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ON A q-ANALOGUE OF THE ONE-DIMENSIONAL HEAT EQUATION

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Abstract. In this paper, a q-analogue of the one-dimensional heat equation associated with some q-differential operators is considered and a q-analogue of the theory of the heat equation introduced by P. C. Rosenbloom and D. V. Widder is developed.

1. Introduction

The solution of the heat equation arises as a modeling task in heat transfer and a variety of engineering, scientific, and financial applications. The best known analytic function theory associated with the heat equation is developed by P. C. Rosenbloom and D. V. Widder in [10, 11] and it is based on the heat polynomials and associated heat functions. The radial heat equation has been investigated by Bragg [1] and more extensively by Haimo [4]. These works have been generalized by Fitouhi [2] for singular operators. For many problems, the exact solution is not available or too complicated to use. Then, a numerical method is necessary for solving the problem. It is well known that the quantum calculus provides a natural discretization of the heat equation. For this discretization, we shall replace the partial derivatives $\dfrac{\partial}{\partial t}$ and $\dfrac{\partial}{\partial x}$ by D_{q^2} derivative [3] and the Rubin's ∂_q-derivative [8, 9] in time and in space, respectively, and we attempt to develop the q-analogue of the theory introduced by P. C. Rosenbloom and D. V. Widder. In this way, at the limit as q tends to 1, one recovers some results related to the heat equation in the continuous model.
We proclaim that, in this paper, we are not in a situation to study or discuss a numerical method, but we show by some examples and graphics that our results coincide with the classical ones when q is near 1.
This paper is organized as follows: in Section 2, we recall some notations and useful

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results. In Section 3, we define the generalized translation associated with the Rubin's ∂_q-operator and we establish some of its properties. In Section 4, we review a few of the most basic solutions of the one-dimensional classical heat equation. Next, we introduce the q-heat equation, and we present the q-source solution $k(x,t;q)$ and study some of its properties. Section 5 is devoted to construct and study two basic sets of solutions of the q-heat equation: the set $\{v_n(x,t;q)\}_{n=0}^{\infty}$ of q-heat polynomials and the q-associated functions set $\{w_n(x,t;q)\}_{n=0}^{\infty}$. In particular, we show that the q-heat polynomials and the q-associated functions are closely related to the discrete q-Hermite I polynomials and the discrete q-Hermite II polynomials, respectively. Furthermore, we introduce two systems of biorthogonal polynomials related to the q-source solution $k(x,t;q)$. In Section 6, we discuss an asymptotic estimations for the functions $v_n(x,t;q)$ and $w_n(x,t;q)$ for large n. Next, we establish some results related to the series expansion of solutions of q-heat equation. Finally, we discuss from an analytic and a graphic point of view how these q-difference operators can be used to solve approximately the heat equation and illustrate the performance of this approach with some examples.

2. Preliminaries

For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper. We refer the reader to the general references [3] and [5], for the definitions, notations and properties of the q-shifted factorials and the q-hypergeometric functions.
Throughout this paper, we assume $q \in]0,1[$ and we write

$$\mathbb{R}_q = \{\pm q^n \ : \ n \in \mathbb{Z}\}, \quad \mathbb{R}_{q,+} = \{q^n \ : \ n \in \mathbb{Z}\} \quad \text{and} \quad \widetilde{\mathbb{R}}_q = \mathbb{R}_q \cup \{0\}.$$

2.1. **Basic symbols.** For a complex number a, the q-shifted factorials are defined by:

$$(a;q)_0 = 1; \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), n = 1,2,...; \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

We also denote

$$[x]_q = \frac{1-q^x}{1-q}, \quad x \in \mathbb{C} \ \text{ and } \ n!_q = \frac{(q;q)_n}{(1-q)^n}, n \in \mathbb{N}.$$

It is easy to verify that

$$n!_{q^{-1}} = q^{-\frac{n(n-1)}{2}} n!_q. \tag{1}$$

Using the Gauss q-binomial coefficients (see [3])

$$\left(\begin{array}{c} n \\ k \end{array}\right)_q = \frac{n!_q}{k!_q (n-k)!_q},$$

the q-binomial theorem is given by

$$(-z;q)_n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right)_q q^{k(k-1)/2} z^k. \tag{2}$$

2.2. **Operators and elementary q-special functions.**
The Jackson's q-derivative is defined by (see [3, 5])

$$D_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & if \quad z \neq 0 \\ \lim\limits_{x \to 0} D_q f(x) & if \quad z = 0. \end{cases}$$

The Rubin's q-differential operator is defined in [8, 9] by

$$\partial_q(f)(z) = \begin{cases} \dfrac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{2(1-q)z} & if \quad z \neq 0 \\ \lim\limits_{x \to 0} \partial_q(f)(x) & if \quad z = 0. \end{cases} \tag{3}$$

Note that if f is differentiable at z, then $\partial_q(f)(z)$ and $D_q(f)(z)$ tend to $f'(z)$ as q tends to 1.

We state the following easily proved result:

Proposition 2.1. *For all $n \in \mathbb{N}$, we have*

(1) $\partial_q^{2n} f = q^{-n(n+1)} \left(D_q^{2n} f_e\right) o\Lambda_q^n + q^{-n^2} \left(D_q^{2n} f_o\right) o\Lambda_q^n,$

(2) $\partial_q^{2n+1} f = q^{-(n+1)^2} \left(D_q^{2n+1} f_e\right) o\Lambda_q^{(n+1)} + q^{-n(n+1)} \left(D_q^{2n+1} f_o\right) o\Lambda_q^n,$

where f_e and f_o are, respectively, the even and the odd parts of f, and Λ_q^n is the function defined by $\Lambda_q^n(x) = q^{-n}x$.

The q-Jackson integral (see [3]) is defined by

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x, \tag{4}$$

where

$$\int_0^a f(x) d_q x = a(1-q) \sum_{n=-\infty}^{\infty} f(aq^n) q^n, \tag{5}$$

and from 0 to $+\infty$ and from $-\infty$ to $+\infty$ are defined by

$$\int_0^\infty f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n, \tag{6}$$

$$\int_{-\infty}^\infty f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n + (1-q) \sum_{n=-\infty}^{\infty} f(-q^n) q^n, \tag{7}$$

provided the sums converge absolutely.
Note that when f is continuous on $[0, a]$, it can be shown that

$$\lim_{q \to 1} \int_0^a f(x) d_q x = \int_0^a f(x) dx. \tag{8}$$

The following results hold by direct computation.

Lemma 2.1.

(1) *If* $\displaystyle\int_{-\infty}^\infty f(x) d_q x$ *exists, then*

(a) *for all integer n,* $\displaystyle\int_{-\infty}^\infty f(q^n t) d_q t = q^{-n} \int_{-\infty}^\infty f(t) d_q t.$

(b) *if f is odd, then* $\int_{-\infty}^{\infty} f(t)d_qt = 0.$

(c) *if f is even, then* $\int_{-\infty}^{\infty} f(t)d_qt = 2\int_{0}^{\infty} f(t)d_qt.$

(2) *If* $\int_{-\infty}^{\infty} (\partial_q f)(t)g(t)d_qt$ *exists, then*

$$\int_{-\infty}^{\infty} (\partial_q f)(t)g(t)d_qt = -\int_{-\infty}^{\infty} f(t)(\partial_q g)(t)d_qt. \tag{9}$$

Notation. Using the Jackson's q-integral, we we denote by $L_q^p = L_q^p(\mathbb{R}_q)$, $p > 0$, the space of all complex functions defined on $\mathbb{R}_q$ induced by the norm

$$\|f\|_{p,q} = \left(\int_{-\infty}^{\infty} |f(x)|^p d_qx\right)^{\frac{1}{p}}.$$

Two q-analogues of the exponential function are given by (see [3])

$$E_q(z) := \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}}}{(q;q)_k} z^k = (-z;q)_\infty, \tag{10}$$

$$e_q(z) := \sum_{k=0}^{\infty} \frac{1}{(q;q)_k} z^k = \frac{1}{(z;q)_\infty} \quad |z| < 1. \tag{11}$$

E_q is entire on $\mathbb{C}$. But, for the convergence of the second series, we need $|z| < 1$; however, because of its product representation, e_q is continuable to a meromorphic function on $\mathbb{C}$ and has simple poles at $z = q^{-n}$, $n \in \mathbb{N}$.
We denote by

$$exp_q(z) := e_q((1-q)z) = \sum_{n=0}^{\infty} \frac{z^n}{n!_q} \tag{12}$$

and

$$Exp_q(z) := E_q((1-q)z) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} z^n}{n!_q}. \tag{13}$$

It follows from (1) that

$$Exp_{q^2}(z) = exp_{q^{-2}}(z). \tag{14}$$

We have (see [3])

$$\lim_{q\to 1^-} exp_q(z) = \lim_{q\to 1^-} Exp_q(z) = e^z, \tag{15}$$

where e^z is the classical exponential function.
The q-trigonometric functions (see[7]) are defined on $\mathbb{C}$ by

$$\cos(x;q^2) := \sum_{n=0}^{\infty} (-1)^n b_{2n}(x;q^2) \tag{16}$$

and

$$\sin(x;q^2) := \sum_{n=0}^{\infty} (-1)^n b_{2n+1}(x;q^2), \tag{17}$$

where

$$b_n(x;q^2) = \frac{q^{[\frac{n}{2}]([\frac{n}{2}]+1)}}{n!_q} x^n \tag{18}$$

and $[x]$ is the integer part of $x \in \mathbb{R}$.
These two functions induce a ∂_q-adapted q-analogue exponential function (see [8, 9]):

$$e(z;q^2) := \cos(-iz;q^2) + i\sin(-iz;q^2) = \sum_{n=0}^{\infty} b_n(z;q^2). \tag{19}$$

$e(z;q^2)$ is absolutely convergent for all z in the plane, and we have $\lim_{q\to 1^-} e(z;q^2) = e^z$ point-wise and uniformly on compacta. Note that we have

Lemma 2.2. *(see [8])*

$$\textit{For all}\quad \lambda \in \mathbb{C},\;\; \partial_q e(\lambda z;q^2) = \lambda e(\lambda z;q^2). \tag{20}$$

$$\textit{For all}\quad x \in \mathbb{R}_q,\quad |e(ix;q^2)| \leq \frac{2}{(q;q)_\infty}. \tag{21}$$

2.3. **The Fourier-Rubin transform.**

In [8] and [9], R. L. Rubin defined the Fourier-Rubin transform as

$$\mathcal{F}_q(f)(x) := K\int_{-\infty}^{\infty} f(t)e(-itx;q^2)d_qt, \quad x \in \widetilde{\mathbb{R}}_q, \tag{22}$$

where

$$K = \frac{(q;q^2)_\infty}{2(q^2;q^2)_\infty(1-q)^{\frac{1}{2}}}. \tag{23}$$

Letting $q \uparrow 1$ subject to the condition

$$\frac{Log(1-q)}{Log(q)} \in 2\mathbb{Z}, \tag{24}$$

gives, at least formally, the classical Fourier transform. In the remainder of this paper, we assume that the condition (24) holds.
It was shown in [8] and [9] that the Fourier-Rubin transform $\mathcal{F}_q$ satisfies the following properties:

Theorem 2.1.

