lim_{n\to+\infty} x_n, \lim_{n\to+\infty} y_n) = F(y, x, y),$$

and

$$z = \lim_{n\to+\infty} z_{n+1} = \lim_{n\to+\infty} F(z_n, y_n, x_n) = F(\lim_{n\to+\infty} z_n, \lim_{n\to+\infty} y_n, \lim_{n\to+\infty} x_n) = F(z, y, x).$$

We have proved that F has a triple fixed point.

Suppose now the assumption (b) holds. Since $\{x_n\}$, $\{z_n\}$ are non-decreasing with $x_n \to x$, $z_n \to z$ and also $\{y_n\}$ is non-increasing with $y_n \to y$, then by assumption (b) we have

$$x_n \leq x, \quad y_n \geq y \quad \text{and} \quad z_n \leq z,$$

for all n. Consider now

$$\begin{aligned} d(Tx, TF(x,y,z)) \leq \;& d(Tx, Tx_{n+1}) + d(Tx_{n+1}, TF(x,y,z) \\ =\;& d(Tx, Tx_{n+1}) + d(TF(x_n, y_n, z_n), TF(x,y,z)) \\ &\leq d(Tx, Tx_{n+1}) + \phi(\max\{d(Tx_n, Tx), d(Ty_n, Ty), d(Tz_n, Tz)\}). \end{aligned} \tag{28}$$

Taking $n \to \infty$ and using (27), the right-hand side of (28) tends to 0, so we get that $d(Tx, TF(x,y,z)) = 0$. Thus, $Tx = TF(x,y,z)$ and since T is injective, we get that $x = F(x,y,z)$. Analogously, we find that

$$F(y,x,y) = y \quad \text{and} \quad F(z,y,x) = z.$$

Thus, we proved that F has a triple fixed point. This completes the proof of Theorem 2.1.

Corollary 2.1. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $T : X \to X$ is an ICS mapping and $F : X^3 \to X$ is such that F has the mixed monotone property. Assume that there exists $\phi \in \Phi$ such that*

$$d(TF(x,y,z), TF(u,v,w)) \leq \phi\left(\frac{d(Tx,Tu) + d(Ty,Tv) + d(Tz,Tw)}{3}\right)$$

for any $x, y, z \in X$ for which $x \leq u$, $v \leq y$ and $z \leq w$. Suppose either

- *(a) F is continuous, or*
- *(b) X has the following property:*
  - *(i) if non-decreasing sequence $x_n \to x$ (respectively, $z_n \to z$), then $x_n \leq x$ (respectively, $z_n \leq z$) for all n,*
  - *(ii) if non-increasing sequence $y_n \to y$, then $y_n \geq y$ for all n.*

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$ and $z_0 \leq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that

$$F(x,y,z) = x, \quad F(y,x,y) = y \quad \text{and} \quad F(z,y,x) = z,$$

that is, F has a triple fixed point.

Proof. It suffices to remark that

$$\frac{d(Tx,Tu) + d(Ty,Tv) + d(Tz,Tw)}{3} \leq \max\{d(Tx,Tu), d(Ty,Tv), d(Tz,Tw)\}.$$

Then, we apply Theorem 2.1 because that ϕ is non-decreasing.

Corollary 2.2. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $T : X \to X$ is an*

ICS mapping and $F : X^3 \to X$ is such that F has the mixed monotone property. Assume that there exists $k \in [0,1)$ such that

$$d(TF(x,y,z), TF(u,v,w)) \leq k \max\{d(Tx,Tu), d(Ty,Tv), d(Tz,Tw)\}$$

for any $x, y, z \in X$ for which $x \leq u$, $v \leq y$ and $z \leq w$. Suppose either

- (a) *F is continuous, or*
- (b) *X has the following property:*
  - (i) *if non-decreasing sequence $x_n \to x$ (respectively, $z_n \to z$), then $x_n \leq x$ (respectively, $z_n \leq z$) for all n,*
  - (ii) *if non-increasing sequence $y_n \to y$, then $y_n \geq y$ for all n.*

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$ and $z_0 \leq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that

$$F(x,y,z) = x, \quad F(y,x,y) = y \quad \text{and} \quad F(z,y,x) = z,$$

that is, F has a triple fixed point.

