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COMMON COUPLED FIXED POINT RESULTS IN PARTIALLY ORDERED G-METRIC SPACES

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Abstract. In this paper we obtain improved versions of some common coupled fixed point results for mappings in (ordered) G-metric spaces. Examples show that these improvements are proper.

1. Introduction

In 2004, Mustafa and Sims introduced a new notion of generalized metric space called G-metric space, where to every triplet of elements a non-negative real number is assigned [28]. Fixed point theory in such spaces was studied in [27, 29, 30]. Particularly, Banach contraction mapping principle was established in these works. After that several fixed point results were proved in these spaces (see, e.g., [2, 3, 4, 5, 10, 35, 36]).

Recently, fixed point theory has developed rapidly in partially ordered metric spaces. The first result in this direction was given by Ran and Reurings [33] who presented its applications to matrix equation. Subsequently, Nieto and Rodríguez-López [31] extended this result and applied it to obtain a unique solution for periodic boundary value problems. Further results were obtained by several authors, we mention [6, 20, 21].

The notion of a coupled fixed point was introduced and studied by Guo and Lakshmikantham [18] and Bhaskar and Lakshmikantham [14]. In subsequent papers several authors proved various coupled and common coupled fixed point theorems in (partially ordered) metric spaces (see, e.g., [19, 24, 25, 37]). These results were applied for investigation of solutions of differential and integral equations. Fixed point and coupled fixed point results in partially ordered G-metric spaces were obtained in, e.g., [8, 9, 15, 16, 26, 34, 38, 39, 40].

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In recent papers [7, 11, 12, 13, 17], a method was developed of reducing coupled fixed point results in (ordered) metric spaces to the respective results for mappings with one variable, even obtaining more general theorems. In this paper, we apply this method and obtain improved versions of some common coupled fixed point results for mappings in (ordered) G-metric spaces. Examples show that some of these improvements are proper.

2. Preliminaries

For more details on the following definitions and results concerning G-metric spaces, we refer the reader to [28].

Definition 2.1. *Let $\mathcal{X}$ be a nonempty set and let $g : \mathcal{X}\times\mathcal{X}\times\mathcal{X} \to \mathbb{R}^+$ be a function satisfying the following properties:*

(G1) $g(x, y, z) = 0$ *if* $x = y = z$*;*
(G2) $0 < g(x, x, y)$ *for all* $x, y \in \mathcal{X}$ *with* $x \neq y$*;*
(G3) $g(x, x, y) \leq g(x, y, z)$, *for all* $x, y, z \in \mathcal{X}$ *with* $z \neq y$*;*
(G4) $g(x, y, z) = g(x, z, y) = g(y, z, x) = \cdots$, *(symmetry in all three variables);*
(G5) $g(x, y, z) \leq g(x, a, a) + g(a, y, z)$, *for all* $x, y, z, a \in \mathcal{X}$ *(rectangle inequality).*

Then the function g is called a G-metric on $\mathcal{X}$ and the pair $(\mathcal{X}, g)$ is called a G-metric space.

Definition 2.2. *Let $(\mathcal{X}, g)$ be a G-metric space and let $\{x_n\}$ be a sequence of points in $\mathcal{X}$.*

(1) *A point $x \in \mathcal{X}$ is said to be the limit of a sequence $\{x_n\}$ if $\lim_{n,m\to\infty} g(x, x_n, x_m) = 0$, and one says that the sequence $\{x_n\}$ is g-convergent to x.*
(2) *The sequence $\{x_n\}$ is said to be a g-Cauchy sequence if, for every $\varepsilon > 0$, there is a positive integer N such that $g(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N$; that is, if $g(x_n, x_m, x_l) \to 0$, as $n, m, l \to \infty$.*
(3) *$(\mathcal{X}, g)$ is said to be g-complete (or a complete G-metric space) if every g-Cauchy sequence in $(\mathcal{X}, g)$ is g-convergent in $\mathcal{X}$.*

It was shown in [28] that the G-metric induces a Hausdorff topology and that the convergence, as described in the above definition, is relative to this topology. The topology being Hausdorff, a sequence can converge to at most one point.

Definition 2.3. *A G-metric space $(\mathcal{X}, g)$ is called symmetric if*

$$g(x, y, y) = g(x, x, y)$$

holds for all $x, y \in \mathcal{X}$.

The following are some examples of G-metric spaces.

Example 2.1. (1) Let $(\mathcal{X}, d)$ be an ordinary metric space. Define g_s by

$$g_s(x, y, z) = d(x, y) + d(y, z) + d(x, z)$$

for all $x, y, z \in \mathcal{X}$. Then it is clear that $(\mathcal{X}, g_s)$ is a symmetric G-metric space.

(2) Let $\mathcal{X} = \{a, b\}$. Define

$$g(a, a, a) = g(b, b, b) = 0, \;\; g(a, a, b) = 1, \;\; g(a, b, b) = 2,$$

and extend g to $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$ by using the symmetry in the variables. Then it is clear that $(\mathcal{X}, g)$ is an (asymmetric) G-metric space.

