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REMARKS ON COUPLED FIXED POINT THEOREMS IN PARTIALLY ORDERED METRIC SPACES

(COMMUNICATED BY PROFESSOR S. MECHERI)

ERDAL KARAPINAR

Abstract. In this manuscript, we prove new coupled fixed point theorems extending some recent results in the literature on this topic. We also present applications of these new results through a number of examples.

1. Introduction and Preliminaries

Since 1922, Banach contraction mapping principle, considered as one of the cornerstones of the fixed point theory, has been in the center of long lasting fascination among the mathematicians interested in many branches of mathematics, especially in nonlinear analysis (see e.g. [1]-[30]). One of the analogs of this celebrated principle in the context of partially ordered metric spaces was first proved by Ran and Reurings [26]. Following this initial paper, many authors produced remarkable results in this direction (see e.g. [1]-[30]). Particularly, Gnana-Bhaskar and Lakshmikantham [5] introduced the notions of mixed monotone property and coupled fixed point in the context of partially ordered metric spaces. In this intriguing paper, they additionally discussed the existence and uniqueness of coupled fixed point and certain applications on periodic boundary value problems. Because of these effective applications on differential equations, their paper attracted considerable attention (see e.g. [2]-[18], [21]-[23],[27]-[30]).

We start with listing some notation and main definitions on these topics that we shall need to convey our theorems.

Definition 1. *(See [5]) Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping. Then F is said to have mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$,*

$$x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y), \quad \text{for } x_1, x_2 \in X, \quad \text{and}$$
$$y_1 \leq y_2 \Rightarrow F(x, y_2) \leq F(x, y_1), \ \text{for } y_1, y_2 \in X.$$

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Definition 2. *(see [5]) An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \to X$ if*

$$F(x, y) = x \text{ and } F(y, x) = y.$$

Throughout this paper, $(X, d, \leq)$ denotes a partially ordered metric space where $(X, \leq)$ is a partially ordered set and (X, d) is a metric space for a given metric d on X. Furthermore, the product space $X \times X$ satisfies the following comparability property:

$$(u, v) \leq (x, y) \Leftrightarrow u \leq x,\ y \geq v; \quad \text{for all } (x, y), (u, v) \in X \times X. \tag{1.1}$$

Hereafter, we assume that $X \neq \emptyset$ and use the notation $X^2 = X \times X$. Then the mapping $\rho_2 : X^2 \times X^2 \to [0, +\infty)$ defined by

$$\rho_2(\mathbf{x}, \mathbf{y}) = \max\{d(x_1, y_1), d(x_2, y_2)\}$$

forms a metric on X^2 where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ are in X^2.

Motivated by Definition 1, the following concept of a g-mixed monotone mapping is introduced by V. Lakshmikantham and L.Ćirić [21].

Definition 3. *Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$ be two mappings. Then F is said to have mixed g-monotone property if $F(x, y)$ is monotone g-non-decreasing in x and is monotone g-non-increasing in y, that is, for any $x, y \in X$,*

$$g(x_1) \leq g(x_2) \Rightarrow F(x_1, y) \leq F(x_2, y), \quad \text{for } x_1, x_2 \in X, \quad \text{and} \tag{1.2}$$

$$g(y_1) \leq g(y_2) \Rightarrow F(x, y_2) \leq F(x, y_1), \ \text{for } y_1, y_2 \in X. \tag{1.3}$$

It is clear that Definition 3 reduces to Definition 1 when g is the identity.

Definition 4. *An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $F : X \times X \to X$ and $g : X \to X$ if*

$$F(x, y) = g(x), \qquad F(y, x) = g(y).$$

Moreover, $(x, y) \in X \times X$ is called a common coupled coincidence point of F and g if

$$F(x, y) = g(x) = x, \qquad F(y, x) = g(y) = y.$$

Definition 5. *Let $F : X \times X \to X$ and $g : X \to X$ be mappings where $X \neq \emptyset$. The mappings F and g are said to commute if*

$$g(F(x, y)) = F(g(x), g(y)), \quad \text{for all } x, y \in X.$$

Definition 6. *(See e.g. [14]) The mappings F and g where $F : X \times X \to X$, $g : X \to X$ are said to be compatible if*

$$\lim_{n\to\infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = 0$$

and

$$\lim_{n\to\infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = 0$$

where $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *such that* $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x$ *and* $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y$ *for all* $x, y \in X$ *are satisfied.*

In this paper, we investigate the existence and uniqueness of a coupled coincidence point of the Meir-Keeler type contraction mappings in the context of partially ordered metric spaces. Our results enrich, improve and generalize some well-known results in the literature. We give some examples to illustrate our results.

2. Existence of a Coupled Fixed Point

In this section, we aim to review a major theorem on coupled fixed points and improve on it. In [27] Samet introduced the definition below to modify the Meir-Keeler contraction and extended its applications.

Definition 7. *Let* $(X, d, \leq)$ *be a partially ordered metric space and* $F : X \times X \to X$ *be a mapping. The operator* F *is said to be a generalized Meir-Keeler type function if for all* $\varepsilon > 0$ *there exists a* $\delta(\varepsilon) > 0$ *such that*

$$\varepsilon \leq \frac{1}{2}[d(x,u) + d(y,v)] < \varepsilon + \delta(\varepsilon) \Rightarrow d(F(x,y), F(u,v)) < \varepsilon, \tag{2.1}$$

for all $x, y, u, v \in X$ *with* $x \leq u$, $y \geq v$.