Proof. It follows by taking $\phi(t) = kt$ in Theorem 2.1.

Corollary 2.3. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $T : X \to X$ is an ICS mapping and $F : X^3 \to X$ is such that F has the mixed monotone property. Assume that there exists $k \in [0,1)$ such that*

$$d(TF(x,y,z), TF(u,v,w)) \leq \frac{k}{3}(d(Tx,Tu) + d(Ty,Tv) + d(Tz,Tw)) \tag{29}$$

for any $x, y, z, u, v, w \in X$ for which $x \leq u$, $v \leq y$ and $z \leq w$. Suppose either

- (a) *F is continuous, or*
- (b) *X has the following property:*
  - (i) *if non-decreasing sequence $x_n \to x$ (respectively, $z_n \to z$), then $x_n \leq x$ (respectively, $z_n \leq z$) for all n,*
  - (ii) *if non-increasing sequence $y_n \to y$, then $y_n \geq y$ for all n.*

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$ and $z_0 \leq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that

$$F(x,y,z) = x, \quad F(y,x,y) = y \quad \text{and} \quad F(z,y,x) = z,$$

that is, F has a triple fixed point.

Proof. It suffices to take $\phi(t) = kt$ in Corollary 2.1.

Remark 1. *Taking $T = Id_X$, the identity on X, in Corollary 2.3, we get Theorem 1.1 of Berinde and Borcut (with $j = l = r = \frac{k}{3}$).*

Now, we shall prove the existence and uniqueness of a triple fixed point. For a product X^3 of a partially ordered set $(X, \leq)$, we define a partial ordering in the following way: For all $(x,y,z), (u,v,r) \in X^3$

$$(x,y,z) \leq (u,v,r) \Leftrightarrow x \leq u, \quad y \geq v \quad \text{and} \quad z \leq r. \tag{30}$$

We say that (x,y,z) and (u,v,w) are comparable if

$$(x,y,z) \leq (u,v,r) \quad \text{or} \quad (u,v,r) \leq (x,y,z).$$

Also, we say that (x,y,z) is equal to (u,v,r) if and only if $x = u$, $y = v$ and $z = r$.

Theorem 2.2. *In addition to hypothesis of Theorem 2.1, suppose that that for all* $(x, y, z), (u, v, r) \in X^3$*, there exists* $(a, b, c) \in X \times X \times X$ *such that* $(F(a, b, c), F(b, a, b), F(c, b, a))$ *is comparable to* $(F(x, y, z), F(y, x, y), F(z, y, x))$ *and* $(F(u, v, r), F(v, u, v), F(r, v, u))$*. Then,* F *has a unique triple fixed point* (x, y, z)*.*

Proof. The set of triple fixed points of F is not empty due to Theorem 2.1. Assume, now, (x, y, z) and (u, v, r) are two triple fixed points of F, that is,

$$\begin{array}{ll} F(x,y,z) = x, & F(u,v,r) = u, \\ F(y,x,y) = y, & F(v,u,v) = v, \\ F(z,y,x) = z, & F(r,v,u) = r. \end{array}$$

We shall show that (x, y, z) and (u, v, r) are equal. By assumption, there exists $(a, b, c) \in X^3$ such that $(F(a, b, c), F(b, a, b), F(c, b, a))$ is comparable to $(F(x, y, z), F(y, x, y), F(z, y, x))$ and $(F(u, v, r), F(v, u, v), F(r, v, u))$.

Define sequences $\{a_n\}, \{b_n\}$ and $\{c_n\}$ such that

$$a_0 = a, \quad b_0 = b, \quad c_0 = c, \quad \text{and for any} \quad n \geq 1$$

$$\begin{aligned} a_n &= F(a_{n-1}, b_{n-1}, c_{n-1}), \\ b_n &= F(b_{n-1}, a_{n-1}, b_{n-1}), \\ c_n &= F(c_{n-1}, b_{n-1}, a_{n-1}), \end{aligned} \tag{31}$$