Remark 1. *If $(\mathcal{X}, g)$ is a G-metric space, then*

$$d_g(x, y) = g(x, y, y) + g(x, x, y)$$

defines a standard metric on $\mathcal{X}$. If the G-metric g is symmetric, this reduces to $d_g(x, y) = 2g(x, y, y)$. It has to be noted that in some cases contractive conditions given in g-metric can be reformulated and used in this standard metric, but there are a lot of situations where it is not possible.

Definition 2.4. [14, 24] *Let $(\mathcal{X}, \preceq)$ be a partially ordered set, $f : \mathcal{X}^2 \to \mathcal{X}$ and $h : \mathcal{X} \to \mathcal{X}$.*

(1) *f is said to have h-mixed monotone property if the following two conditions are satisfied:*

$$(\forall x_1, x_2, y \in \mathcal{X})\ hx_1 \preceq hx_2 \implies f(x_1, y) \preceq f(x_2, y),$$
$$(\forall x, y_1, y_2 \in \mathcal{X})\ hy_1 \preceq hy_2 \implies f(x, y_1) \succeq f(x, y_2).$$

If $h = i_{\mathcal{X}}$ (the identity map), we say that f has the mixed monotone property.

(2) *A point $(x, y) \in \mathcal{X} \times \mathcal{X}$ is said to be a coupled coincidence point of f and h if $f(x, y) = hx$ and $f(y, x) = hy$, and their common coupled fixed point if $f(x, y) = hx = x$ and $f(y, x) = hy = y$.*

If $\mathcal{X}$ is a nonempty set, then the triple $(\mathcal{X}, g, \preceq)$ will be called an ordered G-metric space if:

(i) $(\mathcal{X}, g)$ is a G-metric space, and
(ii) $(\mathcal{X}, \preceq)$ is a partially ordered set.

Definition 2.5. *Let $(\mathcal{X}, g, \preceq)$ be an ordered G-metric space. We say that $(\mathcal{X}, g, \preceq)$ is regular if the following conditions hold:*

(i) if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$,

(ii) if a non-increasing sequence $\{y_n\}$ is such that $y_n \to y$ as $n \to \infty$, then $y_n \succeq y$ for all $n \in \mathbb{N}$.

3. **Auxiliary results**

We will make use of the following auxiliary statement.

Lemma 3.1. *Let $(\mathcal{X}, g, \preceq)$ be a symmetric ordered G-metric space.*

(a) If relation $\sqsubseteq$ is defined on $\mathcal{X}^2$ by

$$X \sqsubseteq U \iff x \preceq u \wedge y \succeq v, \qquad X = (x, y), U = (u, v) \in \mathcal{X}^2,$$

and $G : \mathcal{X}^2 \times \mathcal{X}^2 \times \mathcal{X}^2 \to \mathbb{R}^+$ is given by

$$G(X, U, S) = g(x, u, s) + g(y, v, t), \qquad X = (x, y), Y = (u, v), S = (s, t) \in \mathcal{X}^2,$$

then $(\mathcal{X}^2, G, \sqsubseteq)$ is an ordered G-metric space. The space $(\mathcal{X}^2, G)$ is complete iff $(\mathcal{X}, g)$ is complete.

(b) If $h : \mathcal{X} \to \mathcal{X}$ is a self-map, and a mapping $f : \mathcal{X}^2 \to \mathcal{X}$ has the h-mixed monotone property, then the mapping $F : \mathcal{X}^2 \to \mathcal{X}^2$ given by

$$FX = (f(x, y), f(y, x)), \qquad X = (x, y) \in \mathcal{X}^2.$$

is H-nondecreasing w.r.t. $\sqsubseteq$, i.e.

$$HX \sqsubseteq HU \implies FX \sqsubseteq FU,$$

where $H : \mathcal{X}^2 \to \mathcal{X}^2$ is defined by $H(x,y) = (hx, hy)$.

(c) If h is continuous in $(\mathcal{X}, g)$ then H is continuous in $(\mathcal{X}^2, G)$. If f is continuous from $(\mathcal{X}^2, G)$ to $(\mathcal{X}, g)$ (i.e. $x_n \to x$ and $y_n \to y$ imply $f(x_n, y_n) \to f(x,y)$) then F is continuous in $(\mathcal{X}^2, G)$.

Proof. (a) Relation $\sqsubseteq$ is obviously a partial order on $\mathcal{X}^2$. In order to prove that G is a G-metric on $\mathcal{X}^2$, we will check conditions (G3) and (G5) (the other three are trivial).