Upon this definition, Samet [27] proved the following theorem.

Theorem 8. *Let* $(X, d, \leq)$ *be a partially ordered complete metric space and* $F : X^2 \to X$ *be a mapping satisfying the following hypothesis:*

- (i) F *is continuous,*
- (ii) F *has the mixed strict monotone property,*
- (iii) F *is a generalized Meir-Keeler type function,*
- (iv) *there exist* $x_0, y_0 \in X$ *such that*

$$x_0 < F(x_0, y_0), \quad y_0 \geq F(y_0, x_0). \tag{2.2}$$

Then, there exists $(x, y) \in X \times X$ *such that*

$$F(x,y) = x, \quad F(y,x) = y.$$

We first give a modified version of Definition 7 as follows.

Definition 9. *Let* $(X, d, \leq)$ *be a partially ordered metric space and* $F : X \times X \to X$ *and* $g : X \to X$ *be two mappings. The operator* F *is said to be a generalized* g*-Meir-Keeler type contraction if for any* $\varepsilon > 0$ *there exists a* $\delta(\varepsilon) > 0$ *such that*

$$\begin{aligned} \varepsilon \leq \quad & \max\{d(g(x), g(u)), d(g(y), g(v))\} < \varepsilon + \delta(\varepsilon) \\ & \Rightarrow \max\{d(F(x,y), F(u,v)), d(F(y,x), F(v,u))\} < \varepsilon, \end{aligned} \tag{2.3}$$

for all $x, y, u, v \in X$ *with* $g(x) \leq g(u)$, $g(y) \geq g(v)$.

Remark 10. *If we replace* g *with the identity in (2.3), we get the definition of generalized Meir-Keeler type contraction, that is, for any* $\varepsilon > 0$ *there exists a* $\delta(\varepsilon) > 0$ *such that*

$$\varepsilon \leq \max\{d(x,u), d(y,v)\} < \varepsilon + \delta(\varepsilon) \Rightarrow \max\{d(F(x,y), F(u,v)), d(F(y,x), F(v,u))\} < \varepsilon, \tag{2.4}$$

for all $x, y, u, v \in X$ *with* $x \leq u$, $y \geq v$.

The following fact can be derived easily from Definition 9.

Lemma 11. *Let $(X, d, \leq)$ be a partially ordered metric space and $F : X \times X \to X$, $g : X \to X$. If F is a generalized g-Meir-Keeler type contraction, then we have*

$$\max\{d(F(x,y), F(u,v)), d(F(y,x), F(v,u))\} \leq \max\{d(g(x), g(u)), d(g(y), g(v))\}$$

for all $x, y, u, v \in X$ with $g(x) < g(u), g(y) \geq g(v)$ or $g(x) \leq g(u), g(y) > g(v)$.

Proof. Without loss of generality, we assume that $g(x) < g(u), g(y) \geq g(v)$ where $x, y, u, v \in X$. Thus, we have $\max\{d(g(x), g(u)), d(g(y), g(v))\} > 0$. Set $\varepsilon = \max\{d(g(x), g(u)), d(g(y), g(v))\} > 0$. Since F is a g-Meir-Keeler type contraction, then for this ε, there exits $\delta = \delta(\varepsilon) > 0$ such that

$$\begin{aligned} \varepsilon \leq \quad & \max\{d(g(x_0), g(u_0)), d(g(y_0), g(v_0))\} < \varepsilon + \delta \\ \Rightarrow \quad & \max\{d(F(x_0, y_0), F(u_0, v_0)), d(F(y_0, x_0), F(v_0, u_0))\} < \varepsilon, \end{aligned}$$

for all $x_0, y_0, u_0, v_0 \in X$ with $g(x_0) < g(u_0), g(y_0) \geq g(v_0)$. The result follows by choosing $x = x_0, y = y_0, u = u_0, z = z_0$, that is,

$$\max\{d(F(x,y), F(u,v)), d(F(y,x), F(v,u))\} < \max\{d(g(x), g(u)), d(g(y), g(v))\}.$$

□

Very recently, Gordji at al. [16] replaced the mixed g-monotone property with the mixed strict g-monotone property.

Definition 12. *(See* [16]*) Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$ be mappings. Then F is said to have the mixed strict g-monotone property if $F(x, y)$ is monotone g-increasing in x and is monotone g-decreasing in y, that is, for any $x, y \in X$,*

$$g(x_1) < g(x_2) \Rightarrow F(x_1, y) < F(x_2, y), \quad \text{for } x_1, x_2 \in X, \quad \text{and} \tag{2.5}$$

$$g(y_1) < g(y_2) \Rightarrow F(x, y_2) < F(x, y_1), \ \text{for } y_1, y_2 \in X. \tag{2.6}$$

If we replace g with the identity map in (2.5) and (2.6), we get the definition of the mixed strict monotone property of F.

The following theorem is our first main result.