(1) *If* $f, g \in L^1_q$, *then* $\int_{-\infty}^{\infty} \mathcal{F}_q(f)(x)g(x)d_qx = \int_{-\infty}^{\infty} f(x)\mathcal{F}_q(g)(x)d_qx.$

(2) *If* $f(u),\ uf(u) \in L^1_q$, *then*

$$\partial_q\mathcal{F}_q(f)(x) = \mathcal{F}_q(-iuf(u))(x).$$

(3) *If* $f,\ \partial_q f \in L^1_q$, *then*

$$\mathcal{F}_q(\partial_q f)(x) = ix\mathcal{F}_q(f)(x).$$

(4) $\mathcal{F}_q$ *is an isomorphism of* L^2_q, *satisfying for* $f \in L^2_q$

$$\|\mathcal{F}_q(f)\|_{L^2_q} = \|f\|_{L^2_q},$$

and for $t \in \mathbb{R}_q$,

$$f(t) = K\int_{-\infty}^{\infty} \mathcal{F}_q(f)(x)e(itx;q^2)d_qx. \tag{25}$$

3. The q-generalized translation operator associated with the operator ∂_q

Definition 3.1. *The q-translation operator $\mathcal{T}_{y,q}$, $y \in \mathbb{C}$, related to the q-differential operator ∂_q, is defined by*

$$\mathcal{T}_{y,q}(f)(x) := e(y\partial_q; q^2)f(x) = \sum_{n=0}^{\infty} b_n(y; q^2)\partial_q^n f(x), \tag{26}$$

provided the series converges point wise.

Remarks.

(1) Note that for suitable functions $f(x)$, we have

$$\lim_{q\to 1^-} \mathcal{T}_{q,y}(f)(x) = f(x+y).$$

(2) Since for all $n \in \mathbb{N}$, $b_n(.; q^2)$ is a polynomial of degree n, then for all $k \geq n+1$, we have $\partial_q^k b_n(x; q^2) = 0$. Then

$$p_{n,q}(x, y) := n!_q \mathcal{T}_{y,q} b_n(x; q^2) \tag{27}$$

is a polynomial of degree n, that it will be called q-binomial polynomial.

Proposition 3.1. *Let $x,\ y,\ \lambda \in \mathbb{C}$ and $n \in \mathbb{N}$. Then,*

$$\partial_q^k b_n(x; q^2) = b_{n-k}(x; q^2), \quad k = 0, 1, ...n. \tag{28}$$

$$\mathcal{T}_{y,q} b_n(x; q^2) = \sum_{k=0}^{n} b_k(y; q^2) b_{n-k}(x; q^2) = \frac{1}{n!_q} p_{n,q}(x, y). \tag{29}$$

$$\mathcal{T}_{y,q} e(\lambda x; q^2) = e(\lambda x; q^2) e(\lambda y; q^2). \tag{30}$$

The generating function of $\{p_{n,q}(.,.)\}_{n\in\mathbb{N}}$ is given by

$$e(\lambda x; q^2) e(\lambda y; q^2) = \sum_{n=0}^{\infty} \frac{p_{n,q}(x, y)}{n!_q} \lambda^n. \tag{31}$$

Proof. (28) follows easily from the fact that $\partial_q b_n(x; q^2) = b_{n-1}(x; q^2)$.
(29) is a consequence of the relations (28) and (26).
Using (20), we have

$$\begin{aligned}
\mathcal{T}_{y,q} e(\lambda x; q^2) &= \sum_{n=0}^{\infty} b_n(y; q^2)\partial_q^n e(\lambda x; q^2) \\
&= \sum_{n=0}^{\infty} b_n(y; q^2)\lambda^n e(\lambda x; q^2) \\
&= e(\lambda x; q^2) \sum_{n=0}^{\infty} b_n(\lambda y; q^2) = e(\lambda x; q^2) e(\lambda y; q^2).
\end{aligned}$$

This proves (30). (31) follows from (30) and (27). ∎

Lemma 3.1. *For $n = 0, 1, 2, ...,$ we have*

$$p_{2n,q}(|x|, |y|)| \leq \frac{(2n)!_q}{n!_{q^2}} (1 + |xy|) E_{q^2}(|x|^2) E_{q^2}(|y|^2) \tag{32}$$

and

$$p_{2n+1,q}(|x|,|y|) \leq \frac{(2n+1)!_q}{n!_{q^2}}(|x|+|y|)E_{q^2}(|x|^2)E_{q^2}(|y|^2), \tag{33}$$

where $E_q(.)$ is the q-exponential function given by (10).

Proof. From (29), we have

$$p_{2n,q}(|x|,|y|) = (2n)!_q \sum_{k=0}^{2n} b_k(|x|;q^2)b_{2n-k}(|y|;q^2).$$

But,

$$\begin{aligned}\sum_{k=0}^{2n} b_k(|x|;q^2)b_{2n-k}(|y|;q^2) \quad &= \quad \sum_{k=0}^{n} b_{2k}(|x|;q^2)b_{2(n-k)}(|y|;q^2) \\ &\quad + \sum_{k=0}^{n-1} b_{2k+1}(|x|;q^2)b_{2(n-k)-1}(|y|;q^2)\end{aligned}$$

and using the fact that $(2k)!_q \geq (k!_{q^2})^2$, the following inequalities hold

$$b_{2k}(|x|;q^2) \leq \frac{q^{k(k-1)}|x|^{2k}}{k!_{q^2}}, \quad b_{2k}(|y|;q^2) \leq \frac{E_{q^2}(|y|^2)}{k!_{q^2}}, \quad k=0,1,2,...$$

It follows then, by using the q-binomial theorem (2),

$$\begin{aligned}\sum_{k=0}^{n} b_{2k}(|x|;q^2)b_{2(n-k)}(|y|;q^2) \quad &\leq \quad \frac{E_{q^2}(|y|^2)}{n!_{q^2}} \sum_{k=0}^{n} \frac{n!_{q^2} q^{k(k-1)}}{(n-k)!_{q^2} k!_{q^2}} |x|^{2k} \\ &\leq \quad \frac{E_{q^2}(|y|^2)(-|x|^2;q^2)_n}{n!_{q^2}} \\ &\leq \quad \frac{E_{q^2}(|x|^2)E_{q^2}(|y|^2)}{n!_{q^2}}.\end{aligned} \tag{34}$$

Since $(2k+1)!_q \geq k!_{q^2}(k+1)!_{q^2}$, the following inequalities hold

$$b_{2k+1}(|x|;q^2) \leq \frac{|x|q^{k(k-1)}|x|^{2k}}{k!_{q^2}}, \quad k=0,1,2,...$$

$$b_{2k-1}(|y|;q^2) \leq \frac{|y|E_{q^2}(|y|^2)}{k!_{q^2}}, \quad k=1,2,3,...$$

Consequently,

$$\begin{aligned}\sum_{k=0}^{n-1} b_{2k+1}(|x|;q^2)b_{2(n-k)-1}(|y|;q^2) \quad &\leq \quad \frac{|xy|E_{q^2}(|y|^2)}{n!_{q^2}} \sum_{k=0}^{n} \frac{n!_{q^2} q^{k(k-1)}}{(n-k)!_{q^2} k!_{q^2}} |x|^{2k} \\ &\leq \quad |xy| \frac{E_{q^2}(|x|^2)E_{q^2}(|y|^2)}{n!_{q^2}}.\end{aligned}$$

This inequality together with (34) give (32).
Let us now prove the inequality (33). We have

$$p_{2n+1,q}(|x|,|y|) = (2n+1)!_q \sum_{k=0}^{2n+1} b_k(|x|;q^2)b_{2n+1-k}(|y|;q^2),$$

and

$$\begin{aligned}\sum_{k=0}^{2n+1} b_k(|x|;q^2)b_{2n+1-k}(|y|;q^2) &= \sum_{k=0}^{n} b_{2k}(|x|;q^2)b_{2(n-k)+1}(|y|;q^2)\\ &+\sum_{k=0}^{n} b_{2k+1}(|x|;q^2)b_{2(n-k)}(|y|;q^2).\end{aligned}$$

In the same way as in the proof of inequality (32), we obtain

$$\sum_{k=0}^{2n+1} b_k(|x|;q^2)b_{2n+1-k}(|y|;q^2) \le (|x|+|y|)\frac{E_{q^2}(|x|^2)E_{q^2}(|y|^2)}{n!_{q^2}}.$$

∎

Definition 3.2. *For $\sigma > 0$, we denote by $\mathcal{E}_{\sigma,q}$ the set of all entire functions f satisfying:*

$$\exists \quad M>0 \quad : \forall n\in\mathbb{N}, \qquad \begin{cases} |\partial_q^{2n} f(0)| & \le \dfrac{Mn!_{q^2}}{\sigma^n},\\ \\ |\partial_q^{2n+1} f(0)| \le \dfrac{Mn!_{q^2}}{\sigma^n}. \end{cases} \tag{35}$$

Remark. In the definition of the class $\mathcal{E}_{\sigma,q}$, the constant M is independent of n, depending only on the function f.

Proposition 3.2. *Let $\sigma > 1$ and f be in $\mathcal{E}_{\sigma,q}$. Then*

$$\mathcal{T}_{y,q}f(x) = \sum_{n=0}^{\infty} \frac{\partial_q^n f(0)}{n!_q} p_{n,q}(x,y), \tag{36}$$

where $p_{n,q}(x,y)$ are defined by (27).
The infinite series (36) converges locally uniformly in x and y.