(G3) Let $X = (x,y), U = (u,v), S = (s,t) \in \mathcal{X}^2$ be such that $U \neq S$, i.e., $u \neq s$ or $v \neq t$. If both of these relations hold, the proof is easy. Suppose, for instance, that $u \neq s$ and $v = t$. Then

$$\begin{aligned} G(X,X,U) &= g(x,x,u) + g(y,y,v) \leq g(x,u,s) + g(y,v,v) \\ &\leq g(x,u,s) + g(y,v,t) = G(X,U,S). \end{aligned}$$

(G5) Let $X = (x,y), U = (u,v), S = (s,t), A = (a,b) \in \mathcal{X}^2$. Then

$$\begin{aligned} G(X,U,S) &= g(x,u,s) + g(y,v,t) \\ &\leq g(x,a,a) + g(a,u,s) + g(y,b,b) + g(b,v,t) \\ &= G(X,A,A) + G(A,U,S). \end{aligned}$$

(b,c) The assertions about completeness and continuity follow directly from respective definitions. □

Remark 2. *It was shown in* [28, Theorem 4.1] *that if a G-metric g on $\mathcal{X}$ is asymmetric, then the function G (defined in the previous lemma) might not satisfy property (G3) of a G-metric. In that case a G-metric on $\mathcal{X}^2$ can still be defined, but becomes more involved, e.g. (see* [28, Theorem 4.2]*),*

$$\begin{aligned} G'(X,U,S) = \frac{1}{3}[&g(x,u,u) + g(x,x,u) + g(u,s,s) \\ &+ g(u,u,s) + g(s,x,x) + g(s,s,x)] \\ + \frac{1}{3}[&g(y,v,v) + g(y,y,v) + g(v,t,t) \\ &+ g(v,v,t) + g(t,y,y) + g(t,t,y)], \end{aligned}$$

for $X = (x,y)$, $U = (u,v)$, $S = (s,t) \in \mathcal{X}^2$ defines a symmetric G-metric. For the sake of simplicity, we will stay within the assumptions from the previous lemma, bearing in mind that most examples of G-metrics in applications are symmetric (see, e.g., Example 2.1(1)).

Assertions similar to the following lemma were used in the frame of metric spaces in the course of proofs of several fixed point results in various papers (see, e.g., [32, Lemma 2.1]).

Lemma 3.2. *Let $(\mathcal{X}, g)$ be a G-metric space and let $\{y_n\}$ be a sequence in $\mathcal{X}$ such that $\{g(y_n, y_{n+1}, y_{n+1})\}$ is non-increasing and*

$$\lim_{n\to\infty} g(y_n, y_{n+1}, y_{n+1}) = 0.$$

If $\{y_n\}$ is not a Cauchy sequence in $(\mathcal{X}, g)$, then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and the following four *sequences tend to ε when $k \to \infty$:*

$$g(y_{2m_k}, y_{2n_k}, y_{2n_k}), \quad g(y_{2m_k}, y_{2n_k-1}, y_{2n_k-1}),$$
$$g(y_{2m_k+1}, y_{2n_k}, y_{2n_k}), \quad g(y_{2n_k-1}, y_{2m_k+1}, y_{2m_k+1}).$$

Proof. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence in $(\mathcal{X}, g)$. Then there exists $\varepsilon > 0$ and sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and $g(y_{2m_k}, y_{2n_k}, y_{2n_k}) \geq \varepsilon$, and they can be chosen so that n_k is always the smallest possible, i.e., $g(y_{2m_k}, y_{2n_k-2}, y_{2n_k-2}) < \varepsilon$. Now, applying (G5) we get that

$$\begin{aligned}\varepsilon \leq g(y_{2m_k}, y_{2n_k}, y_{2n_k}) &\leq g(y_{2m_k}, y_{2n_k-2}, y_{2n_k-2}) \\ &\quad + g(y_{2n_k-2}, y_{2n_k-1}, y_{2n_k-1}) + g(y_{2n_k-1}, y_{2n_k}, y_{2n_k}) \\ &< \varepsilon + g(y_{2n_k-2}, y_{2n_k-1}, y_{2n_k-1}) + g(y_{2n_k-1}, y_{2n_k}, y_{2n_k}).\end{aligned}$$

Passing to the limit as $k \to \infty$ we get that $\lim_{k\to\infty} g(y_{2m_k}, y_{2n_k}, y_{2n_k}) = \varepsilon$.

Now, again by (G5), we have that

$$g(y_{2m_k}, y_{2n_k}, y_{2n_k}) \leq g(y_{2m_k}, y_{2n_k-1}, y_{2n_k-1}) + g(y_{2n_k-1}, y_{2n_k}, y_{2n_k}),$$
$$g(y_{2m_k}, y_{2n_k-1}, y_{2n_k-1}) \leq g(y_{2m_k}, y_{2n_k}, y_{2n_k}) + g(y_{2n_k}, y_{2n_k-1}, y_{2n_k-1}).$$

Passing to the limit as $k \to \infty$ we get that $\lim_{k\to\infty} g(y_{2m_k}, y_{2n_k-1}, y_{2n_k-1}) = \varepsilon$

The proof for the remaining two sequences is similar. □

4. **Main results**

Our first main result is the following

Theorem 4.1. *Let $(\mathcal{X}, g, \preceq)$ be a complete partially ordered symmetric G-metric space and let $f : \mathcal{X}^2 \to \mathcal{X}$ satisfies the following conditions:*

