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SOME SERIES IDENTITIES FOR SOME SPECIAL CLASSES OF APOSTOL-BERNOULLI AND APOSTOL-EULER POLYNOMIALS RELATED TO GENERALIZED POWER AND ALTERNATING SUMS

(COMMUNICATED BY R.K. RAINA)

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Abstract. The purpose of this paper is to obtain several series identities involving some classes of generalized Apostol-Bernoulli and Apostol-Euler polynomials introduced lately by Srivastava et al. in [16, 17] as well as the generalized sum of integer powers, the generalized alternating sum and the analogues of the expansions of the hyperbolic tangent and the hyperbolic cotangent. The method used is that of generating functions. It can be found that many identities recently obtained are special cases of our results.

1. Introduction, Definitions and Notations

The generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}$, the generalized Euler polynomials $E_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}$ and the generalized Genocchi polynomials $G_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}$, each of degree n as well as in α, are defined respectively by the following generating functions (see,[4, vol.3, p.253 et seq.], [8, Section 2.8] and [10]):

$$\left(\frac{t}{\mathrm{e}^t-1}\right)^{\alpha}\cdot \mathrm{e}^{xt}=\sum_{k=0}^{\infty}B_k^{(\alpha)}(x)\frac{t^k}{k!} \qquad (|t|<2\pi;1^{\alpha}:=1), \tag{1.1}$$

$$\left(\frac{2}{\mathrm{e}^t+1}\right)^{\alpha}\cdot \mathrm{e}^{xt}=\sum_{k=0}^{\infty}E_k^{(\alpha)}(x)\frac{t^k}{k!} \qquad (|t|<\pi;1^{\alpha}:=1) \tag{1.2}$$

and

$$\left(\frac{2t}{\mathrm{e}^t+1}\right)^{\alpha}\cdot \mathrm{e}^{xt}=\sum_{k=0}^{\infty}G_k^{(\alpha)}(x)\frac{t^k}{k!} \qquad (|t|<\pi;1^{\alpha}:=1). \tag{1.3}$$

The literature contains a large number of interesting properties and relationships involving these polynomials [1, 2, 3, 4, 5, 15]. Q.-M. Luo and Srivastava ([12, 14]) introduced the generalized Apostol-Bernoulli polynomials $\mathfrak{B}_n^{(\alpha)}(x)$ of order α,

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Q.-M. Luo [9] investigated the generalized Apostol-Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x)$ of order α and the generalized Apostol-Genocchi polynomials $\mathfrak{G}_n^{(\alpha)}(x)$ of order α (see also,[10, 11, 13]).

The generalized Apostol-Bernoulli polynomials $\mathfrak{B}_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$, the generalized Apostol-Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$, the generalized Apostol-Genocchi polynomials $\mathfrak{G}_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$ are defined respectively by the following generating functions

$$\left(\frac{t}{\lambda \mathrm{e}^t - 1}\right)^{\alpha} \cdot \mathrm{e}^{xt} = \sum_{k=0}^{\infty} \mathfrak{B}_k^{(\alpha)}(x;\lambda)\frac{t^k}{k!} \qquad (|t + \ln\lambda| < 2\pi; 1^{\alpha} := 1) \tag{1.4}$$

$$\left(\frac{2}{\lambda \mathrm{e}^t + 1}\right)^{\alpha} \cdot \mathrm{e}^{xt} = \sum_{k=0}^{\infty} \mathfrak{E}_k^{(\alpha)}(x;\lambda)\frac{t^k}{k!} \qquad (|t + \ln\lambda| < \pi; 1^{\alpha} := 1) \tag{1.5}$$

and

$$\left(\frac{2t}{\lambda \mathrm{e}^t + 1}\right)^{\alpha} \cdot \mathrm{e}^{xt} = \sum_{k=0}^{\infty} \mathfrak{G}_k^{(\alpha)}(x;\lambda)\frac{t^k}{k!} \qquad (|t + \ln\lambda| < \pi; 1^{\alpha} := 1). \tag{1.6}$$

It is easy to see that

$$B_n^{(\alpha)}(x) = \mathfrak{B}_n^{(\alpha)}(x;1), \quad E_n^{(\alpha)}(x) = \mathfrak{E}_n^{(\alpha)}(x;1) \qquad \text{and} \quad G_n^{(\alpha)}(x) = \mathfrak{G}_n^{(\alpha)}(x;1).$$

Recently, Srivastava et al. in [16, 17] have investigated some new classes of Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials with parameters a, b and c defined by the following generating functions.

Definition 1.1. *Let a, b, $c \in \mathbb{R}^+$, $(a \neq b)$ and $n \in \mathbb{N}_0$. The generalized Apostol-Bernoulli polynomials $\mathfrak{B}_n^{(\alpha)}(x;\lambda;a,b,c)$ of order α, the generalized Apostol-Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)$ of order α and the generalized Apostol-Genocchi polynomials $\mathfrak{G}_n^{(\alpha)}(x;\lambda;a,b,c)$ of order α are defined respectively by the following generating functions*

$$\left(\frac{t}{\lambda b^t - a^t}\right)^{\alpha} c^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(\alpha)}(x;\lambda;a,b,c)\frac{t^n}{n!}, \ \left(\left|t\ln\left(\frac{a}{b}\right) + \ln\lambda\right| < 2\pi; 1^{\alpha} := 1\right), \tag{1.7}$$

$$\left(\frac{2}{\lambda b^t + a^t}\right)^{\alpha} c^{xt} = \sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)\frac{t^n}{n!}, \ \left(\left|t\ln\left(\frac{a}{b}\right) + \ln\lambda\right| < \pi; 1^{\alpha} := 1\right) \tag{1.8}$$

and

$$\left(\frac{2t}{\lambda b^t + a^t}\right)^{\alpha} c^{xt} = \sum_{n=0}^{\infty} \mathfrak{G}_n^{(\alpha)}(x;\lambda;a,b,c)\frac{t^n}{n!}, \ \left(\left|t\ln\left(\frac{a}{b}\right) + \ln\lambda\right| < \pi; 1^{\alpha} := 1\right). \tag{1.9}$$

If we take $a = 1$, $b = c = e$ in (1.7), (1.8) and (1.9) respectively, we have (1.4), (1.5) and (1.6). Obviously, when we set $\lambda = 1$, $\alpha = 1$, $a = 1$, $b = c = e$ in (1.7), (1.8) and (1.9), we have classical Bernoulli polynomials $B_n(x)$, classical Euler polynomials $E_n(x)$ and classical Genocchi polynomials $G_n(x)$.

For each $k \in \mathbb{N}_0$, $S_k(n)$ defined by

$$S_k(n) = \sum_{j=0}^{n} j^k \tag{1.10}$$

is called the sum of integer powers. The exponential generating function for $S_k(n)$ is given by [19]

$$\sum_{k=0}^{\infty} S_k(n)\frac{t^k}{k!} = \frac{\mathrm{e}^{(n+1)t}-1}{\mathrm{e}^t-1}. \tag{1.11}$$

We now define the generalized sum of integer powers as follows.

