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FUNCTIONS OF α-SLOW INCREASE

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Abstract. The main aim of this paper is to generalize the functions of slow increase to α-slow increase for any $\alpha > 0$. We investigate some basic properties of functions of α-slow increase. In addition, the relationship between functions of α-slow increase and those of slow variation are characterized.

1. Introduction

Functions of α-slow increase are defined as follows.

Definition 1.1. *Let $f(x)$ be a function defined on the interval $[a,\infty)$ such that $f(x) > 0$, $\lim_{x\to\infty} f(x) = \infty$ and with continuous derivative $f'(x) > 0$. For $\alpha > 0$, the function $f(x)$ is of α-slow increase if the following condition holds:*

$$\lim_{x\to\infty} \frac{f'(x)}{\frac{f(x)}{x^\alpha}} = 0. \tag{1.1}$$

Note that the special case of 1-slow increase is introduced recently by R. Jakimczuk [3, 4] as a tool to investigate the asymptotic formula of Bell numbers. Further development on the subject can be found in e.g. [1, 5]. Typical examples for functions of α-slow increase are as follows:

- $f(x) = x$ is of α-slow increase with $\alpha < 1$.
- $f(x) = \ln x$ and $f(x) = \ln\ln x$ are of α-slow increase with $\alpha \le 1$.

In the next section, we will study some basic properties for functions of α-slow increase.

2. Some Properties

Theorem 2.1. *Suppose that $0 < \alpha_1 < \alpha_2$. If $f(x)$ is a function of α_2-slow increase, then it is of α_1-slow increase.*

Proof. It is straightforward to see this by using (1.1). □

Theorem 2.2. *Let $\alpha_1, \alpha_2, \beta > 0$ and $C \in \mathbb{R}$. If $f(x)$ and $g(x)$ are functions of α_1-slow and α_2-slow increase, respectively, then the following statements are true.*

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- *$f(x)+C$, $Cf(x)$ and $f(x)^\beta$ are functions of α_1-slow increase.*
- *$f(x^\beta)$ is a function of $((\alpha_1-1)\beta+1)$-slow increase, if $(\alpha_1-1)\beta>-1$.*
- *$f(x)g(x)$ and $f(x)+g(x)$ are functions of $\min\{\alpha_1,\alpha_2\}$-slow increase.*

Proof. We prove the second statement as an example. Others can be proved similarly.

By Definition 1.1, we have

$$\begin{aligned}\lim_{x\to\infty}\frac{x^{(\alpha_1-1)\beta+1}\frac{\mathrm{d}}{\mathrm{d}x}f(x^\beta)}{f(x^\beta)} &= \lim_{x\to\infty}\frac{\beta x^{\alpha_1\beta}f'(x^\beta)}{f(x^\beta)}\\ &= \lim_{y\to\infty}\frac{\beta y^{\alpha_1}f'(y)}{f(y)}=0, \qquad (2.1)\end{aligned}$$

which yields that $f(x^\beta)$ is of $((\alpha_1-1)\beta+1)$-slow increase if $(\alpha_1-1)\beta>-1$. □

Theorem 2.3. *If $f(x)$ is a function of α-slow increase for $\alpha\geq 1$, then the following limits hold.*

(i) $\lim_{x\to\infty}\frac{\ln f(x)}{\ln x}=0$;
(ii) $\lim_{x\to\infty}\frac{f(x)}{x^\beta}=0$ *for any* $\beta>0$;
(iii) $\lim_{x\to\infty}f'(x)=0$.

Proof. To show (i), we obtain by L'Hôspital's rule that

$$\lim_{x\to\infty}\frac{\ln f(x)}{\ln x}=\lim_{x\to\infty}\frac{f'(x)x}{f(x)}\leq\lim_{x\to\infty}\frac{f'(x)x^\alpha}{f(x)}=0, \qquad (2.2)$$

since $\alpha\geq 1$ by our assumption.

To see (ii), let $0<\gamma<\beta$. By virtue of (2.2), we have $f'(x)x/f(x)<\gamma$ for x large enough. Hence,

$$\left(\frac{f(x)}{x^\gamma}\right)'=\frac{f'(x)x^\gamma-\gamma x^{\gamma-1}f(x)}{x^{2\gamma}}<0, \qquad (2.3)$$

for large x. Thus, there exists some $0<M<\infty$ such that $0<f(x)/x^\gamma<M$. We obtain

$$\lim_{x\to\infty}\frac{f(x)}{x^\beta}=\lim_{x\to\infty}\frac{f(x)}{x^\gamma}\cdot\frac{1}{x^{\beta-\gamma}}=0. \qquad (2.4)$$

(iii) is an immediate consequence of (ii) and (1.1). □

Theorem 2.4. *Let $C\in\mathbb{R}$. If $f(x)$ is a function of α-slow increase for $\alpha\geq 1$, then*

$$\lim_{x\to\infty}\frac{f(x+C)}{f(x)}=1. \qquad (2.5)$$

Proof. We only prove the case $C>0$, and the case $C<0$ can be proved likewise. Applying the Lagrange mean value theorem, we have

$$0\leq\frac{f(x+C)-f(x)}{f(x)}=\frac{Cf'(\xi)}{f(x)}, \qquad (2.6)$$

for some $x<\xi<x+C$. Combining (2.6) with (iii) in Theorem 2.3 readily yields the limit (2.5). □

The following result characterizes the relationship between slowly varying functions (see e.g. [2] p. 275) and those of α-slow increase. A function $L(x)$ is said to be slowly varying if

$$\frac{L(tx)}{L(t)} \to 1, \tag{2.7}$$

as $t \to \infty$, for every $x > 0$. An application in scale-free networks can be found in [8].