(1) *f satisfies the mixed monotone property.*
(2) *There exist nonnegative real numbers a, b, c, d such that $a+b+c+d < 1$ and for all $x, u, s, y, v, t \in \mathcal{X}$ satisfying ($x \preceq u \preceq s$ and $y \succeq v \succeq t$) or ($x \succeq u \succeq s$ and $y \preceq v \preceq t$) the following inequality holds*

$$\begin{aligned}g(f(x,y), f(u,v), f(s,t)) &+ g(f(y,x), f(v,u), f(t,s)) \\ &\leq a[g(x,u,s) + g(y,v,t)] \\ &\quad + b[g(x, f(x,y), f(x,y)) + g(y, f(y,x), f(y,x))] \\ &\quad + c[g(u, f(u,v), f(u,v)) + g(v, f(v,u), f(v,u))] \\ &\quad + d[g(s, f(s,t), f(s,t)) + g(t, f(t,s), f(t,s))].\end{aligned}$$

(3) *There exist $x_0, y_0 \in \mathcal{X}$ such that $x_0 \preceq f(x_0, y_0)$ and $y_0 \succeq f(y_0, x_0)$ (or vice versa).*

Then f has a coupled fixed point in $\mathcal{X}^2$.

Proof. Define the order $\sqsubseteq$ and G-metric G on $\mathcal{X}^2$, as well as the mapping $F : \mathcal{X}^2 \to \mathcal{X}^2$ as in Lemma 3.1. Then $(\mathcal{X}, G, \sqsubseteq)$ is a complete ordered G-metric space and:

(1) $F : \mathcal{X}^2 \to \mathcal{X}^2$ is a nondecreasing mapping (w.r.t. $\sqsubseteq$).
(2) For all $X, U, S \in \mathcal{X}^2$ satisfying $X \sqsubseteq U \sqsubseteq S$ or $S \sqsubseteq U \sqsubseteq X$, the following inequality holds ($a + b + c + d < 1$):

$$\begin{aligned}G(FX, FU, FS) \leq aG(X, U, S) &+ bG(X, FX, FX) \\ &+ cG(U, FU, FU) + dG(S, FS, FS).\end{aligned}$$

(3) There exist $X_0 \in \mathcal{X}^2$ such that $X_0 \sqsubseteq FX_0$ (or vice versa).

Hence, F satisfies all the conditions of the "ordered version" of [27, Theorem 2.1], which can be proved similarly as the "unordered version" given in [27] with the usual adaptations for the ordered case (only the proof of uniqueness in this case needs additional assumptions). Thus, the mapping F has a fixed point in $\mathcal{X}^2$, i.e., there exists $\bar{X} = (\bar{x}, \bar{y}) \in \mathcal{X}^2$ such that $f(\bar{x}, \bar{y}) = \bar{x}$ and $f(\bar{y}, \bar{x}) = \bar{y}$, which means that f has a coupled fixed point in $\mathcal{X}^2$. $\square$

In particular, when $b = c = d = 0$, we get

Corollary 4.1. *Let $(\mathcal{X}, g, \preceq)$ be a complete partially ordered symmetric G-metric space and let $f : \mathcal{X}^2 \to \mathcal{X}$ satisfies the following conditions:*

(1) *f satisfies the mixed monotone property.*
(2) *There exists $a \in [0, 1)$ such that for all $x, u, s, y, v, t \in \mathcal{X}$ satisfying ($x \preceq u \preceq s$ and $y \succeq v \succeq t$) or ($x \succeq u \succeq s$ and $y \preceq v \preceq t$) the following inequality holds*

$$\begin{aligned} g(f(x,y), f(u,v), f(s,t)) + g(f(y,x), f(v,u), f(t,s)) \\ \leq a[g(x,u,s) + g(y,v,t)] \end{aligned} \tag{1}$$

(3) *There exist $x_0, y_0 \in \mathcal{X}$ such that $x_0 \preceq f(x_0, y_0)$ and $y_0 \succeq f(y_0, x_0)$ (or vice versa).*

Then f has a coupled fixed point in $\mathcal{X}^2$.

We will show in the next example that there exist situations when this corollary can be used for the proof of existence of a coupled fixed point, while [16, Theorem 3.1] cannot.

Example 4.1. Let $\mathcal{X} = \mathbb{R}$ be equipped with standard order $\leq$ and G-metric given as $g(x, y, z) = |x - y| + |y - z| + |z - x|$. Then $(\mathcal{X}, g, \leq)$ is a complete partially ordered symmetric G-metric space. Define $f : \mathcal{X}^2 \to \mathcal{X}$ as $f(x, y) = \dfrac{x - 4y}{6}$ and take $a = \frac{5}{6}$. Then f has the mixed monotone property and for all $x, y, u, v, s, t \in \mathcal{X}$ satisfying $x \leq u \leq s$ and $y \leq v \leq t$ (or vice versa), the following holds