Theorem 13. *Let $(X, d, \leq)$ be a partially ordered complete metric space and $g : X \to X$, $F : X^2 \to X$ be mappings such that $F(X \times X) \subset X$. Moreover, g is continuous and F and g are compatible mappings. Suppose that F satisfies the following conditions*

(i) F is continuous,
(ii) F has the mixed g-strict monotone property,
(iii) F is a generalized g-Meir-Keeler type contraction,
(iv) there exist $x_0, y_0 \in X$ such that

$$g(x_0) < F(x_0, y_0), \quad g(y_0) \geq F(y_0, x_0). \tag{2.7}$$

Then F and g have a coupled coincidence point, that is, there exist $x, y \in X$ such that

$$F(x,y) = g(x), \quad F(y,x) = g(y).$$

Proof. Let $(x,y)=(x_0,y_0)\in X^2$ such that $g(x_0)<F(x_0,y_0)$ and $g(y_0)\geq F(y_0,x_0)$. We construct the sequence $\{x_n\}$ and $\{y_n\}$ in the following way. Due to assumption (iv), we are able to choose $(x_1,y_1)\in X^2$ such that $g(x_1)=F(x_0,y_0)$ and $g(y_1)=F(y_0,x_0)$. By repeating the same argument, we can choose $(x_2,y_2)\in X^2$ such that $g(x_2)=F(x_1,y_1)$ and $g(y_2)=F(y_1,x_1)$. Inductively, we observe that

$$g(x_{n+1})=F(x_n,y_n) \text{ and } g(y_{n+1})=F(y_n,x_n) \text{ for all } n=0,1,2,\cdots. \tag{2.8}$$

We claim that

$$\begin{aligned} &\cdots>g(x_n)>g(x_{n-1})>\cdots>g(x_2)>g(x_1),\\ &\cdots<g(y_n)<g(y_{n-1})<\cdots<g(y_2)<g(y_1). \end{aligned} \tag{2.9}$$

We shall use mathematical induction to show (2.9). By the assumption (iv), we have

$$g(x_0)<F(x_0,y_0)=g(x_1),\quad g(y_0)\geq F(y_0,x_0)=g(y_1). \tag{2.10}$$

Since F has a mixed strict g-monotone property, (2.10) implies that

$$\begin{aligned} g(x_1)=F(x_0,y_0)<F(x_1,y_0)<F(x_1,y_1)=g(x_2) \text{ and}\\ g(y_2)=F(y_1,x_1)>F(y_1,x_0)>F(y_0,x_0)=g(y_1). \end{aligned}$$

Suppose that the inequalities in (2.9) hold for some $n\geq 2$. Regarding the mixed g-strict monotone property of F, we have

$$g(x_{n-1})<g(x_n)\Rightarrow \begin{cases} F(x_{n-1},y_{n-1})<F(x_n,y_{n-1}),\\ F(y_{n-1},x_{n-1})>F(y_{n-1},x_n). \end{cases}$$

By repeating the same arguments, we observe that

$$g(y_{n-1})>g(y_n)\Rightarrow \begin{cases} F(x_n,y_{n-1})<F(x_n,y_n),\\ F(y_{n-1},x_n)>F(y_n,x_n). \end{cases}$$

Combining the above inequalities, together with (2.8), we get that

$$\begin{aligned} g(x_n)=F(x_{n-1},y_{n-1})<F(x_n,y_n)=g(x_{n+1}),\\ g(y_n)=F(y_{n-1},x_{n-1})>F(y_n,x_n)=g(y_{n+1}). \end{aligned} \tag{2.11}$$

So, (2.9) holds for all $n\geq 1$. Set

$$\Delta_n=\max\{d(g(x_n),g(x_{n+1})),d(g(y_n),g(y_{n+1}))\}. \tag{2.12}$$

Taking Lemma 11 and (2.9) into the account, we get that

$$\begin{aligned} &\max\{d(g(x_n),g(x_{n+1})),d(g(y_n),g(y_{n+1}))\}\\ &\qquad=\max\{d(F(x_{n-1},y_{n-1}),F(x_n,y_n)),d(F(y_{n-1},x_{n-1}),F(y_n,x_n))\}\\ &\qquad<\max\{d(g(x_{n-1}),g(x_n)),d(g(y_{n-1}),g(y_n))\}. \end{aligned} \tag{2.13}$$

If we add the previous two inequalities side by side, we obtain that $\Delta_n<\Delta_{n-1}$. Hence, $\{\Delta_n\}$ is a monotone decreasing sequence in $\mathbb{R}$. Since the sequence $\{\Delta_n\}$ is bounded below, there exists $\theta\geq 0$ such that $\lim\limits_{n\to\infty}\Delta_n=\theta$.

We prove $\theta=0$. Suppose on the contrary that $\theta\neq 0$. Thus, there is a positive integer k such that

$$\varepsilon\leq\Delta_k=\max\{d(g(x_k),g(x_{k+1})),d(g(y_k),g(y_{k+1}))\}<\varepsilon+\delta(\varepsilon) \tag{2.14}$$

where $\varepsilon=\theta$.

Regarding the assumption (iii) we have

$$\max\{d(F(x_k,y_k),F(x_{k+1},y_{k+1})),d(F(y_k,x_k),F(y_{k+1},x_{k+1}))\}<\varepsilon$$

which by (2.8) is equivalent to

$$\max\{d(g(x_{k+1}),g(x_{k+2})),d(g(y_{k+1}),g(y_{k+2}))\}<\varepsilon.$$

Hence, we obtain

$$\Delta_{k+1}<\varepsilon\le\Delta_k,$$

which gives

$$\theta<\theta\le\theta$$

upon taking limit as $k\to\infty$. Thus, we deduce that $\theta=0$. In other words,

$$\lim_{n\to\infty}\Delta_n=\lim_{n\to\infty}\max\{d(g(x_n),g(x_{n+1})),d(g(y_n),g(y_{n+1}))\}=0. \tag{2.15}$$