Theorem 2.5. *Let $C \in \mathbb{R}$. If $f(x)$ is a function of α-slow increase for $\alpha \geq 1$ and $f'(x)$ is decreasing, then*

$$\lim_{x\to\infty} \frac{f(Cx)}{f(x)} = 1, \tag{2.8}$$

that is, $f(x)$ is slowly varying. On the other hand, if $f(x)$ is a slowly varying function with $\lim_{x\to\infty} f(x) = \infty$ and continuous derivative $f'(x) > 0$ and $f'(x)$ is increasing, then $f(x)$ is of α-slow increase for $\alpha \leq 1$.

Proof. Suppose that $f(x)$ is of α-slow increase and that $C > 1$. Applying the Lagrange mean value theorem, we have

$$\begin{aligned} 0 \leq \frac{f(Cx) - f(x)}{f(x)} &= \frac{(Cx - x)f'(\xi)}{f(x)} \\ &\leq \frac{(C-1)xf'(x)}{f(x)} \\ &\leq \frac{(C-1)x^\alpha f'(x)}{f(x)}, \end{aligned} \tag{2.9}$$

for some $x < \xi < Cx$. Combining (2.9) with Definition 1.1 gives the limit (2.8).

Now suppose that $C < 1$. Similarly, we can derive

$$\begin{aligned} 0 \leq \frac{f(x) - f(Cx)}{f(Cx)} &= \frac{(x - Cx)f'(\xi)}{f(Cx)} \\ &\leq \frac{1-C}{C} \cdot \frac{Cxf'(Cx)}{f(Cx)} \\ &\leq \frac{1-C}{C^\alpha} \cdot \frac{(Cx)^\alpha f'(Cx)}{f(Cx)}, \end{aligned} \tag{2.10}$$

for some $Cx < \xi < x$. Combining (2.10) with Definition 1.1 gives the limit (2.8).

On the other hand, assume that $f(x)$ satisfies (2.8), then by taking $C > 1$, we obtain

$$\begin{aligned} 0 \leq \frac{(C-1)x^\alpha f'(x)}{f(x)} &\leq \frac{(C-1)xf'(x)}{f(x)} \\ &\leq \frac{(C-1)xf'(\xi)}{f(x)} \\ &= \frac{f(Cx) - f(x)}{f(x)} \to 0, \end{aligned} \tag{2.11}$$

for some $x < \xi < Cx$ and $\alpha \leq 1$. Hence, $f(x)$ is a function of α-slow increase. □

Now recall a well-known lemma (see e.g. [6] p. 332).

Lemma 2.6. *If s_n is a sequence of positive numbers with limit s, then the sequence*

$$\sqrt[n]{s_1 s_2 \cdots s_n}$$

has also limit s.

We conclude the paper by presenting an analogous result for functions of α-slow increase.

Theorem 2.7. *If $f(x)$ is a function of α-slow increase on the interval $[a, \infty)$ then the following asymptotic formula holds*

$$\sqrt[n]{f(a)f(a+1)\cdots f(n)} \sim f(n), \tag{2.12}$$

where a is a positive number.

Proof. Without loss of generality, we assume $f(x) > 1$ on the interval $[a, \infty)$. Since $\ln f(x)$ is increasing and positive, we have by integration by parts

$$\begin{aligned} \sum_{i=a}^{n} \ln f(i) &= \int_a^n \ln f(x)\mathrm{d}x + O(\ln f(n)) \\ &= n \ln f(n) - \int_a^n \frac{xf'(x)}{f(x)}\mathrm{d}x + O(\ln f(n)). \end{aligned} \tag{2.13}$$

From (1.1) and the L'Hôspital rule, we derive that

$$\lim_{x\to\infty} \frac{\ln f(x)}{x} = \lim_{x\to\infty} \frac{f'(x)}{f(x)} = 0, \tag{2.14}$$

and hence

$$\ln f(n) = o(n). \tag{2.15}$$

If the integral $\int_a^x \frac{tf'(t)}{f(t)}\mathrm{d}t$ converges, we obtain

$$\lim_{x\to\infty} \frac{\int_a^x \frac{tf'(t)}{f(t)}\mathrm{d}t}{x} = 0. \tag{2.16}$$

On the other hand, if the integral $\int_a^x \frac{tf'(t)}{f(t)}\mathrm{d}t$ diverges, we have from (1.1) and the L'Hôspital rule that

$$\lim_{x\to\infty} \frac{\int_a^x \frac{tf'(t)}{f(t)}\mathrm{d}t}{x} = \lim_{x\to\infty} \frac{xf'(x)}{f(x)} = o(x^{1-\alpha}). \tag{2.17}$$

Accordingly, from (2.16) and (2.17) we obtain

$$\int_a^n \frac{xf'(x)}{f(x)}\mathrm{d}x = o(n^{1-\alpha}). \tag{2.18}$$

Eqs. (2.13), (2.15) and (2.18) imply that

$$\sum_{i=a}^{n} \ln f(i) = n \ln f(n) + o(n), \tag{2.19}$$

which is equivalent to

$$\frac{1}{n}\sum_{i=a}^{n} \ln f(i) = \ln f(n) + o(1). \tag{2.20}$$

The proof of the theorem is then complete. □

We mention that another generalization of Lemma 2.6 for prime numbers is provided in the work [7].

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