$$\begin{aligned} &g(f(x,y), f(u,v), f(s,t)) + g(f(y,x), f(v,u), f(t,s)) \\ &\quad = g\left(\frac{x-4y}{6}, \frac{u-4v}{6}, \frac{s-4t}{6}\right) + g\left(\frac{y-4x}{6}, \frac{v-4u}{6}, \frac{t-4s}{6}\right) \\ &\quad = \left|\frac{x-4y}{6} - \frac{u-4v}{6}\right| + \left|\frac{u-4v}{6} - \frac{s-4t}{6}\right| + \left|\frac{s-4t}{6} - \frac{x-4y}{6}\right| \\ &\qquad + \left|\frac{y-4x}{6} - \frac{v-4u}{6}\right| + \left|\frac{v-4u}{6} - \frac{t-4s}{6}\right| + \left|\frac{t-4s}{6} - \frac{y-4x}{6}\right| \\ &\quad = \left|\frac{1}{6}(x-u) - \frac{4}{6}(y-v)\right| + \left|\frac{1}{6}(u-s) - \frac{4}{6}(v-t)\right| + \left|\frac{1}{6}(s-x) - \frac{4}{6}(t-y)\right| \\ &\qquad + \left|\frac{1}{6}(y-v) - \frac{4}{6}(x-u)\right| + \left|\frac{1}{6}(v-t) - \frac{4}{6}(u-s)\right| + \left|\frac{1}{6}(t-y) - \frac{4}{6}(s-x)\right| \\ &\quad \leq \frac{1}{6}[|x-u| + |y-v|] + \frac{4}{6}[|x-u| + |y-v|] + \frac{1}{6}[|u-s| + |v-t|] \\ &\qquad + \frac{4}{6}[|u-s| + |v-t|] + \frac{1}{6}[|s-x| + |t-y|] + \frac{4}{6}[|s-x| + |t-y|] \end{aligned}$$

$$= \frac{5}{6}\{[|x-u|+|u-s|+|s-x|]+[|y-v|+|v-t|+|t-y|]\}$$
$$= a[g(x,u,s)+g(y,v,t)].$$

Hence, condition (1) is satisfied and, by Corollary 4.1, f has a coupled fixed point (which is $(0,0)$).

On the other hand, suppose that condition

$$g(f(x,y),f(u,v),f(s,t)) \le \frac{a}{2}[g(x,u,s)+g(y,v,t)] \tag{2}$$

(condition (3.1) from [16]) holds for some $a \in [0,1)$ and all $x,y,u,v,s,t \in \mathcal{X}$ satisfying $x \le u \le s$ and $y \le v \le t$ (or vice versa) and $u \ne s$ or $v \ne t$ (assume, e.g., $v \ne t$). This means that

$$\left|\frac{x-4y}{6}-\frac{u-4v}{6}\right|+\left|\frac{u-4v}{6}-\frac{s-4t}{6}\right|+\left|\frac{s-4t}{6}-\frac{x-4y}{6}\right|$$
$$\le \frac{a}{2}[|x-u|+|u-s|+|s-x|]+[|y-v|+|v-t|+|t-y|].$$

Putting $x=u=s$ in this inequality, we get that

$$\frac{2}{3}[|y-v|+|v-t|+|t-y|] \le \frac{a}{2}[|y-v|+|v-t|+|t-y|],$$

and since $v \ne t$, we get that $\frac{2}{3} \le \frac{a}{2}$, i.e. $a \ge \frac{4}{3}$, a contradiction.

Since, obviously, condition (2) implies condition (1), we conclude that (in the case of a symmetric G-metric) Corollary 4.1 is a strict improvement of [16, Theorem 3.1].

We will consider now mappings satisfying so-called weak contractive conditions, involving also an additional mapping. For the sake of simplicity, we will treat the "unordered" case, although a similar result can be obtained in ordered spaces, as well. We will use the following notion.

Definition 4.1. [1, 2] *Mappings $h:\mathcal{X}\to\mathcal{X}$ and $f:\mathcal{X}^2\to\mathcal{X}$ are called w-compatible if they commute at their coincidence points, i.e., $hx=f(x,y)$ and $hy=f(y,x)$ imply that $hf(x,y)=f(hx,hy)$.*

Theorem 4.2. *Let $(\mathcal{X},g)$ be a symmetric G-metric space, and let two mappings $h:\mathcal{X}\to\mathcal{X}$ and $f:\mathcal{X}^2\to\mathcal{X}$ satisfy the following conditions:*

1. *h and f are w-compatible,*
2. *$f(\mathcal{X}^2)\subset h\mathcal{X}$ and $h\mathcal{X}$ is a complete subset of $\mathcal{X}$,*
3. *for all $x,y,u,v,s,t\in\mathcal{X}$ the following condition holds*

$$\begin{aligned}
&g(f(x,y),f(u,v),f(s,t))+g(f(y,x),f(v,u),f(t,s))\\
&\quad\le \frac{1}{3}[g(hx,f(u,v),f(u,v))+g(hy,f(v,u),f(v,u))\\
&\qquad\quad+g(hu,f(s,t),f(s,t))+g(hv,f(t,s),f(t,s))\\
&\qquad\quad+g(hs,f(x,y),f(x,y))+g(ht,f(y,x),f(y,x))]\\
&\qquad-\varphi(g(hx,f(u,v),f(u,v))+g(hy,f(v,u),f(v,u)),\\
&\qquad\quad g(hu,f(s,t),f(s,t))+g(hv,f(t,s),f(t,s)),\\
&\qquad\quad g(hs,f(x,y),f(x,y))+g(ht,f(y,x),f(y,x))),
\end{aligned}$$

where $\varphi:[0,+\infty)^3\to[0,+\infty)$ is a continuous function such that $\varphi(p,q,r)=0$ iff $p=q=r=0$.