However, this is possible only when

$$\lim_{n\to\infty}d(g(x_n),g(x_{n+1}))=0=\lim_{n\to\infty}d(g(y_n),g(y_{n+1})). \tag{2.16}$$

We claim that the sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences. Assume the contrary, that is, at least one of them is not Cauchy. Then there exist $\varepsilon>0$ and two sequences of integers, say, $\{k_m\}$ and $\{l_m\}$ such that $l_m>k_m>m$ and

$$d(g(x_{k_m}),g(x_{l_m}))\ge\varepsilon \text{ and } d(g(y_{k_m}),g(y_{l_m}))\ge\varepsilon$$

for all $m\ge 1$. This implies that

$$t_m=\max\{d(g(x_{l_m}),g(x_{k_m})),d(g(y_{l_m}),g(y_{k_m}))\}\ge\varepsilon, \tag{2.17}$$

for all $m\ge1$. Take k_m as the smallest number exceeding l_m satisfying (2.17). Then we have

$$\max\{d(g(x_{k_m-1}),g(x_{l_m})),d(g(y_{k_m-1}),g(y_{l_m}))\}<\varepsilon. \tag{2.18}$$

Using (2.18),(2.17), and the definition (2.12) we get

$$\begin{aligned}\varepsilon\quad&\le t_m=\max\{d(g(x_{l_m}),g(x_{k_m})),d(g(y_{l_m}),g(y_{k_m}))\}\\&\le\max\{d(g(x_{l_m}),g(x_{k_m-1}))+d(g(x_{k_m-1}),g(x_{k_m})),\\&\qquad d(g(y_{l_m}),g(y_{k_m-1}))+d(g(y_{k_m-1}),g(y_{k_m}))\}\\&\le\max\{d(g(x_{l_m}),g(x_{k_m-1})),d(g(y_{l_m}),g(y_{k_m-1}))\},\\&\qquad+\max\{d(g(x_{k_m-1}),g(x_{k_m})),d(g(y_{k_m-1}),g(y_{k_m}))\}\\&<\varepsilon+\Delta_{k_m-1}\end{aligned} \tag{2.19}$$

by employing the triangle inequality. Letting $m\to\infty$ in (2.19), we obtain

$$\varepsilon\le\lim_{m\to\infty}t_m\le\lim_{m\to\infty}[\varepsilon+\Delta_{k_m-1}]$$

which gives

$$\lim_{m\to\infty}t_m=\varepsilon \tag{2.20}$$

because of (2.15). By the definitions (2.12) and (2.8) and using the triangle inequality, we have

$$\begin{aligned}t_m\quad&=\max\{d(g(x_{l_m}),g(x_{k_m})),d(g(y_{l_m}),g(y_{k_m}))\}\\&\le\max\{d(g(x_{l_m}),g(x_{l_m+1}))+d(g(x_{l_m+1}),g(x_{k_m+1}))+d(g(x_{k_m+1}),g(x_{k_m})),\\&\qquad d(g(y_{l_m}),g(y_{l_m+1}))+d(g(y_{l_m+1}),g(y_{k_m+1}))+d(g(y_{k_m+1}),g(y_{k_m}))\}\\&=\Delta_{k_m}+\Delta_{l_m}+\max\{d(g(x_{l_m+1}),g(x_{k_m+1})),d(g(y_{l_m+1}),g(y_{k_m+1}))\}\\&=\Delta_{k_m}+\Delta_{l_m}+\{d(F(x_{l_m},y_{l_m}),F(x_{k_m},y_{k_m})),d(F(y_{l_m},x_{l_m}),F(y_{k_m},x_{k_m}))\}.\end{aligned} \tag{2.21}$$

Regarding Lemma 11, we notice that

$$\begin{aligned}&\{d(F(x_{l_m},y_{l_m}),F(x_{k_m},y_{k_m})),d(F(y_{l_m},x_{l_m}),F(y_{k_m},x_{k_m}))\}\\&<\{d(g(x_{l_m}),g(x_{k_m})),d(g(y_{l_m}),g(y_{k_m}))\}=t_m.\end{aligned}$$

Hence, (2.21) turns into

$$t_m<\Delta_{k_m}+\Delta_{l_m}+t_m.$$

Upon letting $m\to\infty$, the inequality above leads to

$$\varepsilon<\varepsilon$$

on account of (2.15). It is a contradiction. Thus, the sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy in $(X,\leq,d)$. Since $(X,\leq,d)$ is complete, there exist $x,y\in X$ such that

$$\lim_{n\to\infty}d(x,g(x_n))=0\iff\lim_{n\to\infty}g(x_n)=\lim_{n\to\infty}F(x_n,y_n)=x,\tag{2.22}$$

$$\lim_{n\to\infty}d(y,g(y_n))=0\iff\lim_{n\to\infty}g(y_n)=\lim_{n\to\infty}F(y_n,x_n)=y.\tag{2.23}$$

Since F and g are compatible mappings, by (2.22) and (2.23), we derived that

$$\lim_{n\to\infty}d(g(F(x_n,y_n),F(g(x_n),g(y_n))=0,\tag{2.24}$$

$$\lim_{n\to\infty}d(g(F(y_n,x_n),F(g(y_n),g(x_n))=0\tag{2.25}$$

Moreover, since the sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ are monotone, we conclude that

$$g(x_n)<x\text{ and }g(y_n)>y,\tag{2.26}$$

for each $n\geq1$. On the other hand, by the continuity of g, and the limits (2.22) and (2.23), we have

$$\lim_{n\to\infty}d(g(x),g(g(x_n)))=\lim_{n\to\infty}d(g(g(x_n)),g(g(x_n)))=d(g(x),g(x))=0$$

and

$$\lim_{n\to\infty}d(g(y),g(g(y_n)))=\lim_{n\to\infty}d(g(g(y_n)),g(g(y_n)))=d(g(y),g(y))=0.$$

