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A GENERALIZATION OF A THEOREM OF BANARJEE AND THAKUR TO TWO WEAKLY COMPATIBLE PAIRS OF SELF-MAPS

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Abstract. In this paper we obtain a generalization of a result of Banarjee and Thakur to two weakly compatible pairs of self-maps with any one of the range spaces complete.

Let (X, d) be a metric space and A and S self-maps on X. We denote by AS for the composition of A and S. Also we write $A(X)$ for the range of A. Self-maps A and S are known to be commuting if $AS = SA$.

According to Gerald Jungck [2], self-maps S and A on X are compatible if

$$\lim_{n\to\infty} d(SAx_n, ASx_n) = 0 \tag{1}$$

whenever $\langle x_n \rangle_{n=1}^{\infty} \subset X$ is such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = t \quad \text{for some} \quad t \in X. \tag{2}$$

We note that every compatible pair is necessarily commuting. The reverse implication is not true in general (For example, see [2]).

Using the notion of compatibility, Banarjee and Thakur [1] proved the following result:

Theorem A. *Let A, B, S and T be self-maps on X satisfying the inclusions*

$$A(X) \subset T(X) \quad \textit{and} \quad B(X) \subset S(X), \tag{3}$$

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and the inequality

$$d^2(Ax, By) \leq a \max\left\{d^2(Ax, Sx), d^2(By, Ty), d^2(Sx, Ty)\right\} + b \max\{d(Ax, Sx)d(Sx, By), d(Ax, Ty)d(By, Ty)\} + cd(Sx, By)d(Ty, Ax) \quad for\ all \quad x, y \in X, \tag{4}$$

where $a, b, c \geq 0$ such that $a + 2b < 1$ and $a + c < 1$.
Suppose that $T(X)$ is complete and

(a) *the pairs (A, S) and (B, T) are compatible.*

Then all the four maps A, B, S and T have a unique common fixed point.

In this paper we obtain a generalization of Theorem [1] by replacing the compatibility in (a) with the notion of weakly compatible maps [3] when any one of the range spaces $S(X)$, $A(X)$, $T(X)$ and $B(X)$ is a complete subspace of X.

We require the following definitions:

A point $x \in X$ is a coincidence point of self-maps S and A on X if $Tx = Sx$, while $y \in X$ is a point of their coincidence with respet to x if $Tx = Sx = y$.

Definition 1 (Jungck, [3])**.** *Self-maps which commute at their coincidence points are weakly compatible maps.*

Thus self-maps S and A on X are weakly compatible if $Tx = Sx$ whenever $x \in X$ is such that $Tx = Sx$. Weakly compatible maps are also called coincidentally commuting or partially commuting [5].

Taking $x_n = x$ for all n in (1) and (2), it follows that $STx = TSx$ whenever $x \in X$ is such that $Tx = Sx$. In other words, is every compatible pair is weakly compatible one. One can refer to [3, 4] to see that the converse is not true. That is every weakly compatible pair need not be compatible.

We prove the following:

Theorem B. *Let A, B, S and T be self-maps on X satisfying the pair of inclusions* (3) *and the inequality* (4)*. Suppose that*

(b) *one of $S(X)$, $A(X)$, $T(X)$ and $B(X)$ is a complete subspace of X*

and

(c) *the pairs (A, S) and (B, T) are weakly compatible.*

Then all the four maps A, B, S and T have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary. We can choose points x_n in X inductively in view of the inclusions given in (3) with the choice

$$y_{2n-1} = Ax_{2n-2} = Tx_{2n-1},\ y_{2n} = Bx_{2n-1} = Sx_{2n} \text{ for all } n \geq 1. \tag{5}$$

From the proof of Theorem A as given in [1], we see that $\langle y_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence in X. Therefore $\langle y_{2n-1} \rangle_{n=1}^{\infty}$ and $\langle y_{2n} \rangle_{n=1}^{\infty}$ being subsequences are also Cauchy sequences in $T(X)$ and $S(X)$.