Then h and f have a unique common coupled fixed point.

Remark 3. *It would be possible to add also an altering distance function ψ (in the sense of Khan et al.* [23]*) into the contractive condition. But, taking into account the results of Jachymski from* [22]*, we will not do it here.*

Proof. Define again the G-metric G on $\mathcal{X}^2$, as well as the mappings $F, H : \mathcal{X}^2 \to \mathcal{X}^2$ as in Lemma 3.1. Then $(\mathcal{X}, G)$ is a G-metric space and:

(1) H and F are weakly compatible (in the sense of Jungck),
(2) $F\mathcal{X}^2 \subset H\mathcal{X}^2$ and the last is a complete subset of $\mathcal{X}^2$,
(3) for all $X, U, S \in \mathcal{X}^2$ the following condition holds

$$\begin{aligned} G(FX, FU, FS) \leq \frac{1}{3}&[G(HX, FU, FU) + G(HU, FS, FS) + G(HS, FX, FX)] \\ &- \varphi(HX, FU, FU), G(HU, FS, FS), G(HS, FX, FX)), \end{aligned}$$

where $\varphi : [0, +\infty)^3 \to [0, +\infty)$ is a continuous function such that $\varphi(p, q, r) = 0$ iff $p = q = r = 0$.

Hence, all the conditions of [10, Theorem 2.2] are fulfilled and it follows that H and F have a unique common fixed point $\bar{X} = (\bar{x}, \bar{y})$. This point is obviously a unique common coupled fixed point of h and f. □

Similarly as in [10], the following corollary is obtained taking $\varphi(p, q, r) = (\frac{1}{3} - \alpha)(p + q + r)$ for $\alpha \in [0, \frac{1}{3})$. We note that this corollary is an improvement of [10, Corollary 2.4] since the contractive condition of [10, Corollary 2.4] implies (3).

Corollary 4.2. *Let $(\mathcal{X}, g)$ be a symmetric G-metric space, and let two mappings $h : \mathcal{X} \to \mathcal{X}$ and $f : \mathcal{X}^2 \to \mathcal{X}$ satisfy the following conditions:*

(1) *h and f are w-compatible,*
(2) *$f(\mathcal{X}^2) \subset h\mathcal{X}$ and $h\mathcal{X}$ is a complete subset of $\mathcal{X}$,*
(3) *for all $x, y, u, v, s, t \in \mathcal{X}$ the following condition holds*

$$\begin{aligned} g(f(x, y), f(u, v), f(s, t)) &+ g(f(y, x), f(v, u), f(t, s)) \\ \leq \alpha[&g(hx, f(u, v), f(u, v)) + g(hy, f(v, u), f(v, u)) \\ &+ g(hu, f(s, t), f(s, t)) + g(hv, f(t, s), f(t, s)) \\ &+ g(hs, f(x, y), f(x, y)) + g(ht, f(y, x), f(y, x))], \end{aligned} \tag{3}$$

where $\alpha \in [0, \frac{1}{3})$.

Then h and f have a unique common coupled fixed point.

In order to obtain the next result, we will prove first the following lemma. We will denote by Φ the set of functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying $\lim_{t \to r} \varphi(t) > 0$ for each $r > 0$ (note that we again do not use an additional altering distance function ψ, see Remark 3).

Lemma 4.1. *Let $(\mathcal{X}, G, \preceq)$ be a complete partially ordered G-metric space and let $T : \mathcal{X} \to \mathcal{X}$ satisfy the following conditions:*

(i) *T is non-decreasing and there exists $x_0 \in \mathcal{X}$ such that $x_0 \preceq Tx_0$,*
(ii) *there exists $\varphi \in \Phi$ such that the following contractive condition is satisfied for all $x, y, z \in \mathcal{X}$ satisfying $x \preceq y \preceq z$ or $x \succeq y \succeq z$:*

$$G(Tx, Ty, Tz) \leq G(x, y, z) - \varphi(G(x, y, z)), \tag{4}$$

(iii) *T is continuous, or*

(iii') $(\mathcal{X}, G, \preceq)$ *is regular in the sense of Definition 2.5.*

Then T has a fixed point in $\mathcal{X}$.

Proof. Let $x_0 \in \mathcal{X}$ be such that $x_0 \preceq Tx_0$. Construct the Picard sequence $\{x_n\}$ by $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$ Using monotonicity of T we conclude by induction that $\{x_n\}$ is non-decreasing. If $x_{n_0+1} = x_{n_0}$ for some n_0, then it is a fixed point of T. Thus we shall assume that $G(x_n, x_{n+1}, x_{n+1}) > 0$ for all $n \geq 0$. We first prove that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{5}$$

Since $x_n \preceq x_{n+1} \preceq x_{n+1}$, we can use (4) for these points, and we get that, for $n \geq 0$,

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+2}) &= G(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ &\leq G(x_n, x_{n+1}, x_{n+1}) - \varphi(G(x_n, x_{n+1}, x_{n+1})). \end{aligned} \tag{6}$$

In particular, $G(x_{n+1}, x_{n+2}, x_{n+2}) \leq G(x_n, x_{n+1}, x_{n+1})$ for all $n \geq 0$. It means that there exists $\rho \geq 0$ such that $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = \rho$. Now suppose that $\rho > 0$. Taking $n \to \infty$ in (6), and using the property of $\varphi \in \Phi$, we obtain $\rho < \rho$, a contradiction. We conclude that (5) holds.