Hence, for all $k\geq1$, there exists a positive integer N such that

$$d(g(x),g(g(x_n)))<\frac{1}{6k}\text{ and }d(g(y),g(g(y_n)))<\frac{1}{6k}\tag{2.27}$$

for all $n\geq N$. Using again the triangle inequality, together with (2.8), we find

$$d(F(g(x_n),g(y_n)),g(x))\quad\leq d(F(g(x_n),g(y_n)),gF(x_n,y_n))+d(gF(x_n,y_n),g(x))\tag{2.28}$$

Similarly, we derive

$$d(F(g(y_n),g(x_n)),g(y))\quad\leq d(F(g(y_n),g(x_n)),gF(y_n,x_n))+d(gF(y_n,x_n),g(y))\tag{2.29}$$

Letting $n\to\infty$ in the above inequalities (2.28), (2.29), (2.24) , and the continuities of F and g, we have

$$\lim_{n\to\infty}g(F(x,y),gx)=0\text{ and }\lim_{n\to\infty}g(F(y,x),gy)=0.$$

Hence, we derive that $gx=F(x,y)$ and $gy=F(y,x)$, that is, $(x,y)\in X^2$ is a coincidence point of F and g.

□

Corollary 14. *Let $(X, d, \leq)$ be a partially ordered complete metric space and $F : X^2 \to X$ be a mapping such that $F(X \times X) \subset X$. Suppose that F satisfies the following conditions*

(i) F is continuous,
(ii) F has the mixed strict monotone property,
(iii) F is a weak Meir-Keeler type contraction,
(iv) there exist $x_0, y_0 \in X$ such that

$$x_0 < F(x_0, y_0), \quad y_0 > F(y_0, x_0). \tag{2.30}$$

Then F has a coupled fixed point, that is, there exist $x, y \in X$ such that

$$F(x, y) = x, \quad F(y, x) = y.$$

The following example illustrates that Theorem 13 is more general than Theorem 8

Example 15. *Let X be the set $[0, \infty)$ and $d(x, y) = |x-y|$. Set $g : X \to X$ and $F : X \times X \to X$ be defined as $g(x) = x^2$ and $F(x, y) = \begin{cases} \frac{x^2-5y^2}{8} & \text{if } x, y \in X, \quad x \geq y \\ 0 & \text{if } x < y \end{cases}$, respectively.*

Then the operator F satisfies the strict mixed g-monotone property. Although Theorem 8 is not applicable, Theorem 13 yields a fixed point. Suppose, to the contrary, that the condition (2.1) is satisfied. Let $x, y, u, v \in X$ with $x \geq u,\ y \leq v$ such that

$$\varepsilon \leq \frac{1}{2}[d(g(x), g(u)) + d(g(y), g(v))] = \frac{1}{2}[|x^2 - u^2| + |y^2 - v^2|] < \varepsilon + \delta(\varepsilon). \tag{2.31}$$

By choosing $x = u$, we derive that

$$\varepsilon \leq \frac{1}{2}[d(g(x), g(u)) + d(g(y), g(v))] = \frac{1}{2}|y^2 - v^2| < \varepsilon + \delta(\varepsilon), \quad y \leq v. \tag{2.32}$$

On the other hand,

$$d(F(x, y), F(u, v)) = \left|\frac{x^2 - 5y^2}{8} - \frac{u^2 - 5v^2}{8}\right| = \left|\frac{5v^2 - 5y^2}{8}\right| = \frac{5}{8}|v^2 - y^2| < \varepsilon \tag{2.33}$$

where $x = u$ and $y \leq v$. Combining (2.32) and (2.33), we get that

$$2\varepsilon \leq |y^2 - v^2| \leq \frac{8}{5}\varepsilon < 2\varepsilon$$

which is a contradiction.

But, F satisfies Theorem 13. Indeed, we have

$$\varepsilon \leq \max\{d(g(x), g(u)), d(g(y), g(v))\} = \max\{|x^2 - u^2|, |y^2 - v^2|\} < \varepsilon + \delta(\varepsilon) \tag{2.34}$$

and also

$$d(F(x, y), F(u, v)) = \left|\frac{x^2 - 5y^2}{8} - \frac{u^2 - 5v^2}{8}\right| \leq \frac{1}{8}|x^2-u^2| + \frac{5}{8}|v^2-y^2|, \quad x \geq u, y \leq v \tag{2.35}$$

$$d(F(y, x), F(v, u)) = \left|\frac{y^2 - 5x^2}{8} - \frac{v^2 - 5u^2}{8}\right| \leq \frac{5}{8}|x^2-u^2| + \frac{1}{8}|v^2-y^2|, \quad x \geq u, y \leq v. \tag{2.36}$$

From (2.35) and (2.36), we obtain that

$$\begin{aligned}
&\max\{d(F(x,y),F(u,v)),d(F(y,x),F(v,u))\}\\
&= \max\{\left|\tfrac{x^2-5y^2}{8}-\tfrac{u^2-5v^2}{8}\right|,\left|\tfrac{y^2-5x^2}{8}-\tfrac{v^2-5u^2}{8}\right|\} \qquad (2.37)\\
&\le \max\{\tfrac{1}{8}|x^2-u^2|+\tfrac{5}{8}|v^2-y^2|,\tfrac{5}{8}|x^2-u^2|+\tfrac{1}{8}|v^2-y^2|\}.
\end{aligned}$$

Without loss of generality, assume that $|v^2-y^2| \le |x^2-u^2|$. *Then*

$$\max\{d(F(x,y),F(u,v)),d(F(y,x),F(v,u))\} = \frac{6}{8}|x^2-u^2| < \frac{6}{8}(\varepsilon+\delta(\varepsilon)).$$

Thus, by choosing $\delta(\varepsilon) < \frac{4}{3}\varepsilon$*, the condition (2.4) is satisfied. Notice that* $(0,0)$ *is the coupled fixed point of* F.