for all n. Further, set $x_0 = x$, $y_0 = y$, $z_0 = z$ and $u_0 = u$, $v_0 = v$, $r_0 = r$, and on the same way define the sequences $\{x_n\}, \{y_n\}$, $\{z_n\}$ and $\{u_n\}, \{v_n\}$, $\{r_n\}$. Then, it is easy that

$$\begin{array}{ll} x_n = F(x,y,z), & u_n = F(u,v,r), \\ y_n = F(y,x,y,), & v_n = F(v,u,v), \\ z_n = F(z,y,x), & r_n = F(r,v,u), \end{array} \tag{32}$$

for all $n \geq 1$. Since $(F(x, y, z), F(y, x, y), F(z, y, x)) = (x_1, y_1, z_1) = (x, y, z)$ is comparable to $(F(a, b, c), F(b, a, b), F(c, b, a)) = (a_1, b_1, c_1)$, then it is easy to show $(x, y, z) \geq (a_1, b_1, c_1)$. Recursively, we get that

$$(x, y, z) \geq (a_n, b_n, c_n) \quad \text{for all } n. \tag{33}$$

By (33) and (2), we have

$$\begin{aligned} d(Tx, Ta_{n+1}) &= d(TF(x,y,z), TF(a_n, b_n, c_n)) \\ &\leq \phi(\max\{d(Tx, Ta_n), d(Ty, Tb_n), d(Tz, Tc_n)\}) \end{aligned} \tag{34}$$

$$\begin{aligned} d(Tb_{n+1}, Ty) &= d(TF(b_n, a_n, b_n), TF(y,x,y)) \\ &\leq \phi(\max\{d(Ta_n, Tx), d(Tb_n, Ty)\}) \\ &\leq \phi(\max\{d(Tb_n, Ty), d(Ta_n, Tx), d(Tc_n, Tz)\}), \end{aligned} \tag{35}$$

and

$$\begin{aligned} d(Tz, Tc_{n+1}) &= d(TF(z,y,x), TF(c_n, b_n, a_n) \\ &\leq \phi(\max\{d(Tz, Tc_n), d(Ty, Tb_n), d(Tx, Ta_n)\}); \end{aligned} \tag{36}$$

It follows from (34)-(36) that

$$\max\{d(Tz, Tc_{n+1}), d(Ty, Tb_{n+1}), d(Tx, Ta_{n+1})\} \leq \phi(\max\{d(Tz, Tc_n), d(Ty, Tb_n), d(Tx, Ta_n)\}).$$

Therefore, for each $n \geq 1$,

$$\max\{d(Tz, Tc_n), d(Ty, Tb_n), d(Tx, Ta_n)\} \leq \phi^n(\max\{d(Tz, Tc_0), d(Ty, Tb_0), d(Tx, Ta_0)\}). \tag{37}$$

It is known that $\phi(t) < t$ and $\lim_{r\to t^+} \phi(r) < t$ imply $\lim_{n\to\infty} \phi^n(t) = 0$ for each $t > 0$. Thus, from (37),

$$\lim_{n\to\infty} \max\{d(Tz, Tc_n), d(Ty, Tb_n), d(Tx, Ta_n)\} = 0.$$

This yields that

$$\begin{aligned} &\lim_{n\to\infty} d(Tx, Ta_n) = 0, \quad \lim_{n\to\infty} d(Ty, Tb_n) = 0, \\ &\lim_{n\to\infty} d(Tz, Tc_n) = 0. \end{aligned} \tag{38}$$

Analogously, we show that

$$\begin{aligned} &\lim_{n\to\infty} d(Tu, Ta_n) = 0, \quad \lim_{n\to\infty} d(Tv, Tb_n) = 0, \\ &\lim_{n\to\infty} d(Tr, Tc_n) = 0. \end{aligned} \tag{39}$$

Combining (38) and (39) yields that (Tx, Ty, Tz) and (Tu, Tv, Tr) are equal. The fact that T is injective gives us $x = u$, $y = v$ and $z = w$.

Now we state some examples showing that our results are effective.