Definition 1.2. *For an arbitrary real or complex λ, the generalized sum of integers powers $S_k(n;\ \lambda)$ is defined by the generating relation*

$$\sum_{k=0}^{\infty} S_k(n;\ \lambda)\frac{t^k}{k!} = \frac{\lambda e^{(n+1)t}-1}{\lambda e^t-1}. \tag{1.12}$$

It is obvious that

$$S_k(n;1) = S_k(1). \tag{1.13}$$

For $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$, $T_k(n)$ defined by

$$T_k(n) = \sum_{k=0}^{n-1} (-1)^k n^k \tag{1.14}$$

is called the alternating sum. The exponential generating function for $T_k(n)$ is given by

$$\sum_{k=0}^{\infty} T_k(n)\frac{t^k}{k!} = \frac{1-(-1)^n\mathrm{e}^{nt}}{1+\mathrm{e}^t}. \tag{1.15}$$

The generalized alternating sum of order α is defined in [7] as follows.

Definition 1.3. *For any arbitrary real or complex parameter λ, the generalized alternating sum of order α, $T_k^{(\alpha)}(n;\ \lambda)$ is defined by the following generating function:*

$$\sum_{k=0}^{\infty} T_k^{(\alpha)}(n;\ \lambda)\frac{t^k}{k!} = \left(\frac{1-\lambda(-1)^n e^{nt}}{1+\lambda e^t}\right)^{\alpha}. \tag{1.16}$$

It is easy to observe that

$$T_k^{(1)}(n;1) = T_k(n). \tag{1.17}$$

In this paper, we present several series identities involving the generalized Apostol-Bernoulli and the generalized Apostol-Euler polynomials defined respectively by (1.7) and (1.8). In Section 2, we obtain several symmetry identities for the generalized Apostol-Bernoulli polynomials a relation between the these polynomials and the generalized sum of integer powers (1.12). In Section 3, we prove several identities involving the generalized Apostol-Euler, the generalized alternating sum and the analogues of the expansions of the hyperbolic tangent and the hyperbolic cotangent. Some identities are also obtained by using the relationships between the generalized Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials.

2. Symmetry identities for the generalized Apostol-Bernoulli polynomials

In this section, we establish some symmetry identities involving the generalized Apostol-Bernoulli polynomials $\mathfrak{B}_n^{(\alpha)}(x;\lambda;a,b,c)$ defined in the first section and the generalized sum of integer powers defined by (1.12). This is done by using the method of generating functions. These results provides generalization of known identities [18, 21, 22, 23]

Theorem 2.1. *For all integers $\mu > 0$, $\nu > 0$, $\alpha \geq 1$, $n \geq 0$, for a, b, $c \in \mathbb{R}^+$ $(a \neq b)$ and for $\lambda \in \mathbb{C}$, we have the following identity:*

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} \mathfrak{B}_{n-k}^{(\alpha)} \left(\nu x + \frac{(\mu-1)\nu \ln a}{\mu \ln c};\ \lambda;\ a,b,c \right) \mu^{n-k}\, \nu^{k+1} \\
&\times \sum_{i=0}^{k} \binom{k}{i} S_{k-i}(\mu-1,\lambda)\, \mathfrak{B}_i^{(\alpha-1)}(\mu y;\ \lambda;\ a,b,c) \left[\ln\left(\frac{b}{a}\right) \right]^{k-i} \\
&= \sum_{k=0}^{n} \binom{n}{k} \mathfrak{B}_{n-k}^{(\alpha)} \left(\mu x + \frac{(\nu-1)\mu \ln a}{\nu \ln c};\ \lambda;\ a,b,c \right) \nu^{n-k}\, \mu^{k+1} \\
&\times \sum_{i=0}^{k} \binom{k}{i} S_{k-i}(\nu-1,\lambda)\, \mathfrak{B}_i^{(\alpha-1)}(\nu y;\ \lambda;\ a,b,c) \left[\ln\left(\frac{b}{a}\right) \right]^{k-i}.
\end{aligned} \tag{2.1}$$

Proof. Considering $g(t) = \dfrac{t^{2\alpha-1} c^{\,\mu\nu x t} (\lambda b^{\mu\nu t} - a^{\mu\nu t}) c^{\,\mu\nu y t}}{(\lambda b^{\mu t} - a^{\mu t})^{\alpha} (\lambda b^{\nu t} - a^{\nu t})^{\alpha}}$. We have to expand the last function into series in two way to prove the theorem. We have

$$\begin{aligned}
g(t) &= \frac{t^{2\alpha-1} c^{\,\mu\nu x t} (\lambda b^{\mu\nu t} - a^{\mu\nu t}) c^{\,\mu\nu y t}}{(\lambda b^{\mu t} - a^{\mu t})^{\alpha} (\lambda b^{\nu t} - a^{\nu t})^{\alpha}} \\
&= \frac{1}{\mu^{\alpha} \nu^{\alpha-1}} \left(\frac{\mu t}{\lambda b^{\mu t} - a^{\mu t}} \right)^{\alpha} c^{\,\nu x \mu t} \left(\frac{\lambda b^{\mu\nu t} - a^{\mu\nu t}}{\lambda b^{\nu t} - a^{\nu t}} \right) \left(\frac{\nu t}{\lambda b^{\nu t} - a^{\nu t}} \right)^{\alpha-1} c^{\,\mu\nu y t} \\
&= \frac{c^{\frac{(\mu-1)\nu t \ln a}{\ln c}}}{\mu^{\alpha} \nu^{\alpha-1}} \left(\frac{\mu t}{\lambda b^{\mu t} - a^{\mu t}} \right)^{\alpha} c^{\,\nu x \mu t} \left(\frac{\lambda \left(\frac{b^{\nu}}{a^{\nu}}\right)^{\mu t} - 1}{\lambda \left(\frac{b^{\nu}}{a^{\nu}}\right)^{t} - 1} \right) \left(\frac{\nu t}{\lambda b^{\nu t} - a^{\nu t}} \right)^{\alpha-1} c^{\,\mu\nu y t} \\
&= \frac{1}{\mu^{\alpha} \nu^{\alpha-1}} \left(\sum_{n=0}^{\infty} \mathfrak{B}_n^{(\alpha)} \left(\nu x + \frac{(\mu-1)\nu \ln a}{\mu \ln c};\ \lambda;\ a,b,c \right) \frac{(\mu t)^n}{n!} \right) \\
&\quad \times \left(\sum_{k=0}^{\infty} S_k(\mu-1,\ \lambda) \left[\nu \ln\left(\frac{b}{a}\right) \right]^{k} \frac{t^k}{k!} \right) \left(\sum_{i=0}^{\infty} \mathfrak{B}_i^{(\alpha-1)}(\mu y;\ \lambda;\ a,b,c) \frac{(\nu t)^i}{i!} \right) \\
&= \frac{1}{\mu^{\alpha} \nu^{\alpha}} \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} \mathfrak{B}_{n-k}^{(\alpha)} \left(\nu x + \frac{(\mu-1)\nu \ln a}{\mu \ln c};\ \lambda;\ a,b,c \right) \mu^{n-k}\, \nu^{k+1} \right. \\
&\quad \left. \times \sum_{i=0}^{k} \binom{k}{i} S_{k-i}(\mu-1,\lambda)\, \mathfrak{B}_i^{(\alpha-1)}(\mu y;\ \lambda;\ a,b,c) \left[\ln\left(\frac{b}{a}\right) \right]^{k-i} \right] \frac{t^n}{n!}.
\end{aligned} \tag{2.2}$$