3. Uniqueness of Coupled Fixed Point

In this section we shall prove the uniqueness of coupled fixed point.

Theorem 16. *In addition to the hypotheses of Theorem 13, assume that for all* (x,y), $(x^*,y^*) \in X^2$*, there exists* $(a,b) \in X^2$ *such that* $(F(a,b),F(b,a))$ *is comparable to both* $(F(x,y),F(y,x))$ *and* $(F(x^*,y^*),F(y^*,x^*))$ *. Then,* F *and* g *have a unique couple common fixed point, that is, there exists* $(x,y) \in X^2$ *such that*

$$x = g(x) = F(x,y) \text{ and } y = g(y) = F(y,x).$$

Proof. The set of coupled coincidence points of F and g is not empty due to Theorem 13. We suppose that $(x,y),(x^*,y^*) \in X^2$ are coupled coincidence points of F and g and distinguish the following two cases.
First case: (x,y) is comparable to (x^*,y^*) with respect to the ordering in X^2, where

$$F(x,y) = g(x), F(y,x) = g(y), F(x^*,y^*) = g(x^*), F(y^*,x^*) = g(y^*).$$

Without loss of the generality, we may assume that

$$g(x) = F(x,y) < F(x^*,y^*) = g(x^*), \qquad g(y) = F(y,x) \ge F(y^*,x^*) = g(y^*).$$

By the definition of ρ_2 and Lemma 11 we have

$$\begin{aligned}
\rho_2((g(x),g(y)),(g(x^*),g(y^*))) &= \max\{d(g(x),g(x^*)),d(g(y),g(y^*))\}\\
&= \max\{d(F(x,y),F(x^*,y^*)),d(F(y,x),F(y^*,x^*))\},\\
&< \max\{d(g(x),g(x^*)),d(g(y),g(y^*))\}
\end{aligned}$$

which is a contradiction. Therefore, we have $(g(x),g(x^*)) = (g(y),g(y^*))$.Hence $g(x) = g(x^*)$ and $g(y) = g(y^*)$.

Second case: (x,y) is not comparable to (x^*,y^*). By the assumption there exists $(a,b) \in X^2$ which is comparable to both (x,y) and (x^*,y^*). Without loss of the generality we may assume that

$$\begin{aligned}
g(x) = F(x,y) < g(a) \quad &\text{and} \quad F(x^*,y^*) = g(x^*) < g(a),\\
g(y) = F(y,x) \ge g(b) \quad &\text{and} \quad F(y^*,x^*) = g(y^*) \ge g(b),
\end{aligned} \qquad (3.1)$$

Setting $x = x_0, y = y_0, a = a_0, b = b_0$, as in the proof of Theorem 13, we get

$$g(x_{n+1}) = F(x_n,y_n) \text{ and } g(y_{n+1}) = F(y_n,x_n) \text{ for all } n = 0,1,2,\cdots, \qquad (3.2)$$

$$g(a_{n+1}) = F(a_n, b_n) \text{ and } g(b_{n+1}) = F(b_n, a_n) \text{ for all } n = 0, 1, 2, \cdots . \tag{3.3}$$

Since $(F(x,y), F(y,x)) = (g(x), g(y)) = (g(x_1), g(y_1))$ is comparable with $(F(a,b), F(a,a)) = (g(a), g(b)) = (g(a_1), g(b_1))$, we have $g(x) < g(a_1)$ and $g(b_1) < g(y)$. Inductively, we observe that $(g(x), g(y))$ is comparable with $(g(a_n), g(b_n))$ for all $n \geq 1$. Thus, by Lemma 11 we get that

$$\begin{aligned} &\max\{d(g(x), g(a_{n+1})), d(g(y), g(b_{n+1}))\} \\ &\quad = \max\{d(F(x,y), F(a_n, b_n)), d(F(y,x), F(b_n, a_n))\} \\ &\quad < \max\{d(g(x), g(a_n)), d(g(y), g(b_n))\}. \end{aligned} \tag{3.4}$$

Inductively, we derive that

$$\max\{d(g(x), g(a_{n+1})), d(g(y), g(b_{n+1}))\} < \max\{d(g(x), g(a_1)), d(g(y), g(b_1))\}.$$