Proof. First, if f is in $\mathcal{E}_{\sigma,q}$, then

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{a_n}{b_n(1;q^2)} b_n(x;q^2).$$

So, by (28), we have for all nonnegative integer k

$$\partial_q^k f(x) = \sum_{n=k}^{\infty} \frac{a_n}{b_n(1;q^2)} b_{n-k}(x;q^2), \tag{37}$$

from which we deduce that

$$\forall n\ge 0, \quad a_n = \partial_q^n f(0) b_n(1;q^2). \tag{38}$$

By (26), (37) and (38), we have

$$\begin{aligned}\mathcal{T}_{y,q}f(x) &= \sum_{k=0}^{\infty} b_k(y;q^2)\sum_{n=k}^{\infty}\frac{a_n}{b_n(1;q^2)}b_{n-k}(x;q^2)\\ &= \sum_{k=0}^{\infty} b_k(y;q^2)\sum_{n=k}^{\infty}\partial_q^n f(0)b_{n-k}(x;q^2)\\ &= \sum_{n=0}^{\infty}\partial_q^n f(0)\sum_{k=0}^{n} b_k(y;q^2)b_{n-k}(x;q^2).\end{aligned}\tag{39}$$

Then, the desired conclusion follows from the relation (29).
Let us now, prove the locally uniformly convergence in x and y of the series (36). From the definition of the function $b_n(.;q^2)$ and the relation (29), we obtain for all $x, y \in \mathbb{C}$,

$$|p_{n,q}(x,y)| \leq p_{n,q}(|x|,|y|).$$

Then, by using Lemma 3.1, we get, since $f \in \mathcal{E}_{\sigma,q}$, that there exists $M > 0$, such that for all $x, y \in \mathbb{C}$ and $n \geq 0$,

$$\left|\frac{\partial_q^{2n}f(0)|}{(2n)!_q}p_{2n,q}(x,y)\right| \leq \frac{M}{\sigma^n}(1+|xy|)E_{q^2}(|x|^2)E_{q^2}(|y|^2)$$

and

$$\left|\frac{\partial_q^{2n+1}f(0)|}{(2n+1)!_q}p_{2n+1,q}(x,y)\right| \leq \frac{M}{\sigma^n}(|x|+|y|)E_{q^2}(|x|^2)E_{q^2}(|y|^2).$$

Finally, these relations, the continuity of the function E_{q^2} and the hypothesis $\sigma > 1$ prove the locally uniformly convergence in x and y of the series (36). ■

4. q-heat Equation and q-source solution

4.1. **Heat Equation.** We restrict our attention to the simplest one-dimensional heat equation on $\mathbb{R}$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x},\tag{40}$$

which has been the object of extensive studies. In particular, we refer the reader to the book of Widder [10]. One of the most important families of solutions of the heat equation (40) is the so-called heat polynomials defined by

$$v_n(x,t) = n!\sum_{k=0}^{[\frac{n}{2}]}\frac{t^k}{k!(n-2k)!}x^{n-2k}, \quad n = 0,\ 1,\ 2, \ldots\tag{41}$$

The heat polynomials are closely related to the Hermite polynomials $H_n(x)$ by (see [11])

$$v_n(x,t) = (-t)^{n/2}H_n\left(\frac{x}{\sqrt{-4\pi t}}\right).$$

The source solution or fundamental solution of (40) is given by

$$k(x,t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}.$$

The associated functions $w_n(x,t)$ are defined by

$$w_n(x,t) = \frac{v_n(x,-t)k(x,t)}{t^n}, \quad n = 0,\ 1,\ 2....$$

They are solutions of the heat equation (40) and they are related to the Hermite polynomials $H_n(x)$ according to

$$w_n(x,t) = t^{-n/2}k(x,t)H_n(\frac{x}{\sqrt{4\pi t}}).$$

We have the following biorthogonality relation

$$\int_{-\infty}^{+\infty} v_n(x,-t)w_m(x,t)dx = 2^n n!\delta_{m,n},$$

where $\delta_{m,n}$ is the Kronecker symbol.
In [10, 11] a complete study of necessary and sufficient conditions for the validity of the expansion of solutions of the heat equation (40) in terms of heat polynomials and associated functions has been developed.

4.2. **The q-Heat Equation.** We consider the following q-heat equation:

$$D_{q^2,t}u = \partial_{q,x}^2 u. \tag{42}$$

Remark. Taking into account that $\lim_{q\to 1} D_{q^2,t}u(t,x) = \frac{\partial u}{\partial t}(t,x)$ and $\lim_{q\to 1}\partial_{q,x}^2 u(t,x) = \frac{\partial^2 u}{\partial^2 x}u(t,x)$, it is clear that (42) is q-analogue of the standard heat equation (40). That is, equation (40) can be recovered when q tends to 1.
Consider now, the following function

$$k(x,t;q) = C(t;q)exp_{q^2}\left(-\frac{qx^2}{t(1+q)^2}\right), \quad t > 0, \tag{43}$$

where $exp_{q^2}(.)$ is defined by (12) and

$$C(t;q) = \frac{\left(-\frac{q(1-q)}{t(1+q)}, -\frac{qt(1+q)}{1-q}, q; q^2\right)_\infty}{2(1-q)\left(-\frac{q^2(1-q)}{t(1+q)}, -\frac{t(1+q)}{1-q}, q^2; q^2\right)_\infty}. \tag{44}$$

Since

$$\partial_{q,x}k(x,t;q) = -\frac{C(t;q)x}{qt(1+q)}exp_{q^2}\left(-\frac{x^2}{tq(1+q)^2}\right) \tag{45}$$

and

$$\partial_{q,x}^2 k(x,t;q) = D_{q^2,t}k(x,t;q) = -\frac{C(t;q)}{qt(1+q)}\left(1-\frac{x^2}{t(1+q)}\right)exp_{q^2}\left(-\frac{x^2}{tq(1+q)^2}\right),$$

then $k(x,t;q)$ is a solution of the q-heat equation (42), that it will be called the q-source solution.
For discussing its properties, we need the following preliminary results.

Proposition 4.1.

(1) *The function $e_{q^2}(-x^2)$ has the rapid decreasing property:*

$$\lim_{x\to\infty} p_n(x)e_{q^2}(-x^2) = 0, \quad n = 0, 1, 2..., \tag{46}$$

for all polynomial $p_n(x)$ is a of degree n.

(2)

$$\lim_{x\to\infty}\frac{e_{q^2}(-qx^2)}{e_{q^2}(-x^2)}=+\infty. \tag{47}$$

(3) *For $\beta>0$, we have*

$$\lim_{x\to\infty}\frac{e_{q^2}(-qx^2)e_{q^2}(-\beta x^2)}{e_{q^2}(-x^2)}=0. \tag{48}$$

Proof. (1) Observe that for $n=0,1,2,...$, we have

$$e_{q^2}(-x^2)=\prod_{k=0}^{\infty}(1+q^{2k}x^2)^{-1}\le(1+q^{2n}x^2)^{-n}=O(x^{-2n})\ \text{ as } x\to\pm\infty.$$

(2) For $n=0,1,2,...$, we have

$$\frac{e_{q^2}(-qx^2)}{e_{q^2}(-x^2)}\ge\prod_{k=0}^{n}\frac{1+q^{2k}x^2}{1+q^{2k+1}x^2},$$

it follows that for all $n\in\mathbb{N}$,

$$\lim_{x\to\infty}\frac{e_{q^2}(-qx^2)}{e_{q^2}(-x^2)}\ge q^{-n},$$

which yields to the result.
(3) Since the function $e_{q^2}(-x^2)$ is decreasing on $[0,\infty[$, we obtain when $\beta\ge1$,

$$e_{q^2}(-\beta x^2)\le e_{q^2}(-x^2),\quad \forall x\in\mathbb{R}$$

and then

$$\lim_{x\to\infty}\frac{e_{q^2}(-qx^2)e_{q^2}(-\beta x^2)}{e_{q^2}(-x^2)}\le\lim_{x\to\infty}e_{q^2}(-qx^2)=0.$$

If $\beta<1$, then there exists $n\in\mathbb{N}$, such that $\beta>q^{2n}$ and so

$$e_{q^2}(-\beta x^2)\le e_{q^2}(-q^{2n}x^2)=(-x^2;q^2)_n e_{q^2}(-x^2).$$

Then, by using (46), we get

$$\lim_{x\to\infty}\frac{e_{q^2}(-qx^2)e_{q^2}(-\beta x^2)}{e_{q^2}(-x^2)}\le\lim_{x\to\infty}(-x^2;q^2)_n e_{q^2}(-qx^2)=0.$$

■

Proposition 4.2. *For $\lambda>0$ and $n=0,1,2,...$, we have*

$$\int_0^\infty e_{q^2}(-\lambda y^2)y^{2n+1}d_qy=\frac{(1-q)q^{-n(n+1)}(q^2;q^2)_n}{\lambda^{n+1}} \tag{49}$$

and

$$\int_0^\infty e_{q^2}(-\lambda y^2)y^{2n}d_qy=c_q(\lambda)\frac{q^{-n^2}(q;q^2)_n}{\lambda^n}, \tag{50}$$

where

$$c_q(\lambda)=\frac{(1-q)(-q\lambda,-q/\lambda,q^2;q^2)_\infty}{(-\lambda,-q^2/\lambda,q;q^2)_\infty}. \tag{51}$$

Proof. (1) From the definition of the Jackson's q-integral (6) and the relation (11), we have

$$\int_0^\infty e_{q^2}(-\lambda y^2)y^{2n+1}d_qy = (1-q)\sum_{k=-\infty}^{\infty}\frac{q^{(2n+2)k}}{(-\lambda q^{2k};q^2)_\infty}.$$

Then, using the Ramanujan identity

$$\sum_{k=-\infty}^{\infty}\frac{z^k}{(bq^k;q)_\infty} = \frac{(bz,q/(bz),q;q)_\infty}{(b,z,q/b;q)_\infty}, \quad b\neq 0, \tag{52}$$

we obtain

$$\int_0^\infty e_{q^2}(-\lambda y^2)y^{2n+1}d_qy = \frac{(1-q)\left(-\lambda q^{2n+2},-q^{-2n}/\lambda,q^2;q^2\right)_\infty}{(-\lambda,q^{2n+2},-q^2/\lambda;q^2)_\infty}.$$

We conclude (49) by using the following two identities

$$\left(-\lambda q^{2n+2};q^2\right)_\infty = \frac{\left(-q^2\lambda;q^2\right)_\infty}{(-q^2\lambda;q^2)_n}$$

and

$$\left(-q^{-2n}/\lambda;q^2\right)_\infty = q^{-n(n+1)}\lambda^{-n}\left(-q^2\lambda;q^2\right)_n\left(-\lambda^{-1};q^2\right)_\infty.$$

(2) A new use of the Ramanujan identity gives

$$\int_0^\infty e_{q^2}(-\lambda y^2)y^{2n}d_qy \quad=\quad (1-q)\frac{\left(-\lambda q^{2n+1},-q^{-2n+1}/\lambda,q^2;q^2\right)_\infty}{(-\lambda,q^{2n+1},-q^2/\lambda;q^2)_\infty}.$$

Then, (50) follows by using the two following facts that

$$\left(-\lambda q^{2n+1};q^2\right)_\infty = \frac{\left(-q\lambda;q^2\right)_\infty}{(-q\lambda;q^2)_n}$$

and

$$\left(-q^{-2n+1}/\lambda;q^2\right)_\infty = q^{-n(n+1)}\left(q/\lambda\right)^n\left(-q\lambda;q^2\right)_n\left(-q/\lambda;q^2\right)_\infty.$$

∎

Proposition 4.3. *For $t>0$ and $n=0,1,2...$, we have*

$$\int_{-\infty}^\infty k(x,t;q)b_{2n}(x;q^2)d_qx \quad= \frac{t^n}{n!_{q^2}} \tag{53}$$

and

$$\int_{-\infty}^\infty k(y,t;q)b_{2n+1}(|y|;q^2)d_qy = \frac{2C(t;q)t^{n+1}(1+q)^{2n+1}n!_{q^2}}{q^{n+1}(2n+1)!_q}, \tag{54}$$

where $C(t;q)$ is defined by (44).