Case (a): Suppose that $T(X)$ is a complete subspace of X.
Then it follows from the choice (5) that $\langle y_{2n-1}\rangle_{n=1}^{\infty}$ is a Cauchy sequence in $T(X)$. Therefore $\lim_{n\to\infty} y_{2n-1} = Tp$ for some p in X. Thus

$$\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} Ax_{2n-2} = \lim_{n\to\infty} Tx_{2n-1} = Tp. \tag{6}$$

Let

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Bx_{2n-1} = \lim_{n\to\infty} Sx_{2n} = q. \tag{7}$$

We now show that $q = Tp$.
Writing $x = x_{2n}$ and $y = x_{2n-1}$ in the inequality (4), we see that

$$\begin{aligned} d^2(Ax_{2n}, Bx_{2n-1}) \le{}& a\max\left\{d^2(Ax_{2n}, Sx_{2n}), d^2(Bx_{2n-1}, Tx_{2n-1}), d^2(Sx_{2n}, Tx_{2n-1})\right\} \\ &+ b\max\{d(Ax_{2n}, Sx_{2n})d(Sx_{2n}, Bx_{2n-1}), \\ &\quad d(Ax_{2n}, Tx_{2n-1})d(Bx_{2n-1}, Tx_{2n-1})\} \\ &+ cd(Sx_{2n}, Bx_{2n-1})d(Tx_{2n-1}, Ax_{2n}). \end{aligned} \tag{8}$$

Applying the limit as $n \to \infty$ in (8) and then using (6) and (7), we get

$d^2(Tp, q) \le a\max\left\{d^2(Tp, q), d^2(q, Tp), d^2(q, Tp)\right\} + b\max\{d(Tp, q).0, 0.d(q, Tp)\} + c.0$

or $d^2(Tp, q) \le ad^2(Tp, q)$ so that $d^2(Tp, q) = 0$ or $Tp = q$, since $a < 1$. Thus

$$\lim_{n\to\infty} Ax_{2n-2} = \lim_{n\to\infty} Tx_{2n-1} = \lim_{n\to\infty} Bx_{2n-1} = \lim_{n\to\infty} Sx_{2n} = Tp. \tag{9}$$

Now p is a coincidence point of (B, T). For, from (4)

$$\begin{aligned} d^2(Ax_{2n}, Bp) \le{}& a\max\left\{d^2(Ax_{2n}, Sx_{2n}), d^2(Bp, Tp), d^2(Sx_{2n}, Tp)\right\} \\ &+ b\max\{d(Ax_{2n}, Sx_{2n})d(Sx_{2n}, Bp), d(Ax_{2n}, Tp)d(Bp, Tp)\} \\ &+ cd(Sx_{2n}, Bp)d(Tp, Ax_{2n}). \end{aligned}$$

Applying the limit as $n \to \infty$ in this, and then using (9), we get

$d^2(Tp, Bp) \le a\max\left\{0, d^2(Bp, Tp), 0\right\} + b\max\{0.d(Tp, Bp), 0.d(Bp, Tp)\} + cd(Tp, Bp)$

or $d^2(Tp, Bp) \le ad^2(Tp, Bp)$ so that $Tp = Bp$. But then the inclusion (3) implies that $Bp = Sr$ for some $r \in X$ and (4) gives

$$\begin{aligned} d^2(Ar, Bp) \le{}& a\max\left\{d^2(Ar, Sr), d^2(Bp, Tp), d^2(Sr, Tp)\right\} \\ &+ b\max\{d(Ar, Sr)d(Sr, Bp), d(Ar, Tp)d(Bp, Tp)\} \\ &+ cd(Sr, Bp)d(Tp, Ar) \end{aligned}$$

so that

$d^2(Ar, Bp) \le a\max\left\{d^2(Ar, Sr), 0, 0\right\} + b\max\{d(Ar, Sr).0, 0.d(Ar, Sr)\} + c.0.d(Sr, Ar)$

or $d^2(Ar, Sr) \le ad^2(Ar, Sr)$ or $Ar = Sr$. Thus

$$Ar = Sr = Bp = Tp = z. \tag{10}$$

Case (b): Suppose that $A(X)$ is complete. Then $\langle y_{2n-1}\rangle_{n=1}^{\infty}$ and $\langle y_{2n}\rangle_{n=1}^{\infty}$ is a Cauchy sequence in $A(X)$ and hence in $T(X)$, since $A(X) \subset T(X)$. Thus (10) follows from Case (a).