The right hand side of the inequality above tends to zero as $n \to \infty$. Hence, $\lim_{n\to\infty} \max\{d(g(x), g(a_{n+1})), d(g(y), g(b_{n+1}))\} = 0$. Analogously, we get that $\lim_{n\to\infty} \max\{d(g(x^*), g(a_{n+1})), d(g(y^*), g(b_{n+1}))\} = 0$. By the triangle inequality, we have

$$\begin{aligned} d(g(x), g(x^*)) &\leq d(g(x), g(a_{n+1})) + d(g(x^*), g(a_{n+1})) - d(g(a_{n+1}), g(a_{n+1})) \\ &\leq d(g(x), g(a_{n+1})) + d(g(x^*), g(a_{n+1})) \to 0 \text{ as } n \to \infty, \end{aligned}$$

$$\begin{aligned} d(g(y), g(y^*)) &\leq d(g(y), g(b_{n+1})) + d(g(y^*), g(b_{n+1})) - d(g(b_{n+1}), g(b_{n+1})) \\ &\leq d(g(y), g(b_{n+1})) + d(g(y^*), g(b_{n+1})) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Combining all of the observations above, we get that $\lim\limits_{n\to\infty} d(g(x^*), g(x)) = 0$ and $\lim\limits_{n\to\infty} d(g(y^*), g(y)) = 0$. So we have

$$g(x) = g(x^*) \text{ and } g(y) = g(y^*). \tag{3.5}$$

Let $g(x) = u$ and $g(y) = v$. By combining the commutativity of F and g with the fact that $g(x) = F(x,y)$ and $F(y,x) = g(y)$, we have

$$g(u) = g(g(x)) = g(F(x,y)) = F(g(x), g(y)) = (u, v), \tag{3.6}$$

$$g(v) = g(g(y)) = g(F(y,x)) = F(g(y), g(x)) = (v, u). \tag{3.7}$$

Thus, (u,v) is a coupled coincidence point of F and g. Setting $u = x^*$ and $v = y^*$ in (3.6), (3.7). Then, by (3.5) we have

$$u = g(x) = g(x^*) = g(u) \text{ and } v = g(y) = g(y^*) = g(v).$$

From (3.6), and (3.7) we get that

$$u = g(u) = F(u,v) \text{ and } v = g(v) = F(v,u).$$

Hence, the pair (u,v) is a coupled common fixed point of F and g.

We claim that (u,v) is the unique coupled common fixed point of F and g. Suppose, on the contrary, that (z,w) is another coupled fixed point of F and g. But then

$$u = g(u) = g(z) = z \text{ and } v = g(v) = g(w) = w$$

follows from (3.5).

□

Corollary 17. *In addition to the hypotheses of Theorem 14, assume that for all (x,y), $(x^*,y^*) \in X^2$, there exists $(a,b) \in X^2$ that is comparable to (x,y) and (x^*,y^*). Then, F has a unique coupled fixed point.*

4. A Short Survey For Coupled Fixed Point Theorems

In addition to our results proved in earlier sections, in this section, we include the newer and improved versions of some of the recent theorems on the topic. We only give the statements of the existence theorems on coupled fixed points and omit the proofs. Because the proof of each of the earlier version applies, mutatis mutandis, to the corresponding newer version. We also illustrate the apparent effectiveness of improved versions by providing a number of examples. We would like to point out that the conclusion of the uniqueness of coupled fixed points in newer versions can be obtained under the condition of comparability used in a way analogous to Theorem 16. By taking this remark into account, we do not phrase the statements of the related uniqueness theorems. This section can be considered as a continuation of work of Berinde [6, 7].

Definition 18. [20] *A function $\varphi : [0,\infty) \to [0,\infty)$ is called an alternating distance function if the following properties are satisfied:*

(*i*) *φ is monotone increasing and continuous,*
(*ii*) *$\varphi(t) = 0$ if and only if $t = 0$.*

In [22] Luong and Thuan consider the following classes of functions. Let Φ denotes the set of all alternating distance functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying

(*iii*) $\varphi(t+s) \leq \varphi(t) + \varphi(s)$ for all $s,t \in [0,\infty)$.

In addition, let Ψ denotes the set of all functions $\psi : [0,\infty) \to [0,\infty)$ which satisfy $\lim_{t\to r} \psi(t) > 0$ for all $r > 0$ and $\lim_{t\to 0^+} \psi(t) = 0$.

In [22] Luong and Thuan proved the following coupled fixed point theorem.

Theorem 19. *Let $(X,d,\leq)$ be a partially ordered complete metric space. Let $F : X \times X \to X$ be a mapping with the mixed monotone property on X. Suppose that there exist a $\varphi \in \Phi$ and a $\psi \in \Psi$ such that*

$$\varphi(d(F(x,y),F(u,v))) \leq \frac{1}{2}\varphi([d(x,u)+d(y,v)]) - \psi(\frac{1}{2}[d(x,u)+d(y,v)]), \tag{4.1}$$

for all $u \leq x,\ y \leq v$. Suppose either

(*a*) *F is continuous, or*
(*b*) *X has the following properties:*
  (*i*) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$, $\forall n$;*
  (*i*) *if a non-increasing sequence $\{y_n\} \to y$, then $y \leq y_n$, $\forall n$.*

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0,y_0)$ and $F(y_0,x_0) \leq y_0$, then, there exist $x,y \in X$ such that $x = F(x,y)$ and $y = F(y,x)$.

We improve Theorem 19 as follows.