Proof. We obtain the result by taking $\lambda = \dfrac{q(1-q)}{t(1+q)}$ in (49) and (50) and using the relation

$$\frac{(q;q^2)_n}{(q;q)_{2n}} = \frac{1}{(q^2;q^2)_n}.$$

∎

Proposition 4.4.
$$\mathcal{F}_q\left[k(.,t;q)\right](x) = Kexp_{q^2}\left(-tx^2\right).$$

Proof. Since $k(.,t;q)$ is even, then by using (53), we obtain

$$\begin{aligned}
\mathcal{F}_q(k(.,t;q))(x) &= K\int_{-\infty}^{\infty} k(y,t;q)\cos(yx;q^2)d_qy \\
&= K\sum_{n=0}^{\infty}(-1)^n x^{2n}\int_{-\infty}^{\infty} k(y,t;q)b_{2n}(y;q^2)d_qy, \\
&= K\sum_{n=0}^{\infty}(-1)^n x^{2n}\frac{t^n}{n!_{q^2}} = Kexp_{q^2}\left(-tx^2\right).
\end{aligned}$$

■

The following result summarizes some other properties of the q-source solution.

Theorem 4.1. *For all $t>0$ and $x\in\mathbb{R}$, we have*
1) $k(x,t;q)>0$.
2) $\lim_{t\to 0^+} k(x,t;q) = \begin{cases} +\infty & if \quad x=0, \\ 0 & if \quad x\neq 0. \end{cases}$
3)
$$\int_{-\infty}^{\infty} k(x,t;q)d_qx = 1. \tag{55}$$

Proof. 1) follows from the definition of the q-source solution.
2) Note, that $k(0,t;q)=C(t;q)$ and put $\lambda = \sqrt{\dfrac{q(1-q)}{t(1+q)}}$.
Using (47), we obtain

$$\lim_{t\to 0^+} C(t;q) = \lim_{\lambda\to+\infty}\frac{(q;q^2)_\infty e_{q^2}(-q\lambda^2)}{2(1-q)(q^2;q^2)_\infty e_{q^2}(-\lambda^2)} = +\infty.$$

If $x\neq 0$, then by using (48), we get

$$\lim_{t\to 0^+} k(x,t;q) = \lim_{\lambda\to+\infty}\frac{(q;q^2)_\infty e_{q^2}(-q\lambda^2)e_{q^2}(-\lambda^2x^2)}{2(1-q)(q^2;q^2)_\infty e_{q^2}(-\lambda^2)} = 0.$$

3) follows from (53) by taking $n=0$. ■

5. q-Heat Polynomials and q-Associated functions

5.1. q-Heat Polynomials.

For $z\in\mathbb{C}$, $t\in\mathbb{R}$ and $x\in\mathbb{R}$, we have

$$\begin{aligned}
exp_{q^2}\left(tz^2\right)\cos(-ixz;q^2) &= \left(\sum_{p=0}^{\infty}\frac{t^p z^{2p}}{p!_{q^2}}\right)\left(\sum_{k=0}^{\infty}\frac{q^{k(k+1)}x^{2k}z^{2k}}{(2k)!_q}\right) \\
&= \sum_{n=0}^{\infty} z^{2n}\sum_{k=0}^{n}\frac{t^{n-k}q^{k(k+1)}x^{2k}}{(n-k)!_{q^2}(2k)!_q}
\end{aligned} \tag{56}$$

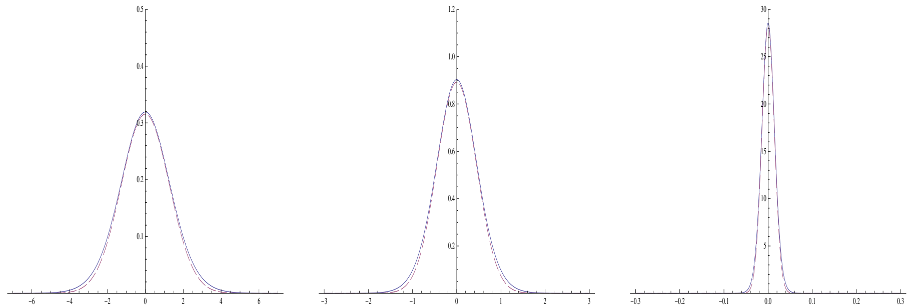

Figure 1: Comparison of the classical heat kernel $k(x,t)$ (solid line) and the q-heat kernel $k(x,t;q)$ (dashed line) at $q=0.9$ for $t=0.8$, $t=10^{-1}$ and $t=10^{-4}$.

and

$$\begin{aligned} iexp_{q^2}\left(tz^2\right)\sin(-ixz;q^2) &= \left(\sum_{p=0}^{\infty}\frac{t^p z^{2p}}{p!_{q^2}}\right)\left(\sum_{k=0}^{\infty}\frac{q^{k(k+1)}x^{2k+1}z^{2k+1}}{(2k+1)!_q}\right) \\ &= \sum_{n=0}^{\infty} z^{2n+1}\sum_{k=0}^{n}\frac{t^{n-k}q^{k(k+1)}x^{2k+1}}{(n-k)!_{q^2}(2k+1)!_q}. \end{aligned} \tag{57}$$

Then, using the relation

$$e(xz;q^2)=\cos(-ixz;q^2)+i\sin(-ixz;q^2),$$

we obtain

$$exp_{q^2}\left(tz^2\right)e(xz;q^2)=\sum_{n=0}^{\infty}v_n(x,t;q)\frac{z^n}{n!_q}, \tag{58}$$

with, for all nonnegative integer n,

$$v_n(x,t;q)=n!_q\sum_{k=0}^{[\frac{n}{2}]}\frac{t^k}{k!_{q^2}}b_{n-2k}(x;q^2). \tag{59}$$

Remarks

(1) It is clear that for all nonnegative integer n, $v_n(.,t;q)$ is a polynomial of degree n and when q tends to 1, $v_n(.,t;q)$ reduces to the standard heat polynomial (41). So, the polynomials $v_n(.,t;q)$ will be called q-heat polynomials.

(2) It is easy to derive from (28)

$$\partial_{q,x}v_n(x,t;q)=[n]_q v_{n-1}(x,t;q) \tag{60}$$

and

$$D_{q^2,t}v_n(x,t;q)=[n]_q[n-1]_q v_{n-2}(x,t;q). \tag{61}$$

Then all the q-heat polynomials $v_n(x,t;q)$ are solutions of the q-heat equation (42).

(3) Multiplying the both sides of (58) by $Exp_{q^2}(-tz^2)$ and next comparing the coefficient of z^n, we obtain

$$b_n(x;q^2) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-t)^k q^{k(k-1)} v_{n-2k}(x,t;q)}{k!_{q^2}(n-2k)!_q}, \quad n = 0,\ 1,\ 2.... \tag{62}$$

The q-heat polynomials have the following q-integral representations.

Proposition 5.1. *For $t > 0$, $x \in \mathbb{R}$ and $n = 0, 1, 2...$, we have*

$$v_n(x,t;q) = \int_{-\infty}^{\infty} k(y,t;q) p_{n,q}(x,y) d_q y. \tag{63}$$

Proof. Let $t > 0$, $x \in \mathbb{R}$ and n be a nonnegative integer. Then, using the parity of the function $k(.,t;q)$ and the relation (53), we get

$$\begin{aligned} \int_{-\infty}^{\infty} k(y,t;q) p_{n,q}(x,y) d_q y &= n!_q \sum_{k=0}^{n} b_{n-k}(x;q^2) \int_{-\infty}^{\infty} k(y,t,q) b_k(y;q^2) d_q y \\ &= n!_q \sum_{k=0}^{[\frac{n}{2}]} b_{n-2k}(x;q^2) \int_{-\infty}^{\infty} k(y,t;q) b_{2k}(y;q^2) d_q y \\ &= n!_q \sum_{k=0}^{[\frac{n}{2}]} b_{n-2k}(x;q^2) \frac{t^k}{k!_{q^2}} = v_n(x,t;q). \end{aligned}$$

∎

The following easily proved result shows that the q-heat polynomials are closely related to the discrete q-Hermite I polynomials defined by (see [6])

$$h_n(x;q) = (q;q)_n \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k q^{k(k-1)} x^{n-2k}}{(q^2;q^2)_k (q;q)_{n-2k}}.$$

Proposition 5.2. *For all nonnegative integer n, $t > 0$ and $x \in \mathbb{R}$ we have*

$$v_{2n}(x,t;q) = \frac{q^{-n(n-1)}}{(i\beta)^{2n}} h_{2n}(i\beta q^n x;q) \tag{64}$$

and

$$v_{2n+1}(x,t;q) = \frac{q^{-n^2}}{(i\beta)^{2n+1}} h_{2n+1}(i\beta q^n x;q), \tag{65}$$

where

$$\beta = \beta(t,q) = \sqrt{\frac{1-q}{t(1+q)}}. \tag{66}$$

5.2. q-Associated functions.

Definition 5.1. *For $n = 0, 1, 2, ...$ and $t > 0$, the q-associated functions $w_n(x,t;q)$ is the function defined by*

$$w_n(x,t;q) = (-(1+q))^n \partial_{q,x}^n k(x,t;q). \tag{67}$$

Remark. It is easy to see that the q-associated functions $w_n(x,t;q),\ n=0,1,2,...$ are solutions of the q-heat equation (42).

We recall that the discrete q-Hermite II polynomials are given by (see [6])

$$\tilde{h}_n(x;q)=(q;q)_n\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k q^{-2nk}q^{k(2k+1)}x^{n-2k}}{(q^2;q^2)_k(q;q)_{n-2k}}$$

and satisfy the following Rodrigues-type formula

$$\omega(x;q)\tilde{h}_n(x;q)=(q-1)^n q^{-\frac{n(n-1)}{2}}D_q^n[\omega(x;q)],$$

where

$$\omega(x;q)=\frac{1}{(-x^2;q^2)_\infty}.$$

The following result gives some relations between the q-source solution and the discrete q-Hermite II polynomials.