By expanding in a different way, we have

$$g(t) = \frac{1}{\nu^{\alpha}\mu^{\alpha}} \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} \mathfrak{B}_{n-k}^{(\alpha)} \left(\mu x + \frac{(\nu-1)\mu \ln a}{\nu \ln c};\ \lambda;\ a,b,c \right) \nu^{n-k}\, \mu^{k+1} \right.$$
$$\left. \times \sum_{i=0}^{k} \binom{k}{i} S_{k-i}(\nu-1,\lambda)\, \mathfrak{B}_{i}^{(\alpha-1)}(\nu y;\ \lambda;\ a,b,c) \left[\ln\left(\frac{b}{a}\right) \right]^{k-i} \right] \frac{t^n}{n!}. \tag{2.3}$$

□

By setting $a = 1$, $b = c = \mathrm{e}$ in Theorem 1, we obtain one of the results exhibited by Zhang and Yang [23, Eq. 8]:

Corollary 2.2. *For all integers $\mu > 0$, $\nu > 0$, $\alpha \geq 1$, $n \geq 0$ and for $\lambda \in \mathbb{C}$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} \mathfrak{B}_{n-k}^{(\alpha)}(\nu x;\ \lambda)\ \mu^{n-k}\, \nu^{k+1} \sum_{i=0}^{k} \binom{k}{i} S_{k-i}(\mu-1,\lambda)\, \mathfrak{B}_{i}^{(\alpha-1)}(\mu y;\ \lambda)$$
$$= \sum_{k=0}^{n} \binom{n}{k} \mathfrak{B}_{n-k}^{(\alpha)}(\mu x;\ \lambda)\ \nu^{n-k}\, \mu^{k+1} \sum_{i=0}^{k} \binom{k}{i} S_{k-i}(\nu-1,\lambda)\, \mathfrak{B}_{i}^{(\alpha-1)}(\nu y;\ \lambda). \tag{2.4}$$

Putting $x = 0$, $y = 0$ and $\alpha = 1$ in Theorem 1, we have:

Corollary 2.3. *For all integers $\mu > 0$, $\nu > 0$, $n \geq 0$ and for a, b, $c \in \mathbb{R}^{+}$ $(a \neq b)$ and $\lambda \in \mathbb{C}$, we have the following relation:*

$$\sum_{k=0}^{n} \binom{n}{k} \mathfrak{B}_{k}\left(\frac{(\mu-1)\nu \ln a}{\mu \ln c}; \lambda; a,b,c \right) \mu^{k-1}\nu^{n-k}\, S_{n-k}(\mu-1,\lambda) \left[\ln\left(\frac{b}{a}\right) \right]^{n-k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} \mathfrak{B}_{k}\left(\frac{(\nu-1)\mu \ln a}{\nu \ln c}; \lambda; a,b,c \right) \nu^{k-1}\mu^{n-k}\, S_{n-k}(\nu-1,\lambda) \left[\ln\left(\frac{b}{a}\right) \right]^{n-k}. \tag{2.5}$$

Finally, substituting $\lambda = 1$, $a = 1$, $b = c = \mathrm{e}$ in (2.5), we find

$$\sum_{k=0}^{n} \binom{n}{k} \mu^{k-1}\nu^{n-k} B_k S_{n-k}(\mu-1) = \sum_{k=0}^{n} \binom{n}{k} \nu^{k-1}\mu^{n-k} B_k S_{n-k}(\nu-1) \tag{2.6}$$

a result given by Tuenter [18].

Theorem 2.4. *For all integers $\mu > 0$, $\nu > 0$, $\alpha \geq 1$, $n \geq 0$, for a, b, $c \in \mathbb{R}^{+}$, $(a \neq b)$ and for $\lambda \in \mathbb{C}$, we have the following identity:*

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{\mu-1} \sum_{j=0}^{\nu-1} \lambda^{i+j} \mu^{n-k} \nu^{k} \mathfrak{B}_{n-k}^{(\alpha)} \left(\nu x + \frac{i\nu \ln\left(\frac{b}{a}\right) + (2\mu\nu - \nu - \mu) \ln a}{\mu \ln c}; \lambda; a,b,c \right)$$
$$\times\, \mathfrak{B}_{k}^{(\alpha)} \left(\mu y + \frac{j\mu \ln\left(\frac{b}{a}\right)}{\nu \ln c}; \lambda; a,b,c \right)$$
$$= \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{\nu-1} \sum_{j=0}^{\mu-1} \nu^{n-k} \mu^{k} \mathfrak{B}_{n-k}^{(\alpha)} \left(\mu x + \frac{i\mu \ln\left(\frac{b}{a}\right) + (2\mu\nu - \nu - \mu) \ln a}{\nu \ln c}; \lambda; a,b,c \right)$$
$$\times\, \lambda^{i+j}\, \mathfrak{B}_{k}^{(\alpha)} \left(\nu y + \frac{j\nu \ln\left(\frac{b}{a}\right)}{\mu \ln c}; \lambda; a,b,c \right). \tag{2.7}$$

Proof. Let the function $h(t)$ be given by

$$h(t)=\frac{t^{2\alpha}c^{\,\mu\nu xt}(\lambda^{\mu}b^{\mu\nu t}-a^{\mu\nu t})(\lambda^{\nu}b^{\mu\nu t}-a^{\mu\nu t})c^{\,\mu\nu yt}}{(\lambda b^{\mu t}-a^{\mu t})^{\alpha+1}(\lambda b^{\nu t}-a^{\nu t})^{\alpha+1}}, \tag{2.8}$$

which can be expanded as follows:

$$\begin{aligned}
h(t)&=\frac{1}{(\mu\nu)^{\alpha}}\left(\frac{\mu t}{\lambda b^{\mu t}-a^{\mu t}}\right)^{\alpha}c^{\,\mu\nu xt}\left(\frac{\lambda^{\mu}b^{\mu\nu t}-a^{\mu\nu t}}{\lambda b^{\nu t}-a^{\nu t}}\right)\left(\frac{\nu t}{\lambda b^{\nu t}-a^{\nu t}}\right)^{\alpha}\\
&\quad\times\left(\frac{\lambda^{\nu}b^{\mu\nu t}-a^{\mu\nu t}}{\lambda b^{\mu t}-a^{\mu t}}\right)c^{\,\mu\nu yt}\\
&=\frac{a^{2\mu\nu t-\nu t-\mu t}}{(\mu\nu)^{\alpha}}\left(\frac{\mu t}{\lambda b^{\mu t}-a^{\mu t}}\right)^{\alpha}c^{\,\mu\nu xt}\left(\frac{\lambda^{\mu}\mathrm{e}^{\,\mu\nu t\ln\left(\frac{b}{a}\right)}-1}{\lambda\mathrm{e}^{\,\nu t\ln\left(\frac{b}{a}\right)}-1}\right)\left(\frac{\nu t}{\lambda b^{\nu t}-a^{\nu t}}\right)^{\alpha}\\
&\quad\times\left(\frac{\lambda^{\nu}\mathrm{e}^{\,\nu\mu t\ln\left(\frac{b}{a}\right)}-1}{\lambda\mathrm{e}^{\,\mu t\ln\left(\frac{b}{a}\right)}-1}\right)c^{\,\mu\nu yt}\\
&=\frac{1}{(\mu\nu)^{\alpha}}c^{\frac{(2\mu\nu-\nu-\mu)t\ln a}{\ln c}}\left(\frac{\mu t}{\lambda b^{\mu t}-a^{\mu t}}\right)^{\alpha}c^{\,\mu\nu xt}\sum_{i=0}^{\mu-1}\lambda^{i}\mathrm{e}^{\,i\nu t\ln\left(\frac{b}{a}\right)}\\
&\quad\times\left(\frac{\nu t}{\lambda b^{\nu t}-a^{\nu t}}\right)^{\alpha}\sum_{j=0}^{\nu-1}\lambda^{j}\mathrm{e}^{\,j\mu t\ln\left(\frac{b}{a}\right)}c^{\,\mu\nu yt}\\
&=\frac{1}{(\mu\nu)^{\alpha}}\sum_{i=0}^{\mu-1}\lambda^{i}\left(\frac{\mu t}{\lambda b^{\mu t}-a^{\mu t}}\right)^{\alpha}c^{\,\mu\nu xt+\frac{i\nu t\ln\left(\frac{b}{a}\right)}{\ln c}+\frac{(2\mu\nu-\nu-\mu)t\ln a}{\ln c}}\\
&\quad\times\sum_{j=0}^{\nu-1}\lambda^{j}\left(\frac{\nu t}{\lambda b^{\nu t}-a^{\nu t}}\right)^{\alpha}c^{\,\mu\nu yt+\frac{j\mu t\ln\left(\frac{b}{a}\right)}{\ln c}}\\
&=\frac{1}{(\mu\nu)^{\alpha}}\sum_{i=0}^{\mu-1}\lambda^{i}\sum_{n=0}^{\infty}\mathfrak{B}_{n}^{(\alpha)}\left(\nu x+\frac{i\nu\ln\left(\frac{b}{a}\right)+(2\mu\nu-\nu-\mu)\ln a}{\mu\ln c};\lambda;a,b,c\right)\frac{(\mu t)^{n}}{n!}\\
&\quad\times\sum_{j=0}^{\nu-1}\lambda^{j}\sum_{k=0}^{\infty}\mathfrak{B}_{k}^{(\alpha)}\left(\mu y+\frac{j\mu\ln\left(\frac{b}{a}\right)}{\nu\ln c};\lambda;a,b,c\right)\frac{(\nu t)^{k}}{k!}\\
&=\frac{1}{(\mu\nu)^{\alpha}}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{\mu-1}\sum_{j=0}^{\nu-1}\lambda^{i+j}\mu^{n-k}\nu^{k}\mathfrak{B}_{k}^{(\alpha)}\left(\mu y+\frac{j\mu\ln\left(\frac{b}{a}\right)}{\nu\ln c};\lambda;a,b,c\right)\right.\\
&\quad\left.\times\mathfrak{B}_{n-k}^{(\alpha)}\left(\nu x+\frac{i\nu\ln\left(\frac{b}{a}\right)+(2\mu\nu-\nu-\mu)\ln a}{\mu\ln c};\lambda;a,b,c\right)\right]\frac{t^{n}}{n!}.
\end{aligned} \tag{2.9}$$

Since $h(t)$ is symmetric in μ and ν, we can also expand $h(t)$ as follows:

$$\begin{aligned}
h(t)&=\frac{1}{(\mu\nu)^{\alpha}}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{\nu-1}\sum_{j=0}^{\mu-1}\lambda^{i+j}\nu^{n-k}\mu^{k}\mathfrak{B}_{k}^{(\alpha)}\left(\nu y+\frac{j\nu\ln\left(\frac{b}{a}\right)}{\mu\ln c};\lambda;a,b,c\right)\right.\\
&\quad\left.\times\mathfrak{B}_{n-k}^{(\alpha)}\left(\mu x+\frac{i\mu\ln\left(\frac{b}{a}\right)+(2\mu\nu-\nu-\mu)\ln a}{\nu\ln c};\lambda;a,b,c\right)\right]\frac{t^{n}}{n!}.
\end{aligned} \tag{2.10}$$

By equating the coefficient of $\frac{t^n}{n!}$ on the right-hand sides of these last two (2.9) and (2.10), we get the identity (2.7). □

Setting $a = 1$, $b = c = \mathrm{e}$ in Theorem 2 yields a result given recently by Zhang and Yang [23, Eq. 18] :

Corollary 2.5. *For all integers $\mu > 0$, $\nu > 0$, $\alpha \geq 1$, $n \geq 0$ and for $\lambda \in \mathbb{C}$, we have*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{\mu-1}\sum_{j=0}^{\nu-1}\lambda^{i+j}\,\mu^{n-k}\,\nu^{k}\,\mathfrak{B}_{n-k}^{(\alpha)}\left(\nu x+\frac{i\nu}{\mu};\lambda\right)\,\mathfrak{B}_{k}^{(\alpha)}\left(\mu y+\frac{j\mu}{\nu};\lambda\right)\\
&\quad=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{\nu-1}\sum_{j=0}^{\mu-1}\lambda^{i+j}\,\nu^{n-k}\,\mu^{k}\mathfrak{B}_{n-k}^{(\alpha)}\left(\mu x+\frac{i\mu}{\nu};\lambda\right)\,\mathfrak{B}_{k}^{(\alpha)}\left(\nu y+\frac{j\nu}{\mu};\lambda\right).
\end{aligned}\tag{2.11}$$

Putting $\nu = 1$ and $y = 0$ in Theorem 2 gives the next corollary:

Corollary 2.6. *For all integers $\mu > 0$, $\alpha \geq 1$, $n \geq 0$, for a, b, $c \in \mathbb{R}^{+}$, $(a \neq b)$ and for $\lambda \in \mathbb{C}$, we have the following identity:*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{\mu-1}\lambda^{i}\mu^{n-k}\mathfrak{B}_{n-k}^{(\alpha)}\left(x+\frac{i\ln\left(\frac{b}{a}\right)+(\mu-1)\ln a}{\mu\ln c};\lambda;a,b,c\right)\mathfrak{B}_{k}^{(\alpha)}\left(0;\lambda;a,b,c\right)\\
&=\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{\mu-1}\lambda^{j}\mu^{k}\mathfrak{B}_{n-k}^{(\alpha)}\left(\mu x+\frac{(\mu-1)\ln a}{\nu\ln c};\lambda;a,b,c\right)\mathfrak{B}_{k}^{(\alpha)}\left(\frac{j\ln\left(\frac{b}{a}\right)}{\mu\ln c};\lambda;a,b,c\right).
\end{aligned}\tag{2.12}$$