We will prove now that $\{x_n\}$ is a G-Cauchy sequence in $\mathcal{X}$. Suppose this is not the case. Then, by Lemma 3.2, there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the sequences

$$\begin{gathered} G(x_{2m_k}, x_{2n_k}, x_{2n_k}), \quad G(x_{2m_k}, x_{2n_k-1}, x_{2n_k-1}), \\ G(x_{2m_k+1}, x_{2n_k}, x_{2n_k}), \quad G(x_{2n_k-1}, x_{2m_k+1}, x_{2m_k+1}) \end{gathered}$$

tend to ε when $k \to \infty$. Putting $x = x_{2m_k}$, $y = x_{2n_k-1}$, $z = x_{2n_k-1}$ in (4) (which can be done since the sequence $\{x_n\}$ is monotone) we have

$$\begin{aligned} G(x_{2m_k+1}, x_{2n_k}, x_{2n_k})) &= G(Tx_{2m_k}, Tx_{2n_k-1}, Tx_{2n_k-1})) \\ &\leq G(x_{2m_k}, x_{2n_k-1}, x_{2n_k-1}) - \varphi(G(x_{2m_k}, x_{2n_k-1}, x_{2n_k-1})). \end{aligned}$$

Letting $k \to \infty$, utilizing obtained limits and the property of function $\varphi \in \Phi$, we get $\varepsilon < \varepsilon$, which is a contradiction. We have proved that $\{x_n\}$ is a Cauchy sequence in $(\mathcal{X}, G)$. Hence, there exists $z \in \mathcal{X}$ such that $\lim_{n\to\infty} x_n = z$.

Suppose that (i) holds. Passing to the limit as $n \to \infty$ in $x_{n+1} = Tx_n$ we readily get that $z = Tz$.

Suppose that (ii) holds. Then $x_n \preceq z \preceq z$ and we can apply (4) for these points to get

$$G(x_{n+1}, Tz, Tz) = G(Tx_n, Tz, Tz) \leq G(x_n, z, z) - \varphi(G(x_n, z, z)).$$

Letting $n \to \infty$ we obtain that $G(z, Tz, Tz) = 0$, i.e., $z = Tz$. $\square$

Applying the method used in the proofs of Theorems 4.1 and 4.2, and using Lemma 4.1, we obtain the following result.

Theorem 4.3. *Let $(\mathcal{X}, g, \preceq)$ be a complete partially ordered symmetric G-metric space. Suppose that $f : \mathcal{X}^2 \to \mathcal{X}$ satisfies the following conditions:*

(1) *f has the mixed monotone property,*
(2) *there exist $x_0, y_0 \in \mathcal{X}$ such that $x_0 \preceq f(x_0, y_0)$ and $y_0 \succeq f(y_0, x_0)$,*

(3) *there is a function* $\varphi \in \Phi$ *such that*

$$g(f(x,y), f(u,v), f(s,t)) + g(f(y,x), f(v,u), f(t,s)) \\ \leq g(x,u,s) + g(y,v,t) - \varphi(g(x,u,s) + g(y,v,t)) \tag{7}$$

holds for all $x, u, s, y, v, t \in \mathcal{X}$ *satisfying* ($x \preceq u \preceq s$ *and* $y \succeq v \succeq t$) *or* ($x \succeq u \succeq s$ *and* $y \preceq v \preceq t$),

(4) f *is continuous, or*

(4') $(\mathcal{X}, g, \preceq)$ *is regular.*

Then f *has a coupled fixed point in* $\mathcal{X}$.

This theorem is obviously an improvement of [26, Corollary 2.3], since condition (2.27) from [26], namely

$$g(f(x,y), f(u,v), f(s,t)) \leq \frac{1}{2}[g(x,u,s) + g(y,v,t)] - \psi(g(x,u,s) + g(y,v,t)) \tag{8}$$

implies our condition (7), when one takes $\varphi = 2\psi$. The following example shows that this improvement is proper.

Example 4.2. Similarly as in Example 4.1, let $\mathcal{X} = \mathbb{R}$ be equipped with standard order $\leq$ and G-metric given as $g(x,y,z) = |x-y| + |y-z| + |z-x|$. Then $(\mathcal{X}, g, \leq)$ is a complete partially ordered symmetric G-metric space. Define $f : \mathcal{X}^2 \to \mathcal{X}$ as $f(x,y) = \dfrac{x-2y}{4}$ and take $\varphi(t) = \frac{1}{4}t$. Then f has the mixed monotone property and for all $x, y, u, v, s, t \in \mathcal{X}$ satisfying $x \leq u \leq s$ and $y \leq v \leq t$ (or vice versa), the following holds

$$\begin{aligned} &g(f(x,y), f(u,v), f(s,t)) + g(f(y,x), f(v,u), f(t,s)) \\ &\quad \leq \frac{3}{4}\{[|x-u| + |u-s| + |s-x|] + [|y-v| + |v-t| + |t-y|]\} \\ &\quad = g(x,u,s) + g(y,v,t) - \varphi(g(x,u,s) + g(y,v,t)). \end{aligned}$$

Hence, condition (7) is satisfied and, by Theorem 4.3, f has a coupled fixed point (which is $(0,0)$).