Proposition 5.3. *For $n=0,1,2,...,$ we have*

$$\partial_{q,x}^{2n}k(x,t;q)=\frac{q^{n(n-2)}\gamma^{2n}}{(q-1)^{2n}}\tilde{h}_{2n}(\gamma q^{-n}x;q)k(q^{-n}x,t;q)$$

and

$$\partial_{q,x}^{2n+1}k(x,t;q)=\frac{q^{n(n-1)}\gamma^{2n+1}}{q(q-1)^{2n+1}}\tilde{h}_{2n+1}(\gamma q^{-(n+1)}x;q)k(q^{-(n+1)}x,t;q),$$

where

$$\gamma=\gamma(t,q)=q^{\frac{1}{2}}\beta(t,q)=\sqrt{\frac{q(1-q)}{t(1+q)}}. \tag{68}$$

Proof. Using Proposition 2.1, we obtain

$$\partial_{q,x}^{2n}k(x,t;q)=q^{-n(n+1)}D_q^{2n}[k(.,t;q)]o\Lambda_q^n(x)$$

and

$$\partial_{q,x}^{2n+1}k(x,t;q)=q^{-(n+1)^2}D_q^{2n+1}[k(.,t;q)]o\Lambda_q^{n+1}(x).$$

But, the the fact that

$$k(x,t;q)=C(t;q)w(\gamma x;q)$$

gives

$$D_{q,x}^n[k(x,t;q)]=C(t;q)D_q^n[\omega(\gamma x;q)]=C(t;q)\gamma^n[D_q^n\omega](\gamma x;q).$$

So, from the Rodrigues-type formula, we get

$$D_{q,x}^n[k(x,t;q)]=\gamma^n(q-1)^{-n}q^{\frac{n(n-1)}{2}}\tilde{h}_n(\gamma x;q)k(x,t;q),$$

which yields to the desired results. ∎

Proposition 5.4. *For $n=0,1,2,...,$ we have*

$$\partial_{q,x}^{2n}k(x,t;q)=\frac{v_{2n}(q^{\frac{1}{2}}x,-t;q^{-1})}{t^{2n}(1+q)^{2n}}k(q^{-n}x,t;q)$$

and

$$\partial_{q,x}^{2n+1}k(x,t;q)=-\frac{q^{-\frac{1}{2}}v_{2n+1}(q^{-\frac{1}{2}}x,-t;q^{-1})}{t^{2n+1}(1+q)^{2n+1}}k(q^{-(n+1)}x,t;q).$$

Proof. It is easy to verify that the discrete q-Hermite II polynomials are related to the discrete q-Hermite I polynomials by

$$\tilde{h}_n(x;q) = i^{-n} h_n(ix;q^{-1}) \quad n = 0,\ 1,\ 2, ... \tag{69}$$

Then, for $n = 0,\ 1,\ 2, ...$, we have

$$\tilde{h}_{2n}(\gamma q^{-n}x;q) = i^{-2n} h_{2n}(i\gamma q^{-n}x;q^{-1}) = i^{-2n} h_{2n}(i\beta q^{-n} q^{\frac{1}{2}} x;q^{-1})$$

and

$$\begin{aligned}\tilde{h}_{2n+1}(\gamma q^{-(n+1)}x;q) &= i^{-(2n+1)} h_{2n+1}(i\gamma q^{-(n+1)}x;q^{-1}) \\ &= i^{-(2n+1)} h_{2n+1}(i\beta q^{-n} q^{-\frac{1}{2}} x;q^{-1}).\end{aligned}$$

Thus, since

$$\beta(t,q) = \beta(-t,q^{-1}), \tag{70}$$

the first equality follows from (64) and the fact that

$$\frac{i^{-2n} q^{n(n-2)} \gamma^{2n}}{(1-q)^{2n}} = \frac{q^{n(n-1)}}{(i\beta)^{2n} t^{2n} (1+q)^{2n}},$$

and the second equality follows from (65) and the relation

$$\frac{i^{-(2n+1)} q^{n(n-1)} \gamma^{2n+1}}{q(1-q)^{2n+1}} = \frac{q^{n^2-\frac{1}{2}}}{(i\beta)^{2n+1} t^{2n+1} (1+q)^{2n+1}}.$$

■

Definition 5.2. *We define the polynomials $\tilde{v}_k(x,t;q)$, $n = 0, 1, 2, ...$, by*

$$\begin{aligned}\tilde{v}_{2n}(x,t;q) &= v_{2n}(q^{\frac{1}{2}}x,t;q^{-1}), \\ \tilde{v}_{2n+1}(x,t;q) &= q^{-\frac{1}{2}} v_{2n+1}(q^{-\frac{1}{2}}x,t;q^{-1}).\end{aligned} \tag{71}$$

Note that

$$\lim_{q \to 1} \tilde{v}_n(x,t;q) = v_n(x,t).$$

The following result summarizes some properties of the polynomials $\tilde{v}_n(x,t;q)$.

Proposition 5.5.
1) Operation of the operator ∂_q on $\tilde{v}_n(x,t;q)$:

$$\partial_{q,x} \tilde{v}_n(x,t;q) = q^{-1}[n]_{q^{-1}} \tilde{v}_{n-1}(x,t;q), \quad n = 1,\ 2\ , ... \tag{72}$$

2) The generating function for $\tilde{v}_n(x,t;q)$ is given by

$$Exp_{q^2}\left(tz^2\right) e(q^{-1}xz;q^2) = \sum_{n=0}^{\infty} \tilde{v}_n(x,t;q) \frac{z^n}{n!_{q^{-1}}}. \tag{73}$$

3) The $\tilde{v}_n(x,t;q)$ polynomials are related to the discrete q-Hermite II polynomials by

$$\begin{aligned}\tilde{v}_{2n}(x,-t;q) &= \frac{q^{n(n-1)}}{\beta^{2n}} \tilde{h}_{2n}(\gamma q^{-n}x;q), \quad n = 0, 1, 2, ... \\ \tilde{v}_{2n+1}(x,-t;q) &= \frac{q^{n^2-\frac{1}{2}}}{\beta^{2n+1}} \tilde{h}_{2n+1}(\gamma q^{-(n+1)}x;q), \quad n = 0, 1, 2, ...\end{aligned} \tag{74}$$

where β and γ are, respectively, defined by (66) and (68).

Proof. 1) By (1), we have

$$(2k)!_{q^{-1}} = q^{-2k^2+k}(2k)!_q \quad \text{and} \quad (2k+1)!_{q^{-1}} = q^{-2k^2-k}(2k+1)!_q. \tag{75}$$

So, for all $k = 0,\ 1,\ 2, ...,$

$$b_{2k}(q^{\frac{1}{2}}x; q^{-2}) = b_{2k}(q^{-1}x; q^2), \tag{76}$$

and

$$q^{-\frac{1}{2}}b_{2k+1}(q^{-\frac{1}{2}}x; q^{-2}) = b_{2k+1}(q^{-1}x; q^2), \tag{77}$$

where $b_n(x; q^2)$ is defined by (18).
Then, from (28), we obtain for all $k = 1,\ 2,\ ...,$

$$\partial_q b_{2k}(q^{\frac{1}{2}}x; q^{-2}) = q^{-1}b_{2k-1}(q^{-1}x; q^2) = q^{-\frac{3}{2}}b_{2k-1}(q^{-\frac{1}{2}}x; q^{-2})$$

and

$$q^{-\frac{1}{2}}\partial_q b_{2k+1}(q^{-\frac{1}{2}}x; q^{-2}) = q^{-1}b_{2k}(q^{-1}x; q^2) = q^{-1}b_{2k}(q^{\frac{1}{2}}x; q^{-2}).$$

Thus, (72) follows from these two equalities and the definition of $\tilde{v}_n(x, t; q)$.

2) By (76) and (77), we have

$$\cos(q^{-1}z; q^2) = \cos(q^{\frac{1}{2}}z; q^{-2}) \quad \text{and} \quad \sin(q^{-1}z; q^2) = q^{-\frac{1}{2}}\sin(q^{-\frac{1}{2}}z; q^{-2}).$$

Then,

$$Exp_{q^2}\left(tz^2\right)\cos(-iq^{-1}xz; q^2) = exp_{q^{-2}}\left(tz^2\right)\cos(-iq^{\frac{1}{2}}xz; q^{-2})$$

and

$$Exp_{q^2}\left(tz^2\right)\sin(-iq^{-1}xz; q^2) = q^{-\frac{1}{2}}exp_{q^{-2}}\left(tz^2\right)\sin(-iq^{-\frac{1}{2}}xz; q^{-2}).$$

Finally, (73) follows by replacing x, q by $q^{\frac{1}{2}}x$, q^{-1} in (56) and by $q^{-\frac{1}{2}}x$, q^{-1} in (57).

3) Using (70), (64) and (65), we obtain

$$v_{2n}(q^{\frac{1}{2}}x, -t; q^{-1}) = \frac{q^{n(n-1)}}{(i\beta)^{2n}}h_{2n}(iq^{\frac{1}{2}}\beta q^{-n}x; q^{-1}) = \frac{q^{n(n-1)}}{(i\beta)^{2n}}h_{2n}(i\gamma q^{-n}x; q^{-1})$$

and

$$\begin{aligned} v_{2n+1}(q^{-\frac{1}{2}}x, -t; q^{-1}) &= \frac{q^{n^2}}{(i\beta)^{2n+1}}h_{2n+1}(iq^{-\frac{1}{2}}\beta q^{-n}x; q^{-1}) \\ &= \frac{q^{n^2}}{(i\beta)^{2n+1}}h_{2n+1}(i\gamma q^{-(n+1)}x; q^{-1}), \end{aligned}$$

which together with (69) give the result. ■

The following result follows easily from Proposition 5.4, and the relations (71) and (74). It shows that the q-associated functions $w_n(x, t; q)$ are closely related to the discrete q-Hermite II polynomials and to the polynomials $\tilde{v}_n(x, -t; q)$.

Proposition 5.6. *For $t > 0$ and $n = 0, 1, 2, ...,$ we have*

$$\begin{aligned} w_{2n}(x,t;q) &= \frac{\tilde{v}_{2n}(x,-t;q)}{t^{2n}} k(q^{-n}x,t;q) \\ &= \frac{q^{n(n-1)}}{(t\beta)^{2n}} \tilde{h}_{2n}(\gamma q^{-n}x;q) k(q^{-n}x,t;q), \\ w_{2n+1}(x,t;q) &= \frac{\tilde{v}_{2n+1}(x,-t;q)}{t^{2n+1}} k(q^{-(n+1)}x,t;q) \\ &= \frac{q^{n^2-\frac{1}{2}}}{(t\beta)^{2n+1}} \tilde{h}_{2n+1}(\gamma q^{-(n+1)}x;q) k(q^{-n}x,t;q), \end{aligned} \tag{78}$$

where β and γ are, respectively, defined by (66) and (68).