Theorem 2.7. *For all integers $\mu > 0$, $\nu > 0$, $\alpha \geq 1$, $n \geq 0$, for a, b, $c \in \mathbb{R}^{+}$, $(a \neq b)$ and for $\lambda \in \mathbb{C}$, we have the following identity:*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{\mu-1}\sum_{j=0}^{\nu-1}\lambda^{i+j}\,\mu^{n-k}\,\nu^{k}\,\mathfrak{B}_{k}^{(\alpha)}\left(\mu y;\lambda;a,b,c\right)\\
&\quad\times\mathfrak{B}_{n-k}^{(\alpha)}\left(\nu x+\frac{i\nu\ln\left(\frac{b}{a}\right)+(2\mu\nu-\nu-\mu)\ln a+j\mu\ln\left(\frac{b}{a}\right)}{\mu\ln c};\lambda;a,b,c\right)\\
&=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{\nu-1}\sum_{j=0}^{\mu-1}\lambda^{i+j}\,\nu^{n-k}\,\mu^{k}\,\mathfrak{B}_{k}^{(\alpha)}\left(\nu y;\lambda;a,b,c\right)\\
&\quad\times\mathfrak{B}_{n-k}^{(\alpha)}\left(\mu x+\frac{i\mu\ln\left(\frac{b}{a}\right)+(2\mu\nu-\nu-\mu)\ln a+j\nu\ln\left(\frac{b}{a}\right)}{\nu\ln c};\lambda;a,b,c\right).
\end{aligned}\tag{2.13}$$

Proof. The proof of Theorem 3 is similar to that of Theorems 1 and 2. In the proof of Theorem 3, we first make use of (1.7) in order to expand the function $h(t)$ defined by (2.8) and then apply the symmetry of $h(t)$ in μ and ν to obtain a second expansion of $h(t)$. The details involved are straightforward and we leave them as an exercise. □

If we set $a = 1$, $b = c = \mathrm{e}$ in Theorem 3, we recover a result given recently by Zhang and Yang [23, Eq. 23] :

Corollary 2.8. *For all integers $\mu > 0$, $\nu > 0$, $\alpha \geq 1$, $n \geq 0$ and for $\lambda \in \mathbb{C}$, we have*

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{\mu-1}\sum_{j=0}^{\nu-1}\lambda^{i+j}\,\mu^{n-k}\,\nu^{k}\,\mathfrak{B}_{n-k}^{(\alpha)}\left(\nu x+\frac{\nu}{\mu}i+j;\lambda\right)\,\mathfrak{B}_{k}^{(\alpha)}\left(\mu y;\lambda\right)\\&\quad=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{\nu-1}\sum_{j=0}^{\mu-1}\lambda^{i+j}\,\nu^{n-k}\,\mu^{k}\,\mathfrak{B}_{n-k}^{(\alpha)}\left(\mu x+\frac{\mu}{\nu}i+j;\lambda\right)\,\mathfrak{B}_{k}^{(\alpha)}\left(\nu y;\lambda\right).\end{aligned}\tag{2.14}$$

Putting $\nu = 1$ and $y = 0$ in Theorem 3 gives the next corollary:

Corollary 2.9. *For all integers $\mu > 0$, $\alpha \geq 1$, $n \geq 0$, for a, b, $c \in \mathbb{R}^{+}$, $(a \neq b)$ and for $\lambda \in \mathbb{C}$, we have the following identity:*

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{\mu-1}\lambda^{i}\mu^{n-k}\mathfrak{B}_{n-k}^{(\alpha)}\left(x+\frac{i\ln\left(\frac{b}{a}\right)+(\mu-1)\ln a}{\mu\ln c};\lambda;a,b,c\right)\mathfrak{B}_{k}^{(\alpha)}\left(0;\lambda;a,b,c\right)\\&=\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{\mu-1}\lambda^{j}\mu^{k}\mathfrak{B}_{n-k}^{(\alpha)}\left(\mu x+\frac{(\mu-1)\ln a+j\ln\left(\frac{b}{a}\right)}{\ln c};\lambda;a,b,c\right)\mathfrak{B}_{k}^{(\alpha)}\left(0;\lambda;a,b,c\right).\end{aligned}\tag{2.15}$$

3. Some identities related to generalized Apostol-Euler polynomials

In this section, we derive some identities concerning the generalized Apostol-Euler polynomials $\mathfrak{E}_{n}^{(\alpha)}(x;\lambda;a,b,c)$ defined by (1.8), the generalized alternating sum (1.16) and the analogues of the expansions of hyperbolic cotangent and hyperbolic tangent introduced in [20]. These results extend some known formulas [6, 7, 21]. We conclude this section by giving some identities based on relationships between the generalized Apostol-Euler polynomials $\mathfrak{E}_{n}^{(\alpha)}(x;\lambda;a,b,c)$ and the generalized Apostol-Bernoulli polynomials $\mathfrak{B}_{n}^{(\alpha)}(x;\lambda;a,b,c)$ and between the generalized Apostol-Bernoulli polynomials $\mathfrak{B}_{n}^{(\alpha)}(x;\lambda;a,b,c)$ and the generalized Apostol-Genocchi polynomials $\mathfrak{G}_{n}^{(\alpha)}(x;\lambda;a,b,c)$ defined by (1.9).

Theorem 3.1. *For $n \in \mathbb{N}_0$, $\mu, \nu \in \mathbb{N}$, $\alpha \geq 1$, a, b, $c \in \mathbb{R}^{+}$, $(a \neq b)$ and for $\lambda \in \mathbb{C}$. If μ and ν have the same parity, then the following identity holds true:*

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}\mu^{n-k}\nu^{k}\left(\ln\left(\frac{b}{a}\right)\right)^{k}\mathfrak{E}_{n-k}^{(\alpha)}\left(\nu x-\frac{\alpha\nu\ln a}{\mu\ln c};\;\lambda;a,b,c\right)\,T_{k}^{(\alpha)}(\mu;\lambda)\\&=\sum_{k=0}^{n}\binom{n}{k}\nu^{n-k}\mu^{k}\left(\ln\left(\frac{b}{a}\right)\right)^{k}\mathfrak{E}_{n-k}^{(\alpha)}\left(\mu x-\frac{\alpha\mu\ln a}{\nu\ln c};\;\lambda;a,b,c\right)\,T_{k}^{(\alpha)}(\nu;\lambda).\end{aligned}\tag{3.1}$$