Example 2.1. *Let $X = [\frac{1}{2}, 64]$ with the metric $d(x, y) = |x - y|$, for all $x, y \in X$ and the usual ordering $\leq$. Clearly, (X, d) is a complete metric space.*

Let $T : X \to X$ and $F : X^3 \to X$ be defined by

$$Tx = \ln(x) + 1 \quad \text{and} \quad F(x, y, z) = 8\left(\frac{\sqrt{xz}}{y}\right)^{\frac{1}{6}}, \quad \forall\, x, y, z \in X.$$

It is clear that T is an ICS mapping, F has the mixed monotone property and continuous.

Set $k = \frac{1}{2}$. Taking $x, y, z, u, v, w \in X$ for which $x \leq u$, $v \leq y$ and $z \leq w$, we have

$$\begin{aligned} d(TF(x, y, z), TF(u, v, w)) =& \frac{1}{12}|(\ln x + \ln z - 2\ln y) - (\ln u + \ln w - 2\ln v)| \\ \leq& \frac{1}{12}|\ln x - \ln u| + \frac{1}{6}|\ln y - \ln v| + \frac{1}{12}|\ln z - \ln w| \\ \leq& \frac{1}{6}\Bigg(|\ln x - \ln u| + |\ln y - \ln v| + |\ln z - \ln w|\Bigg) \\ =& \frac{k}{3}(d(Tx, Tu) + d(Ty, Tv) + d(Tz, Tw)), \end{aligned}$$

which is the contractive condition (29). Moreover, taking $x_0 = 1 = z_0$ and $y_0 = 64$, we have

$$x_0 \leq F(x_0, y_0, z_0), \quad y_0 \geq F(y_0, x_0, y_0) \quad \text{and} \quad z_0 \leq F(z_0, y_0, x_0).$$

Therefore, all the conditions of Corollary 2.3 hold and $(8, 8, 8)$ is the unique triple fixed point of F, since also the hypotheses of Theorem 2.2 hold.

On the other hand, we can not apply Theorem 1.1 to this example because the condition (1) does not hold (for $j = l = r = \frac{k}{3}$ where k is arbitrary in $[0, 1)$). Indeed, for $x = z = \frac{1}{2}$, $y = v = 64$ and $u = w = 1$, (1) becomes

$$\begin{aligned} d(F(x, y, z), F(u, v, w)) =& 8|(\frac{1}{2})^{\frac{7}{6}} - \frac{1}{2}| \\ \leq& \frac{k}{3}(d(x, u) + d(y, v) + d(z, w)) = \frac{k}{3}, \end{aligned}$$

that is, $k \geq 12(1 - (\frac{1}{2})^{\frac{1}{6}}) \sim 1,309 > 1$, which is a contradiction because of $k < 1$. We conclude that our results generalize the result of Berinde and Borcut given by Theorem 1.1.

Example 2.2. *Let $X = \mathbb{R}$ with $d(x,y) = |x-y|$ and natural ordering. Let $T : X \to X$ and $F : X^3 \to X$ be defined by $Tx = \frac{x}{12}$ and $F(x,y,z) = \frac{2}{9}(x-y+z)$. It is obvious that T is an ICS mapping, F has the mixed monotone property and continuous. Set $\phi(t) = \frac{2t}{3} \in \Phi$. Clearly, all conditions of Theorem 2.1 are satisfied and $(0,0,0)$ is the desired triple fixed point.*

Finally, following Example 2.9 in [21], we give a simple example which shows that if T is not an ICS mapping then the conclusion of Theorem 2.1 fails.

Example 2.3. *Let $X = \mathbb{R}$ with the usual metric and the usual ordering. Let $F : X^3 \to X$ be defined by*

$$F(x,y,z) = 2x - y + 1, \quad \text{for all}, \ \ x,y,z \in X.$$

then F has the mixed monotone property and F is continuous. Also, there exist $x_0 = 1$, $y_0 = 0$ and $z_0 = 1$ such that

$$x_0 \leq F(x_0,y_0,z_0), \quad y_0 \geq F(y_0,x_0,y_0) \quad \text{and} \quad z_0 \leq F(z_0,y_0,x_0).$$

Let $T : X \to X$ be defined by $T(x) = 1$ for all $x \in X$, then T is not an ICS mapping. It is obvious that the condition (2) holds for any $\phi \in \Phi$. However, F has no triple fixed point.

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