5.3. **Biorthogonal relation.** To establish a biorthogonal relation between the $\tilde{v}_n(qx,-t;q)$ and $w_n(x,t;q)$, we need the following result.

Lemma 5.1. *For $t > 0$ and $n = 1,\ 2,\ 3, ..,$ we have*

$$\int_{-\infty}^{\infty} \tilde{v}_{2n}(qx,-t;q) k(x,t;q) d_q x = 0. \tag{79}$$

Proof. Let $t > 0$ and $n \geq 1$. Using the relation (1), we obtain

$$\begin{aligned} \tilde{v}_{2n}(qx,-t;q) &= v_{2n}(q^{\frac{3}{2}}x,-t;q^{-1}) = (2n)!_{q^{-1}} \sum_{k=0}^{n} \frac{(-t)^{n-k} q^{-k^2+2k} x^{2k}}{(n-k)!_{q^{-2}} (2k)!_{q^{-1}}} \\ &= q^{-n^2} (2n)!_q \sum_{k=0}^{n} \frac{(-t)^{n-k} q^{2k^2-2nk+2k} x^{2k}}{(n-k)!_{q^2} (2k)!_q}. \end{aligned}$$

The lefthand side of (79) is then equal to

$$C(t;q) q^{-n^2} (2n)!_q \sum_{k=0}^{n} \frac{(-t)^{n-k} q^{2k^2-2nk+2k}}{(n-k)!_{q^2} (2k)!_q} \int_{-\infty}^{\infty} exp_{q^2}\left(-\frac{qx^2}{t(1+q)^2}\right) x^{2k} d_q x.$$

But, taking $\lambda = \dfrac{q(1-q)}{t(1+q)}$ in (50), we obtain

$$\int_{-\infty}^{\infty} exp_{q^2}\left(-\frac{qx^2}{t(1+q)^2}\right) x^{2k} d_q x = 2c_q(\lambda) \frac{q^{-k^2-k} (q;q^2)_k (1+q)^k t^k}{(1-q)^k},$$

where $c_q(\lambda)$ is defined by (51).

Then, since $C_q(t,\lambda) = \dfrac{1}{2c_q(\lambda)}$, the q-integral in (79) is equal to

$$q^{-n^2} (2n)!_q (-t)^n (1-q^2)^n \sum_{k=0}^{n} \frac{(-1)^k q^{k^2-2nk+k}}{(q^2;q^2)_{n-k} (q^2;q^2)_k},$$

which is equal to

$$\frac{q^{-n^2} (2n)!_q (-t)^n}{n!_{q^2}} \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right)_{q^2} q^{k(k-1)} \left(-q^{-2n+2}\right)^k.$$

Hence, from the q-binomial theorem (2), we get

$$\int_{-\infty}^{\infty} \tilde{v}_{2n}(qx,-t;q) k(x,t;q) d_q x = \frac{q^{-n^2} (2n)!_q (-t)^n}{n!_{q^2}} (q^{-2n+2};q^2)_n = 0.$$

■

Theorem 5.1. *For $0 < t < \infty$, $m, n = 0, 1, 2, ..,$ we have*

$$\int_{-\infty}^{\infty} \tilde{v}_n(qx, -t; q) w_m(x, t; q) d_q x = (1+q)^n n!_{q^{-1}} \delta_{m,n}.$$

Proof. First, from (72), we get

$$\partial_{q,x} \tilde{v}_n(qx, t; q) = [n]_{q^{-1}} \tilde{v}_{n-1}(qx, t; q). \tag{80}$$

In what follows, we will use the letter A for an unessential constant that may vary from equation to another.
If $m > n$, we have by q-integration by parts (9) and the relation (80),

$$\begin{aligned}
\int_{-\infty}^{\infty} \tilde{v}_n(qx, -t; q) w_m(x, t; q) d_q x &= A \int_{-\infty}^{\infty} \tilde{v}_n(qx, -t; q) \partial_q^m k(x, t; q) d_q x \\
&= A \int_{-\infty}^{\infty} \tilde{v}_0(qx, -t; q) \partial_q^{m-n} k(x, t; q) d_q x \\
&= 0.
\end{aligned}$$

If $m < n$, we have

$$\int_{-\infty}^{\infty} \tilde{v}_n(qx, -t; q) w_m(x, t; q) d_q x = A \int_{-\infty}^{\infty} \tilde{v}_{n-m}(qx, -t; q) k(x, t; q) d_q x.$$

In case $n - m$ is odd, the q-integrand is an odd function. So, the above q-integral vanishes.
In case $n - m$ is even, by (79) the above q-integral is also null.
If $m = n$, we have

$$\int_{-\infty}^{\infty} \tilde{v}_n(qx, -t; q) w_n(x, t; q) d_q x = (-(1+q))^n \int_{-\infty}^{\infty} \tilde{v}_n(qx, -t; q) \partial_q^n k(x, t; q) d_q x.$$

Then, using the q-integration by parts (9) and the relations (80), and (55), we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} \tilde{v}_n(qx, -t; q) w_n(x, t; q) d_q x &= (1+q)^n n!_{q^{-1}} \int_{-\infty}^{\infty} \tilde{v}_0(qx, -t; q) k(x, t; q) d_q x \\
&= (1+q)^n n!_{q^{-1}} \int_{-\infty}^{\infty} k(x, t; q) d_q x \\
&= (1+q)^n n!_{q^{-1}}.
\end{aligned}$$

■

6. Expansions in terms of the q-heat polynomials and q-associated functions

6.1. **Some useful asymptotic estimations.** For further study, it is essential to know the behavior of the functions $v_n(x, t; q)$ and $w_n(x, t; q)$ for large integer n.

Proposition 6.1. *For $x \in \mathbb{R}$, $t > 0$, and $n = 0,\ 1,\ 2, ...,$ we have*

$$|v_{2n}(x,t;q)| \leq \frac{(2n)!_q t^n}{n!_{q^2}} E_{q^2}\left(\frac{x^2}{t}\right) \tag{81}$$

and

$$|v_{2n+1}(x,t;q)| \leq \frac{(2n+1)!_q t^n}{n!_{q^2}} |x| E_{q^2}\left(\frac{x^2}{t}\right). \tag{82}$$

Proof. Let $x \in \mathbb{R}$, $t > 0$, and $n = 0,\ 1,\ 2, ...$
Using the fact that $(2k)!_q \geq k!_{q^2}$, we obtain

$$|v_{2n}(x,t;q)| \leq \frac{(2n)!_q t^n}{n!_{q^2}} \sum_{k=0}^{n} \frac{n!_{q^2} q^{k(k-1)}}{(n-k)!_{q^2} k!_{q^2}} \left(\frac{x^2}{t}\right)^k.$$

Then, (81) follows from the q-binomial theorem (2) and (82) follows from (81) together with the following easily verified inequality

$$|v_{2n+1}(x,t;q)| \leq |x|[2n+1]_q v_{2n}(x,t;q).$$

■

Lemma 6.1. *Let $t_0 > 0$ and $x_0 \neq 0$. If the series $\sum_n a_n v_n(x_0,t_0;q)$ converges absolutely, then*

$$a_{2n} = O\left(\frac{n!_{q^2}}{(2n)!_q t_0^n}\right) \quad \textit{and} \quad a_{2n+1} = O\left(\frac{n!_{q^2}}{(2n+1)!_q t_0^n}\right), \textit{ as } n \to \infty. \tag{83}$$

Proof. Let $t_0 > 0$ and $x_0 \neq 0$. For all $n \in \mathbb{N}$, we have

$$|v_{2n}(x_0,t_0;q)| \geq |v_{2n}(0,t_0;q)| = \frac{(2n)!_q t_0^n}{n!_{q^2}} \tag{84}$$

and

$$|v_{2n+1}(x_0,t_0;q)| \geq |x_0| \frac{(2n+1)!_q t_0^n}{n!_{q^2}}. \tag{85}$$

If the series $\sum_n a_n v_n(x_0,t_0;q)$ converges absolutely, then its general term $a_n v_n(x_0,t_0;q)$ tends to 0 as $n \to \infty$. Hence, by (84) and (85), we get the desired conclusion. ■

Replacing λ by $(1-q^2)t$ in (49) and in (50), we obtain:

Lemma 6.2. *For $n = 0, 1, 2, ...,$ we have*

$$\int_{-\infty}^{\infty} exp_{q^2}(-ty^2) y^{2n+1} d_q y = \frac{2q^{-n(n+1)} n!_{q^2}}{(1+q)t^{n+1}} \tag{86}$$

and

$$\int_{-\infty}^{\infty} exp_{q^2}(-ty^2) y^{2n} d_q y = 2c_q((1-q^2)t) \frac{q^{-n^2}(q;q^2)_n}{(1-q^2)^n t^n}, \tag{87}$$

where $c_q(.)$ is given by (51).

Proposition 6.2. *For all $x \in \mathbb{R}_q$, $t > 0$ and $n = 0, 1, 2, \ldots$, we have*

$$|w_{2n}(x,t;q)| \le M \frac{(2n)!_{q^{-1}}}{n!_{q^{-2}} t^n} \tag{88}$$

and

$$|w_{2n+1}(x,t;q)| \le M \frac{(2n+1)!_{q^{-1}}}{n!_{q^{-2}} t^n}, \tag{89}$$

where M is a constant independent of x and n.