Proof. Let the function $g(t)$ be given by

$$g(t)=\frac{c^{\,\mu\nu xt}\left(1-\lambda(-1)^{\mu}\mathrm{e}^{\mu\nu t\ln\left(\frac{b}{a}\right)}\right)^{\alpha}}{\left(\lambda b^{\mu t}+a^{\mu t}\right)^{\alpha}\left(\lambda b^{\nu t}+a^{\nu t}\right)^{\alpha}}\tag{3.2}$$

Making use of (1.8) and (1.16) to expand $g(t)$ to obtain first:

$$\begin{aligned} g(t) &= \frac{1}{2^{\alpha}} c^{\frac{-\alpha\nu t\ln a}{\ln c}} \left(\frac{2}{\lambda b^{\mu t}+a^{\mu t}}\right)^{\alpha} c^{\,\nu x\mu t} \left(\frac{1-\lambda(-1)^{\mu}\mathrm{e}^{\mu\nu t\ln\left(\frac{b}{a}\right)}}{\lambda \mathrm{e}^{\nu t\ln\left(\frac{b}{a}\right)}+1}\right)^{\alpha} \\ &= \frac{1}{2^{\alpha}} \sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}\left(\nu x-\frac{\alpha\nu\ln a}{\mu\ln c};\ \lambda; a, b, c\right) \frac{(\mu t)^n}{n!} \sum_{k=0}^{\infty} T_k^{(\alpha)}(\mu;\lambda) \frac{\left(\nu t\ln\left(\frac{b}{a}\right)\right)^k}{k!} \\ &= \frac{1}{2^{\alpha}} \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} \mu^{n-k}\nu^k \left(\ln\left(\frac{b}{a}\right)\right)^k \mathfrak{E}_{n-k}^{(\alpha)}\left(\nu x-\frac{\alpha\nu\ln a}{\mu\ln c};\lambda; a, b, c\right) T_k^{(\alpha)}(\mu;\lambda)\frac{t^n}{n!}. \end{aligned} \tag{3.3}$$

Now, since μ and ν have the same parity, then the function $g(t)$ is symmetric in μ and ν. Therefore, we can expand $g(t)$ as follows:

$$g(t) = \frac{1}{2^{\alpha}} \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} \nu^{n-k}\mu^k \left(\ln\left(\frac{b}{a}\right)\right)^k \mathfrak{E}_{n-k}^{(\alpha)}\left(\mu x-\frac{\alpha\mu\ln a}{\nu\ln c};\lambda; a, b, c\right) T_k^{(\alpha)}(\nu;\lambda)\frac{t^n}{n!}. \tag{3.4}$$

By equating the coefficient of $\frac{t^n}{n!}$ on the right-hand side of the last two equations (3.3) and (3.4), we thus recover the identity (3.1) asserted by Theorem 4. □

As a special case, if we set $a = 1$, $b = c = \mathrm{e}$ in Theorem 4, we obtain the following corollary given recently by Lu and Srivastava in [7, Eq. 30].

Corollary 3.2. *For $n \in \mathbb{N}_0$, $\mu, \nu \in \mathbb{N}$, $\alpha \geq 1$ and for $\lambda \in \mathbb{C}$. If μ and ν have the same parity, then the following identity holds true:*

$$\sum_{k=0}^{n} \binom{n}{k} \mu^{n-k}\nu^k \mathfrak{E}_{n-k}^{(\alpha)}(\nu x;\lambda)\ T_k^{(\alpha)}(\mu;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \nu^{n-k}\mu^k \mathfrak{E}_{n-k}^{(\alpha)}(\mu x;\ \lambda)\ T_k^{(\alpha)}(\nu;\lambda). \tag{3.5}$$

Now, letting $\alpha = \lambda = 1$, we recover the result given by Yang and Qiao in [21, Eq. 18]:

Corollary 3.3. *For $n \in \mathbb{N}_0$ and $\mu, \nu \in \mathbb{N}$. If μ and ν have the same parity, then we have*

$$\sum_{k=0}^{n} \binom{n}{k} \mu^{n-k}\nu^k E_{n-k}(\nu x)\ T_k(\mu) = \sum_{k=0}^{n} \binom{n}{k} \nu^{n-k}\mu^k E_{n-k}(\mu x)\ T_k(\nu). \tag{3.6}$$

Theorem 3.4. *For $n \in \mathbb{N}_0$, $\mu, \nu \in \mathbb{N}$, $\alpha \geq 1$, a, b, $c \in \mathbb{R}^+$, $(a \neq b)$ and for $\lambda \in \mathbb{C}$. If μ and ν have the same parity, then the following identity holds true:*

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{\mu-1}\sum_{j=0}^{\nu-1} (-\lambda)^{i+j} \mu^{n-k}\nu^k \mathfrak{E}_{n-k}^{(\alpha)}\left(\nu x+\frac{(i\nu+j\mu)\ln\left(\frac{b}{a}\right)-(\mu+\nu)\ln a}{\mu\ln c};\lambda; a, b, c\right) \\ &\quad\times \mathfrak{E}_k^{(\alpha)}(\mu y;\lambda; a, b, c) \\ &= \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{\nu-1}\sum_{j=0}^{\mu-1} (-\lambda)^{i+j} \nu^{n-k}\mu^k \mathfrak{E}_{n-k}^{(\alpha)}\left(\mu x+\frac{(i\mu+j\nu)\ln\left(\frac{b}{a}\right)-(\mu+\nu)\ln a}{\nu\ln c};\lambda; a, b, c\right) \\ &\quad\times \mathfrak{E}_k^{(\alpha)}(\nu y;\lambda; a, b, c). \end{aligned} \tag{3.7}$$

Proof. Let the function $g(t)$ be given by

$$g(t)=\frac{\mathrm{c}^{\mu\nu xt}\mathrm{c}^{\mu\nu yt}\left[1-\left(-\lambda\left(\frac{b}{a}\right)^{\nu t}\right)^{\mu}\right]\left[1-\left(-\lambda\left(\frac{b}{a}\right)^{\mu t}\right)^{\nu}\right]}{\left(\lambda b^{\mu t}+a^{\mu t}\right)^{\alpha+1}\left(\lambda b^{\nu t}+a^{\nu t}\right)^{\alpha+1}} \tag{3.8}$$

which can be expanded, with the help of (1.8), as follows:

$$\begin{aligned}
g(t)&=\frac{1}{2^{2\alpha}}\left(\frac{2}{\lambda b^{\mu t}+a^{\mu t}}\right)^{\alpha}\mathrm{c}^{\mu\nu xt}\,\frac{1}{a^{\nu t}}\left(\frac{1-\left(-\lambda\left(\frac{b}{a}\right)^{\nu t}\right)^{\mu}}{\lambda\left(\frac{b}{a}\right)^{\nu t}+1}\right)\\
&\quad\times\frac{1}{a^{\mu t}}\left(\frac{1-\left(-\lambda\left(\frac{b}{a}\right)^{\mu t}\right)^{\nu}}{\lambda\left(\frac{b}{a}\right)^{\mu t}+1}\right)\left(\frac{2}{\lambda b^{\nu t}+a^{\nu t}}\right)^{\alpha}\mathrm{c}^{\mu\nu yt}\\
&=\frac{a^{-(\mu+\nu)t}}{2^{2\alpha}}\left(\frac{2}{\lambda b^{\mu t}+a^{\mu t}}\right)^{\alpha}\mathrm{c}^{\mu\nu xt}\,\left(\sum_{i=0}^{\mu-1}(-\lambda)^{i}\left(\frac{b}{a}\right)^{i\nu t}\right)\\
&\quad\times\left(\sum_{j=0}^{\nu-1}(-\lambda)^{j}\left(\frac{b}{a}\right)^{j\mu t}\right)\left(\frac{2}{\lambda b^{\nu t}+a^{\nu t}}\right)^{\alpha}\mathrm{c}^{\mu\nu yt}\\
&=\frac{1}{2^{2\alpha}}\sum_{i=0}^{\mu-1}\sum_{j=0}^{\nu-1}(\lambda)^{i+j}\left(\frac{2}{\lambda b^{\mu t}+a^{\mu t}}\right)^{\alpha}c^{\mu\nu xt+\frac{i\nu t\ln\left(\frac{b}{a}\right)}{\ln c}+\frac{j\mu t\ln\left(\frac{b}{a}\right)}{\ln c}-\frac{(\mu+\nu)t\ln a}{\ln c}}\\
&\quad\times\left(\frac{2}{\lambda b^{\nu t}+a^{\nu t}}\right)^{\alpha}\mathrm{c}^{\mu\nu yt}\\
&=\frac{1}{2^{2\alpha}}\sum_{i=0}^{\mu-1}\sum_{j=0}^{\nu-1}(\lambda)^{i+j}\sum_{n=0}^{\infty}\mathfrak{E}_{n}^{(\alpha)}\left(\nu x+\frac{(i\nu+\mu j)\ln\left(\frac{b}{a}\right)-(\mu+\nu)\ln a}{\mu\ln c};\lambda;a,b,c\right)\,\frac{(\mu t)^{n}}{n!}\\
&\quad\times\sum_{k=0}^{\infty}\mathfrak{E}_{k}^{(\alpha)}(\mu y;\lambda;a,b,c)\,\frac{(\nu t)^{k}}{k!}\\
&=\frac{1}{2^{2\alpha}}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{\mu-1}\sum_{j=0}^{\nu-1}(-\lambda)^{i+j}\mu^{n-k}\nu^{k}\mathfrak{E}_{k}^{(\alpha)}(\mu y;\lambda;a,b,c)\right.\\
&\quad\left.\times\,\mathfrak{E}_{n-k}^{(\alpha)}\left(\nu x+\frac{(i\nu+\mu j)\ln\left(\frac{b}{a}\right)-(\mu+\nu)\ln a}{\mu\ln c};\lambda;a,b,c\right)\right]\,\frac{t^{n}}{n!}.
\end{aligned}$$

Using the fact that $g(t)$ is symmetric since μ and ν have the same parity, we can also expand $g(t)$ in the following the way:

$$\begin{aligned}
g(t)=\frac{1}{2^{2\alpha}}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{\nu-1}\sum_{j=0}^{\mu-1}(-\lambda)^{i+j}\nu^{n-k}\mu^{k}\mathfrak{E}_{k}^{(\alpha)}(\nu y;\lambda;a,b,c)\right.\\
\left.\times\mathfrak{E}_{n-k}^{(\alpha)}\left(\mu x+\frac{(i\mu+\nu j)\ln\left(\frac{b}{a}\right)-(\mu+\nu)\ln a}{\nu\ln c};\lambda;a,b,c\right)\right]\,\frac{t^{n}}{n!}.
\end{aligned} \tag{3.9}$$

Equating coefficients of $\frac{t^n}{n!}$ in the right-hand side of the last two equations gives the identity of the Theorem 5. □

Letting $a=1$, $b=c=\mathrm{e}$ in Theorem 5, we find the following corollary given recently by Lu and Srivastava in [7, Eq. 43].

Corollary 3.5. *For $n \in \mathbb{N}_0$, $\mu, \nu \in \mathbb{N}$, $\alpha \geq 1$ and for $\lambda \in \mathbb{C}$. If μ and ν have the same parity, then the following identity holds true:*

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{\mu-1}\sum_{j=0}^{\nu-1}(-\lambda)^{i+j}\mu^{n-k}\nu^{k}\mathfrak{E}_{n-k}^{(\alpha)}\left(\nu x+\frac{\nu}{\mu}i+j;\ \lambda\right)\mathfrak{E}_{k}^{(\alpha)}(\mu y;\ \lambda)\\&=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{\nu-1}\sum_{j=0}^{\mu-1}(-\lambda)^{i+j}\nu^{n-k}\mu^{k}\mathfrak{E}_{n-k}^{(\alpha)}\left(\mu x+\frac{\mu}{\nu}i+j;\ \lambda\right)\mathfrak{E}_{k}^{(\alpha)}(\nu y;\ \lambda).\end{aligned}\tag{3.10}$$

According to [20], we have the following analogues of the expansions of hyperbolic cotangent and hyperbolic tangent, respectively:

$$\frac{\lambda \mathrm{e}^{2z}+1}{\lambda \mathrm{e}^{2z}-1}=\sum_{n=0}^{\infty}\frac{\mathfrak{B}_n(0;\lambda)+\lambda\mathfrak{B}_n(1;\lambda)}{n!}(2z)^{n-1},\tag{3.11}$$

$$\frac{\lambda \mathrm{e}^{2z}-1}{\lambda \mathrm{e}^{2z}+1}=-\sum_{n=0}^{\infty}\mathfrak{E}_n(0;\lambda)\frac{(2z)^n}{n!}+1.\tag{3.12}$$

From (3.11), we can obtain the following theorem.

Theorem 3.6. *For $n \in \mathbb{N}_0$, $\mu, \nu \in \mathbb{N}$, $\alpha \geq 1$, a, b, $c \in \mathbb{R}^+$, $(a \neq b)$ and for $\lambda \in \mathbb{C}$. Let $\delta_{i,j}$ denotes the Kronecker delta defined by $\delta_{i,i}=1$ and $\delta_{i,j}=0$ for $i \neq j$. If μ is odd and ν is even then the following identity holds true:*

$$\begin{aligned}&-\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{[\delta_{n+1-k,1}+2\mathfrak{B}_{n+1-k}(0;\lambda)]}{n+1}\mu^{n-k}\sum_{i=0}^{k}\binom{k}{i}\mu^{i}\nu^{n-i}\\&\times\mathfrak{E}_{k-i}^{(\alpha)}\left(\mu x-\frac{\mu\ln a}{\nu\ln c};\lambda;a,b,c\right)\sum_{j=0}^{i}\binom{i}{j}\left[\ln\left(\frac{b}{a}\right)\right]^{n-k+i-j}T_{i-j}^{(1)}(\nu;\lambda)\mathfrak{E}_{j}^{(\alpha-1)}(\nu y;\lambda;a,b,c)\\&=\sum_{k=0}^{n}\binom{n}{k}\mu^{n-k}\nu^{k}\mathfrak{E}_{n-k}^{(\alpha)}\left(\nu x-\frac{\nu\ln a}{\mu\ln c};\ \lambda;a,b,c\right)\sum_{i=0}^{k}\binom{k}{i}\left[\ln\left(\frac{b}{a}\right)\right]^{k-i}T_{k-i}^{(1)}(\mu;\lambda)\\&\quad\times\mathfrak{E}_{i}^{(\alpha-1)}(\mu y;\ \lambda;a,b,c).\end{aligned}\tag{3.13}$$