Theorem 20. *(cf. [12]) Let $(X,d,\leq)$ be a partially ordered complete metric space. Let $F : X\times X \to X$ be a mapping with the mixed monotone property on X. Suppose that there exists a $\varphi \in \Phi$ and $\psi \in \Psi$ such that either*

$$\varphi(d(F(x,y),F(u,v))+d(F(y,x),F(v,u))) \leq \varphi([d(x,u)+d(y,v)])-\psi([d(x,u)+d(y,v)]), \tag{4.2}$$

or

$$\varphi(d(F(x,y),F(u,v))) \leq \varphi(\max d(x,u), d(y,v)) - \psi(\max d(x,u), d(y,v)), \tag{4.3}$$

is satisfied for all $u \leq x,\ y \leq v$. Suppose either

(a) *F is continuous, or*

(b) *X has the following properties:*

(i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \le x$, $\forall n$;*

(i) *if a non-increasing sequence $\{y_n\} \to y$, then $y \le y_n$, $\forall n$.*

If there exist $x_0, y_0 \in X$ such that $x_0 \le F(x_0, y_0)$ and $F(y_0, x_0) \le y_0$, then, there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Theorem 20 is more general than Theorem 19. The following example illustrates this claim.

Example 21. *Let X be a real line and $d(x, y) = |x-y|$. Suppose that $F : X \times X \to X$ is defined as $F(x, y) = \frac{x-5y}{8}$ for $x, y \in X$.*

It is clear that the operator F satisfies the mixed monotone property. Theorem 20 yields a fixed point for $\varphi(t) = \frac{15}{16}t$ and $\psi(t) = \frac{1}{16}t$. However Theorem 19 is not applicable. Notice that $(0, 0)$ is the coupled fixed point of F.

The main result of [21] is the following.

Theorem 22. *Let $(X, \le)$ be a partially ordered set and (X, d) be a complete metric space. Assume that there exists a function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(t) < t$ and $\lim\limits_{r \to r^+} \varphi(r) < t$ for each $t > 0$. Suppose that $F : X \times X \to X$ and $g : X \to X$ be two mappings where $X \neq \emptyset$. Also suppose that F has the mixed g-monotone property and*

$$d(F(x, y), F(u, v)) \le \varphi\left(\frac{[d(g(x), g(u)) + d(g(y), g(v))]}{2}\right) \tag{4.4}$$

for all $x, y, u, v \in X$ for which $g(x) \le g(u)$ and $g(v) \le g(y)$. Suppose $F(X \times X) \subset g(X)$, g is sequentially continuous and commutes with F and also suppose either F is continuous or X has the following property:

$$\text{if a non-decreasing sequence } \{x_n\} \to x, \text{ then } x_n \le x, \text{ for all } n, \tag{4.5}$$

$$\text{if a non-increasing sequence } \{y_n\} \to y, \text{ then } y \le y_n, \text{ for all } n. \tag{4.6}$$

If there exist $x_0, y_0 \in X$ such that $g(x_0) \le F(x_0, y_0)$ and $g(y_0) \le F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$, that is, F and g have a couple coincidence.

We improve Theorem 22 in the following way:

Theorem 23. *(cf.[7]) Let $(X, \le)$ be a partially ordered set and (X, d) be a complete metric space. Assume that there exists a function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(t) < t$ and $\lim\limits_{r \to r^+} \varphi(r) < t$ for each $t > 0$. Also assume that $F : X \times X \to X$ and $g : X \to X$ be two maps where $X \neq \emptyset$. Suppose that F has the mixed g-monotone property and either*

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \le \varphi\left([d(g(x), g(u)) + d(g(y), g(v))]\right) \tag{4.7}$$

or

$$\max\{d(F(x, y), F(u, v)), d(F(y, x), F(v, u))\} \le \varphi\left(\{d(g(x), g(u)), d(g(y), g(v))\}\right) \tag{4.8}$$

is satisfied for all $x, y, u, v \in X$ for which $g(x) \le g(u)$ and $g(v) \le g(y)$. Suppose $F(X \times X) \subset g(X)$, g is sequentially continuous and commutes with F. Additionally suppose either F is continuous or X has the following property:

$$\text{if a non-decreasing sequence } \{x_n\} \to x, \text{ then } x_n \le x, \text{ for all } n, \tag{4.9}$$

$$\text{if a non-increasing sequence } \{y_n\} \to y, \text{ then } y \leq y_n, \text{ for all } n. \tag{4.10}$$

If there exist $x_0, y_0 \in X$ *such that* $g(x_0) \leq F(x_0, y_0)$ *and* $g(y_0) \leq F(y_0, x_0)$, *then there exist* $x, y \in X$ *such that* $g(x) = F(x, y)$ *and* $g(y) = F(y, x)$, *that is,* F *and* g *have a couple coincidence.*

Theorem 23 is more general than Theorem 22. The following example illustrates this claim.

Example 24. *Let* X *be a real line and* $d(x, y) = |x - y|$. *Set* $g : X \to X$ *and* $F : X \times X \to X$ *be defined as* $g(x) = \frac{7x}{8}$ *and* $F(x, y) = \frac{x-5y}{8}$ *for* $x, y \in X$, *respectively.*

It is easy to see that the operator F *satisfies the mixed* g*-monotone property. Theorem 23 yields a fixed point but Theorem 22 is not applicable. Notice that* $(0, 0)$ *is the coupled fixed point of* F.

Remark 25. *Theorem 23 and Theorem 3 in* [7] *are equivalent to each other. Indeed, this equivalence follows from the fact that*

$$\frac{a+b}{2} \leq \max\{a, b\} \leq a + b$$

where $a, b \in [0, \infty)$.

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ERDAL KARAPINAR
Department of Mathematics,
Atilim University,
İncek, 06836, Ankara
Turkey,

E-mail address: `erdalkarapinar@yahoo.com`
E-mail address: `ekarapinar@atilim.edu.tr`