Proof. From Proposition 4.4, we have

$$k(x,t;q) = K^2 \int_{-\infty}^{\infty} exp_{q^2}\left(-ty^2\right) e(ixy;q^2) d_q y, \quad x \in \mathbb{R}_q.$$

Then, from (67) and (21), we get for $n = 0, 1, 2, \ldots$ and $x \in \mathbb{R}_q$,

$$\begin{aligned} |w_n(x,t;q)| &= K^2 \left| \int_{-\infty}^{\infty} exp_{q^2}\left(-ty^2\right) e(ixy;q^2) \left[-i(1+q)y\right]^n d_q y \right| \\ &\le \frac{4K^2(1+q)^n}{(q;q)_\infty} \int_0^\infty exp_{q^2}\left(-ty^2\right) y^n d_q y. \end{aligned} \tag{90}$$

But, It follows from (1) that

$$\frac{(2n)!_{q^{-1}}}{n!_{q^{-2}}} = q^{-n^2} \frac{(2n)!_q}{n!_{q^2}} = \frac{q^{-n^2}(1+q)^n (q;q^2)_n}{(1-q)^n} \tag{91}$$

and

$$\frac{(2n+1)!_{q^{-1}}}{n!_{q^{-2}}} = q^{-n^2-2n} \frac{(2n+1)!_q}{n!_{q^2}} \ge q^{-n(n+1)}(1+q)^{2n} n!_{q^2}. \tag{92}$$

Then, by (87) and (86), we obtain

$$(1+q)^{2n} \int_{-\infty}^{\infty} exp_{q^2}(-ty^2) y^{2n} d_q y \le M \frac{(2n)!_{q^{-1}}}{n!_{q^{-2}} t^n} \tag{93}$$

and

$$(1+q)^{2n+1} \int_{-\infty}^{\infty} exp_{q^2}(-ty^2) y^{2n+1} d_q y \le M \frac{(2n+1)!_{q^{-1}}}{n!_{q^{-2}} t^n}, \tag{94}$$

with $M = \dfrac{4K^2}{(q;q)_\infty} \max \left[c_q\left((1-q^2)t\right), \dfrac{1}{t} \right]$. ∎

6.2. **Expansion in series of q-heat polynomials.** Functions in class $\mathcal{E}_{\sigma,q}$ defined by (35) serve well in q-integral representation of solution of (42), at least if the solution is obtained by a power series. For this purpose, we need a preliminary result.

Lemma 6.3. *For $x \in \mathbb{R}$, $t > 0$ and $n = 0,\ 1,\ 2, \ldots$, we have*

$$\int_{-\infty}^{\infty} k(y,t;q) p_{n,q}\left(|x|,|y|\right) d_q y \le v_n(|x|,t;q) + \frac{2C(t;q)}{(1-q)[n+1]_q} v_{n+1}\left(|x|, \frac{t}{q}; q\right), \tag{95}$$

where $C(t;q)$ is defined by (44).

Proof. Two cases arise according to n is even or odd.
If n is even,

$$\int_{-\infty}^{\infty} k(y,t;q)p_{2n,q}(|x|,|y|)d_qy$$
$$= (2n)!_q \left[\sum_{k=0}^{n} b_{2n-2k}(|x|;q^2)\int_{-\infty}^{\infty} k(y,t;q)b_{2k}(|y|;q^2)d_qy\right.$$
$$\left.+\sum_{k=0}^{n-1} b_{2n-1-2k}(|x|;q^2)\int_{-\infty}^{\infty} k(y,t;q)b_{2k+1}(|y|;q^2)d_qy\right].$$

But, from (53), we have

$$(2n)!_q\sum_{k=0}^{n} b_{2n-2k}(|x|;q^2)\int_{-\infty}^{\infty} k(y,t;q)b_{2k}(|y|;q^2)d_qy = v_{2n}(|x|,t;q)$$

and from (54), we have

$$\int_{-\infty}^{\infty} k(y,t;q)b_{2k+1}(|y|;q^2)d_qy = \frac{2C(t;q)t^{k+1}(1+q)^{2k+1}k!_{q^2}}{q^{k+1}(2k+1)!_q}.$$

Then, using the fact that

$$\frac{(1+q)^{2k+1}k!_{q^2}}{(2k+1)!_q} = \frac{(1-q^2)^{k+1}}{(q;q^2)_{k+1}} \le \frac{1}{(1-q)(k+1)!_{q^2}},$$

we obtain

$$(2n)!_q\sum_{k=0}^{n-1} b_{2n-1-2k}(|x|;q^2)\int_{-\infty}^{\infty} k(y,t;q)b_{2k+1}(|y|;q^2)d_qy$$
$$\le \frac{2C(t;q)(2n)!_q}{(1-q)}\sum_{k=0}^{n-1} b_{2(n-(k+1))+1}(|x|;q^2)\frac{t^{k+1}}{q^{k+1}(k+1)!_{q^2}},$$
$$\le \frac{2C(t;q)(2n)!_q}{(1-q)}\sum_{k=0}^{n} b_{2(n-k)+1}(|x|;q^2)\frac{t^{k}}{q^{k}k!_{q^2}}$$
$$= \frac{2C(t;q)}{(1-q)[2n+1]_q}v_{2n+1}\left(|x|,\frac{t}{q};q\right).$$

In the same way, we prove the inquality (95) when n is odd. ∎

Theorem 6.1. *Let $\sigma > 1$ and $f \in \mathcal{E}_{\sigma,q}$. Then, the q-integral*

$$u(x,t;q) = \int_{-\infty}^{\infty} k(y,t;q)\mathcal{T}_{y,q}(f)(x)d_qy, \tag{96}$$

converges absolutely in the strip $0 < t < q\sigma$, and

$$u(x,t;q) = \sum_{n=0}^{\infty}\frac{\partial_q^n f(0)}{n!_q}v_n(x,t;q). \tag{97}$$

Proof. Let $f \in \mathcal{E}_{\sigma,q}$, $\sigma > 1$. Then, from Proposition 3.2, $\mathcal{T}_{y,q}f(x)$ is well defined and we have

$$\begin{aligned}\int_{-\infty}^{\infty} k(y,t;q)\mathcal{T}_{y,q}(f)(x)d_qy &= \int_{-\infty}^{\infty} k(y,t;q)\sum_{n=0}^{\infty}\frac{\partial_q^n f(0)}{n!_q}p_{n,q}(x,y)d_qy\\ &= \sum_{n=0}^{\infty}\frac{\partial_q^n f(0)}{n!_q}\int_{-\infty}^{\infty} k(y,t;q)p_{n,q}(x,y)d_qy\\ &= \sum_{n=0}^{\infty}\frac{\partial_q^n f(0)}{n!_q}v_n(x,t;q).\end{aligned}$$

by using (63).

The absolute convergence of the q-integral and the interchange of the sum and the q-integral, in the previous equation, is legitimated by the Fubini's theorem: since $f \in \mathcal{E}_{\sigma,q}$, then by the relations (95), (81) and (82), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{|\partial_q^n f(0)|}{n!_q}\int_{-\infty}^{\infty} k(y,t;q)p_{n,q}(|x|,|y|)d_qy\\ \leq\ & A\sum_{n=0}^{\infty}\frac{n!_{q^2}}{(2n)!_q\sigma^n}\int_{-\infty}^{\infty} k(y,t;q)p_{2n,q}(|x|,|y|)d_qy\\ +\ & A\sum_{n=0}^{\infty}\frac{n!_{q^2}}{(2n+1)!_q\sigma^n}\int_{-\infty}^{\infty} k(y,t;q)p_{2n+1,q}(|x|,|y|)d_qy,\\ \leq\ & A\sum_{n=0}^{\infty}\left(\frac{t}{q\sigma}\right)^n < \infty, \qquad \text{for all} \quad 0 < t < \sigma q.\end{aligned}$$

Here, A is an unessential constant that may vary from an equation to another. ■

Remark. Note that since the general terms of the series (97) are solutions of the q-heat equation (42), then the function defined by this series is also a solution.

Theorem 6.2. *Suppose that f is defined near 0 by a power series of positive radius of convergence and the series (97) converges absolutely in the strip $0 < t \leq \sigma$. Then, $u(x,t;q)$ has the q-integral representation (96) in the strip $0 < t < q\sigma$, and $f \in \mathcal{E}_{\sigma,q}$.*

Proof. From the hypothesis of the theorem, f is infinitely differentiable at 0. So, it is infinitely q-differentiable at 0 and by Proposition 2.1, we have for all nonnegative integer n,

$$q^{[\frac{n}{2}]([\frac{n}{2}]+1)}\frac{\partial_q^n f(0)}{n!_q} = \frac{f^{(n)}(0)}{n!}.$$

If (97) converges absolutely for $0 < t \leq \sigma$. Then, in particular, it converges for $t_0 = \sigma$. By Lemma 6.1, we have

$$\frac{\partial_q^{2n} f(0)}{(2n)!_q} = O\left(\frac{n!_{q^2}}{(2n)!_q\sigma^n}\right) \quad \text{and} \quad \frac{\partial_q^{2n+1} f(0)}{(2n+1)!_q} = O\left(\frac{n!_{q^2}}{(2n+1)!_q\sigma^n}\right), \quad n\to\infty. \tag{98}$$

So,

$$\frac{f^{(2n)}(0)}{(2n)!} = O\left(\frac{q^{n(n+1)}n!_{q^2}}{(2n)!\sigma^n}\right) \quad \text{and} \quad \frac{f^{(2n+1)}(0)}{(2n+1)!} = O\left(\frac{q^{n(n+1)}n!_{q^2}}{(2n+1)!\sigma^n}\right), \quad n\to\infty.$$

Then, f is entire and owing to the estimations (98), it belongs to $\mathcal{E}_{\sigma,q}$.
Now, the same arguments as those used in Theorem 6.1 show that $u(x,t;q)$ has the q-integral representation (96) in the strip $0 < t < q\sigma$. ■

Remark. It follows from (60) that if

$$u(x,t;q) = \sum_{n=0}^{\infty} a_n v_n(x,t;q), \quad 0 \leq t < \sigma,$$

then we have

$$a_n = \frac{[\partial_{q,x}^n u](0,0;q)}{n!_q}, \quad n = 0, 1, 2, ...$$

6.3. **Expansions in series of q-associated functions.** In this subsection, we need a new class of functions of real variable, useful for the validity of expansions in terms of the q-associated functions $w_n(x,t;q)$.

Definition 6.1. *For, $\sigma > 0$, we write $\tilde{\mathcal{E}}_{\sigma,q}$ the set of functions f satisfying:*

$$\exists M > 0 \quad : \quad \begin{cases} f(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n \quad \textit{is a power series}, \\ |a_{2n}| \quad \leq \dfrac{M n!_{q^{-2}} (1+q)^{2n} \sigma^n}{(2n)!_{q^{-1}}}, \qquad \forall n \in \mathbb{N}, \\ |a_{2n+1}| \leq \dfrac{M n!_{q^{-2}} (1+q)^{2n+1} \sigma^n}{(2n+1)!_{q^{-1}}}, \quad \forall n \in \mathbb{N}. \end{cases} \tag{99}$$

Note that by the relations (91) and (92), the power series defining the functions $f \in \tilde{\mathcal{E}}_{\sigma,q}$ are of radius of convergence $R = +\infty$.
Now, we give a sufficient condition for the possibility to expand a solution of the q-heat equation in terms of $w_n(x,t;q)$.

Theorem 6.3. *If*

$$u(x,t;q) = K^2 \int_{-\infty}^{\infty} exp_{q^2}(-ty^2) e(ixy;q^2) \phi(y) d_q y, \tag{100}$$

with $\phi(y) = \displaystyle\sum_{n=0}^{\infty} a_n y^n \in \tilde{\mathcal{E}}_{\sigma,q}$, *then*

$$u(x,t;q) = \sum_{n=0}^{\infty} \frac{a_n}{[-i(1+q)]^n} w_n(x,t;q), \quad x \in \widetilde{\mathbb{R}}_q, \quad t > \sigma \geq 0. \tag{101}$$

The infinite series (101) converges absolutely.