On the other hand, suppose that condition (8) is satisfied with $\psi(t) = \frac{1}{2}\varphi(t) = \frac{1}{8}t$, i.e.,

$$\left|\frac{x-2y}{4} - \frac{u-2v}{4}\right| + \left|\frac{u-2v}{4} - \frac{s-2t}{4}\right| + \left|\frac{s-2t}{4} - \frac{x-2y}{4}\right| \\ \leq \frac{3}{8}[|x-u| + |u-s| + |s-x|] + [|y-v| + |v-t| + |t-y|].$$

Putting $x = u = s$ in this inequality, we get that

$$\frac{1}{2}[|y-v| + |v-t| + |t-y|] \leq \frac{3}{8}[|y-v| + |v-t| + |t-y|],$$

and if, e.g., $v \neq t$, we get that $\frac{1}{2} \leq \frac{3}{8}$, which is not true.

The next example shows the importance of order defined on a G-metric space.

Example 4.3. Let $\mathcal{X} = [0, +\infty)$ be equipped with the G-metric $g(x,y,z) = |x-y| + |y-z| + |z-x|$ and the order $\preceq$ defined by

$$x \preceq y \iff x = y \vee (x, y \in [0,1] \wedge x \leq y).$$

Then $(\mathcal{X}, g, \preceq)$ is a complete partially ordered symmetric G-metric space. Consider the (continuous) mapping $f : \mathcal{X}^2 \to \mathcal{X}$ given by

$$f(x, y) = \begin{cases} \frac{1}{6}x, & x \in [0, 1], y \in \mathcal{X}, \\ x - \frac{5}{6}, & x > 1, y \in \mathcal{X}. \end{cases}$$

Obviously, f has the mixed monotone property. Let $x, y, u, v, s, t \in \mathcal{X}$ be such that $x \preceq u \preceq s$ and $y \succeq v \succeq t$. Then the following cases are possible.

1) All of these variables belong to $[0, 1]$ and, hence $x \leq u \leq s$ and $y \geq v \geq t$. If we denote by L and R, respectively, the left-hand and right-hand side of inequality (7), then

$$\begin{aligned} L &= g(\tfrac{1}{6}x, \tfrac{1}{6}u, \tfrac{1}{6}s) + g(\tfrac{1}{6}y, \tfrac{1}{6}v, \tfrac{1}{6}t) \\ &= \frac{1}{6}(|x-u| + |u-s| + |s-x| + |y-v| + |v-t| + |t-y|) \\ &\leq \frac{3}{4}(|x-u| + |u-s| + |s-x| + |y-v| + |v-t| + |t-y|) = R, \end{aligned}$$

if we take, e.g., $\varphi \in \Phi$ to be given by $\varphi(t) = \frac{1}{4}t$.

2) $x, u, s \in [0, 1]$ (and $x \leq u \leq s$) and $y, v, t > 1$ (and $y = v = t$). Then we have

$$\begin{aligned} L &= g(\tfrac{1}{6}x, \tfrac{1}{6}u, \tfrac{1}{6}s) - g(y - \tfrac{5}{6}, y - \tfrac{5}{6}, y - \tfrac{5}{6}) \\ &= \frac{1}{6}(|x-u| + |u-s| + |s-x|) \leq \frac{3}{4}(|x-u| + |u-s| + |s-x|) = R. \end{aligned}$$

The case when $x, u, s > 1$ and $y, v, t \in [0, 1]$ is similar.

3) $x, y, u, v, s, t > 1$. Then $L = R = 0$.

Thus, all the conditions of Theorem 4.3 are fulfilled and f has a coupled fixed point (which is $(0, 0)$).

However, consider the same G-metric space $(\mathcal{X}, g)$ without order. Take $(x, y) = (2, 2)$, $(u, v) = (2, 3)$ and $(s, t) = (3, 3)$. Then we have

$$\begin{aligned} L &= g(f(2, 2), f(2, 3), f(3, 3)) + g(f(2, 2), f(3, 2), f(3, 3)) \\ &= g(\tfrac{7}{6}, \tfrac{7}{6}, \tfrac{13}{6}) + g(\tfrac{7}{6}, \tfrac{13}{6}, \tfrac{13}{6}) = 2 + 2 = 4, \end{aligned}$$

and

$$\begin{aligned} R &= g(2, 2, 3) + g(2, 3, 3) - \varphi(g(2, 2, 3) + g(2, 3, 3)) \\ &= 2 + 2 - \varphi(2 + 2) = 4 - \varphi(4) < 4, \end{aligned}$$

i.e., $L > R$ whatever function $\varphi \in \Phi$ is chosen, and the contractive condition is not satisfied.

Remark 4. *In a very similar way, the results from* [8, 9] *can be modified.*

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