Case (c): Suppose that $S(X)$ is complete. Then again from the choice (5), we obtain that $\langle y_{2n}\rangle_{n=1}^{\infty}$ being a Cauchy sequence in $S(X)$, converges to Su for some $u \in X$. That is

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Bx_{2n-1} = \lim_{n\to\infty} Sx_{2n} = Su. \tag{11}$$

Writing

$$\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} Ax_{2n-2} = \lim_{n\to\infty} Tx_{2n-1} = v$$

and proceeding the limit as $n \to \infty$ in (8), we see that

$$d^2(v, Su) \le a \max\left\{d^2(v, Su), d^2(Su, v), d^2(Su, v)\right\} + b \max\{d(v, Su).0, 0.d(Su, v)\} + cd(Su, v).0$$

which gives $Su = v$ and (9) follows.

Now u is a coincidence point of (A, S). For, from (4)

$$d^2(Au, Bx_{2n-1}) \le a \max\left\{d^2(Au, Su), d^2(Bx_{2n-1}, Tx_{2n-1}), d^2(Su, Tx_{2n-1})\right\} + b \max\{d(Au, Su)d(Su, Bx_{2n-1}), d(Au, Tx_{2n-1})d(Bx_{2n-1}, Tx_{2n-1})\} + cd(Su, Bx_{2n-1})d(Tx_{2n-1}, Au),$$

in which applying the limit as $n \to \infty$ and using (9), we get

$$d^2(Au, Su) \le a \max\{d^2(Au, Su), 0, 0\} + b \max\{d(Au, Su).0, d(Au, Su).0\} + c.0.d(Su, Au)$$

or $d^2(Au, Su) \le ad^2(Au, Su)$ so that $Au = Su$. That is, u is a coincidence point of A and S. But then from (3) again we see that $Au = Tw$ for some $w \in X$, and (4) gives

$$d^2(Au, Bw) \le a \max\left\{d^2(Au, Su), d^2(Bw, Tw), d^2(Su, Tw)\right\} + b \max\{d(Au, Su)d(Su, Bw), d(Au, Tw)d(Bw, Tw)\} + cd(Su, Bw)d(Tw, Au),$$

so that $d^2(Tw, Bw) \le ad^2(Tw, Bw)$ or $Tw = Sw$. Thus

$$Au = Su = Bw = Tw = z. \tag{12}$$

Case (d): Suppose that $B(X)$ is complete. In view of the inclusion (3), we obtain (12) from Case (c).

Thus (10) and (12) reveal that in any case the pairs (A, S) and (B, T) have respective coincidence points and z is a common point of coincidence. Since these pairs are weakly compatible, we see from (10) (or (12)) that $Az = Sz = Bz = Tz$. That is, z is also a common coincidence point of the four maps.

To establish that z is a common fixed point for all the four maps, we again use (4) with $x = u$ and $y = z$ to get

$$d^2(Au, Bz) \le a \max\left\{d^2(Au, Su), d^2(Bz, Tz), d^2(Su, Tz)\right\} + b \max\{d(Au, Su)d(Su, Bz), d(Au, Tz)d(Bz, Tz)\} + cd(Su, Bz)d(Tz, Au),$$

so that

$d^2(z, Bz) \le a \max\{0, 0, d^2(z, Bz)\} + b \max\{0.d(z, Bz), d(z, Bz).0\} + cd(z, Bz)d(z, Bz)$

or $d^2(z, Bz) \le (a + c)d^2(z, Bz)$ or $Bz = z$.

In other words, z is a common fixed point for A, B, S and T, which in fact is a point of their common coincidence.

Uniqueness of the common fixed point follows easily from (4). □

Since every compatible pair is weakly compatible and the completeness of $T(X)$ is included in the condition (b), Theorem A follows as a particular case of Theorem B.

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