Proof. Let $\phi(y) = \displaystyle\sum_{n=0}^{\infty} a_n y^n \in \tilde{\mathcal{E}}_{\sigma,q}$. Then,

$$\int_{-\infty}^{\infty} exp_{q^2}(-ty^2) e(ixy, q^2) \phi(y) d_q y = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} exp_{q^2}(-ty^2) e(ixy, q^2) a_n y^n d_q y. \tag{102}$$

But, using (21), we get by the help of (93) and (94),

$$\begin{aligned}
&\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\left|exp_{q^2}(-ty^2)e(ixy,q^2)a_ny^n\right|d_qy \leq A\sum_{n=0}^{\infty}|a_n|\int_{-\infty}^{\infty}exp_{q^2}(-ty^2)|y|^nd_qy\\
\leq\ & A\sum_{n=0}^{\infty}|a_{2n}|\int_{-\infty}^{\infty}exp_{q^2}(-ty^2)|y|^{2n}d_qy+A\sum_{n=0}^{\infty}|a_{2n+1}|\int_{-\infty}^{\infty}exp_{q^2}(-ty^2)|y|^{2n+1}d_qy\\
\leq\ & A\sum_{n=0}^{\infty}\frac{(2n)!_{q^{-1}}|a_{2n}|}{n!_{q^{-2}}(1+q)^{2n}t^n}+A\sum_{n=0}^{\infty}\frac{(2n+1)!_{q^{-1}}|a_{2n+1}|}{n!_{q^{-2}}(1+q)^{2n+1}t^n}\\
<\ & \infty,
\end{aligned}$$

since $\phi\in\tilde{\mathcal{E}}_{\sigma,q}$.
So, we can exchange the order of the sum and the q-integral signs in (102) and the infinite series (101) converges absolutely. Moreover, by the use of (90), we obtain

$$\begin{aligned}
u(x,t;q) &= K^2\int_{-\infty}^{\infty}exp_{q^2}(-ty^2)e(ixy,q^2)\left[\sum_{n=0}^{\infty}a_ny^n\right]d_qy\\
&= \sum_{n=0}^{\infty}\frac{a_n}{[-i(1+q)]^n}w_n(x,t;q),
\end{aligned}\tag{103}$$

which achieves the proof. ∎

6.4. **Examples.**

Example 1. Take $f(x)=e(i\lambda x;q^2)=\sum_{n=0}^{\infty}b_n(i\lambda x;q^2),\ \lambda\in\mathbb{R}_q,\ x\in\mathbb{R}$.
By (30), we have

$$u(x,t;q)=\int_{-\infty}^{\infty}k(y,t;q)\mathcal{T}_{y,q}f(x)d_qy=e(ix\lambda;q^2)\int_{-\infty}^{\infty}k(y,t;q)e(iy\lambda;q^2)d_qy.$$

By Proposition 4.4, we obtain

$$u(x,t;q)=exp_{q^2}\left(-t\lambda^2\right)e(ix\lambda;q^2)$$

and by Theorem 6.2, we get

$$u(x,t;q)=\sum_{n=0}^{\infty}\frac{(i\lambda)^{2n}v_{2n}(x,t;q)}{(2n)!_q}+\sum_{n=0}^{\infty}\frac{(i\lambda)^{2n+1}v_{2n+1}(x,t;q)}{(2n+1)!_q}.$$

By (81) and (82), these series converge in the strip $0\leq t<\dfrac{1}{(1-q^2)\lambda^2}$.
Consequently,

$$exp_{q^2}\left(-t\right)\cos(x;q^2)=\sum_{n=0}^{\infty}\frac{(-1)^nv_{2n}(x,t;q)}{(2n)!_q},$$

$$exp_{q^2}\left(-t\right)\sin(x;q^2)=\sum_{n=0}^{\infty}\frac{(-1)^nv_{2n+1}(x,t;q)}{(2n+1)!_q}.$$

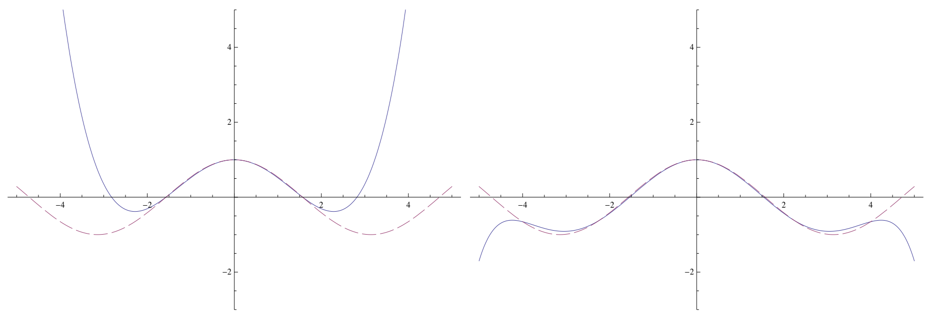

Figure 2: Comparison of the exact solution $u(x,t) = e^{-t}\cos(x)$ (dashed line) and the 5 and 10 q-heat polynomials expansion (solid line) at $q = 0.99$ for $t = 0.001$.

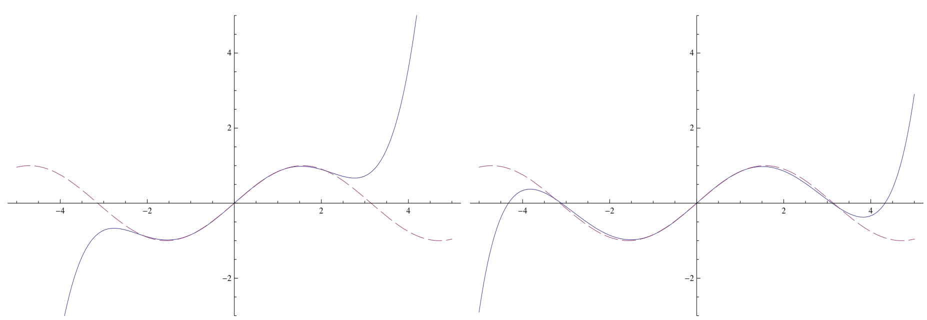

Figure 3: Comparison of the exact solution $u(x,t) = e^{-t}\sin(x)$ (dashed line) and the 5 and 10 q-heat polynomials expansion (solid line) at $q = 0.99$ for $t = 0.001$.

Example 2: Let $a \in \mathbb{R}_q$. The q-translation of the q-heat kernel $k(x,t;q)$ is given by

$$\mathcal{T}_{a,q}k(x,t;q) = \sum_{n=0}^{\infty} b_n(a;q^2)\partial_q^n k(x,t;q)$$

and by (67), we have

$$\mathcal{T}_{a,q}k(x,t;q) = \sum_{n=0}^{\infty} \frac{b_n(ia;q^2)}{[-i(1+q)]^n} w_n(x,t;q). \tag{104}$$

Let σ be a real such that $\sigma > \dfrac{(1-q)a^2}{q(1+q)}$ and consider the function

$$\phi(y) = \sum_{n=0}^{\infty} b_n(ia;q^2)y^n = e(iay;q^2). \tag{105}$$

Using the fact that $0 \leq \dfrac{a^2}{q(1+q)^2\sigma} < \dfrac{1}{1-q^2}$, we get for all nonnegative integer n,

$$\frac{\left(\frac{a^2}{q(1+q)^2\sigma}\right)^n}{n!_{q^2}} < \frac{1}{(q^2;q^2)_\infty}.$$

Then by (91), we obtain

$$\frac{|b_{2n}(ia;q^2)|(2n)!_{q^{-1}}}{n!_{q^{-2}}(1+q)^{2n}\sigma^n} = \frac{\left(\frac{qa^2}{(1+q)^2\sigma}\right)^n}{n!_{q^2}} \leq \frac{\left(\frac{a^2}{q(1+q)^2\sigma}\right)^n}{n!_{q^2}} \leq \frac{1}{(q^2;q^2)_\infty},$$

and by (92), we have

$$\frac{|b_{2n+1}(ia;q^2)|(2n+1)!_{q^{-1}}}{n!_{q^{-2}}(1+q)^{2n+1}\sigma^n} = \frac{|a|}{1+q}.\frac{\left(\frac{a^2}{q(1+q)^2\sigma}\right)^n}{n!_{q^2}} \leq \frac{|a|}{(1+q)(q^2;q^2)_\infty}.$$

We conclude that $\phi(y) \in \tilde{\mathcal{E}}_{\sigma,q}$, and by Theorem 6.3 the series (104) converges absolutely for $x \in \widetilde{\mathbb{R}}_q$, $t > \sigma \geq 0$. So, $\mathcal{T}_{a,q}k(x,t;q)$ is well defined and we have

$$\begin{aligned} \mathcal{T}_{a,q}k(x,t;q) &= K^2\int_{-\infty}^{\infty} exp_{q^2}(-ty^2)e(ixy;q^2)e(iay;q^2)d_qy \\ &= K\int_{-\infty}^{\infty} e(ixy;q^2)\mathcal{F}_q(k(.,t;q))(y)e(iay;q^2)d_qy. \end{aligned}$$

Note that, the q-integral representation of $\mathcal{T}_{a,q}k(x,t;q)$ coincides with the Rubin's one (see [9]). In a formal limit as q goes to 1^-, we obtain the classical result in [10]:

$$k(x+y,t) = \sum_{n=0}^{\infty}\frac{(-y)^n}{2^n n!}w_n(x,t), \quad 0 < t < \infty.$$

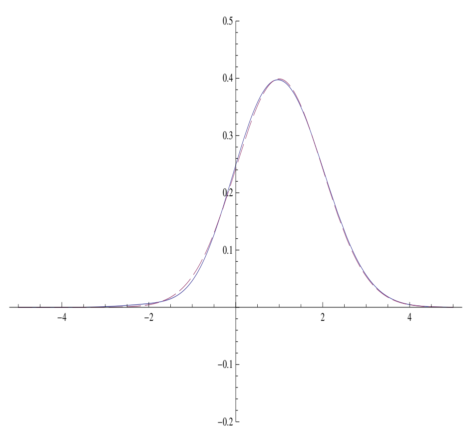

Figure 4: Comparison of the exact solution $k(x-1,t) = \dfrac{e^{-\frac{(x-1)^2}{4t}}}{(4\pi t)^{\frac{1}{2}}}$ (dashed line) and the 5 q-associated functions expansion (solid line) at $q = 0.99$ for $t = 0.5$.

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