Proof. When μ is odd and ν is even, the function $g(t)$, given below, is not symmetric in μ and ν, so we have on the one hand

$$\begin{aligned}g(t)&=\frac{1}{2^{2\alpha-1}}\left(\frac{2}{\lambda b^{\nu t}+a^{\nu t}}\right)^{\alpha}c^{\mu\nu xt}\left(\frac{1-\lambda(-1)^{\nu}\mathrm{e}^{\mu\nu t\ln\left(\frac{b}{a}\right)}}{\lambda b^{\mu t}+a^{\mu t}}\right)\left(\frac{1-\lambda(-1)^{\mu}\mathrm{e}^{\mu\nu t\ln\left(\frac{b}{a}\right)}}{1-\lambda(-1)^{\nu}\mathrm{e}^{\mu\nu t\ln\left(\frac{b}{a}\right)}}\right)\\&\quad\times\left(\frac{2}{\lambda b^{\mu t}+a^{\mu t}}\right)^{\alpha-1}c^{\mu\nu yt}\\&=\frac{1}{2^{2\alpha-1}}\left(\frac{1-\lambda(-1)^{\mu}\mathrm{e}^{\mu\nu t\ln\left(\frac{b}{a}\right)}}{1-\lambda(-1)^{\nu}\mathrm{e}^{\mu\nu t\ln\left(\frac{b}{a}\right)}}\right)\left(\frac{2}{\lambda b^{\nu t}+a^{\nu t}}\right)^{\alpha}c^{\mu\nu xt-\frac{\mu t\ln a}{\ln c}}\\&\quad\times\left(\frac{1-\lambda(-1)^{\nu}\mathrm{e}^{\mu\nu t\ln\left(\frac{b}{a}\right)}}{\lambda\mathrm{e}^{\mu t\ln\left(\frac{b}{a}\right)}+1}\right)\left(\frac{2}{\lambda b^{\mu t}+a^{\mu t}}\right)^{\alpha-1}c^{\mu\nu yt}\\&=\frac{-1}{2^{2\alpha-1}}\left(\sum_{n=0}^{\infty}\frac{\mathfrak{B}_n(0;\lambda)+\lambda\mathfrak{B}_n(1;\lambda)}{n!}\ \left(\mu\nu t\ln\left(\frac{b}{a}\right)\right)^{n-1}\right)\\&\quad\times\left(\sum_{k=0}^{\infty}\mathfrak{E}_{k}^{(\alpha)}\left(\mu x-\frac{\mu\ln a}{\nu\ln c};\ \lambda;a,b,c\right)\frac{(\nu t)^k}{k!}\right)\end{aligned}$$

$$\times \left(\sum_{i=0}^{\infty} T_i^{(1)}(\nu\,;\lambda)\,\frac{\left(\mu t \ln\left(\frac{b}{a}\right)\right)^i}{i!}\right)\left(\sum_{j=0}^{\infty}\mathfrak{E}_j^{(\alpha-1)}(\nu y;\lambda;a,b,c)\frac{(\mu t)^j}{j!}\right)$$

$$= \frac{-1}{2^{2\alpha-1}}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\left[\mathfrak{B}_{n-k}(0;\lambda)+\lambda\mathfrak{B}_{n-k}(1;\lambda)\right]\mu^{n-1-k}\right.$$

$$\times\sum_{i=0}^{k}\binom{k}{i}\mu^i\nu^{n-1-i}\,\mathfrak{E}_{k-i}^{(\alpha)}\left(\mu x-\frac{\mu\ln a}{\nu\ln c};\;\lambda;a,b,c\right)$$

$$\left.\times\sum_{j=0}^{i}\binom{i}{j}\left[\ln\left(\frac{b}{a}\right)\right]^{n-1-k+i-j}T_{i-j}^{(1)}(\nu;\lambda)\mathfrak{E}_j^{(\alpha-1)}(\nu y;\;\lambda;a,b,c)\right]\frac{t^{n-1}}{n!}. \tag{3.14}$$

On the other other hand, we can also expand $g(t)$ as follows:

$$g(t) = \frac{1}{2^{2\alpha-1}}\left(\frac{2}{\lambda b^{\mu t}+a^{\mu t}}\right)^{\alpha} c^{\,\mu\nu x t}\left(\frac{1-\lambda(-1)^{\mu}\mathrm{e}^{\mu\nu t\ln\left(\frac{b}{a}\right)}}{\lambda b^{\nu t}+a^{\nu t}}\right)$$

$$\times\left(\frac{2}{\lambda b^{\nu t}+a^{\nu t}}\right)^{\alpha-1} c^{\,\mu\nu y t}$$

$$= \frac{1}{2^{2\alpha-1}}\left(\frac{2}{\lambda b^{\mu t}+a^{\mu t}}\right)^{\alpha} c^{\,\mu\nu x t-\frac{\nu t\ln a}{\ln c}}\left(\frac{1-\lambda(-1)^{\mu}\mathrm{e}^{\mu\nu t\ln\left(\frac{b}{a}\right)}}{1+\lambda\mathrm{e}^{\nu t\ln\left(\frac{b}{a}\right)}}\right)$$

$$\times\left(\frac{2}{\lambda b^{\nu t}+a^{\nu t}}\right)^{\alpha-1} c^{\,\mu\nu y t}$$

$$= \frac{1}{2^{2\alpha-1}}\left(\sum_{n=0}^{\infty}\mathfrak{E}_n^{(\alpha)}\left(\nu x-\frac{\nu\ln a}{\mu\ln c};\;\lambda;a,b,c\right)\frac{(\mu t)^n}{n!}\right)\left(\sum_{k=0}^{\infty}T_k^{(1)}(\mu;\lambda)\frac{\left[\nu t\ln\left(\frac{b}{a}\right)\right]^k}{k!}\right)$$

$$\times\left(\sum_{i=0}^{\infty}\mathfrak{E}_i^{(\alpha-1)}(\mu y;\;\lambda;a,b,c)\frac{(\nu t)^i}{i!}\right)$$

$$= \frac{1}{2^{2\alpha-1}}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\mu^{n-k}\nu^k\mathfrak{E}_{n-k}^{(\alpha)}\left(\nu x-\frac{\nu\ln a}{\mu\ln c};\;\lambda;a,b,c\right)\right.$$

$$\left.\times\sum_{i=0}^{k}\binom{k}{i}\left[\ln\left(\frac{b}{a}\right)\right]^{k-i}T_{k-i}^{(1)}(\mu;\lambda)\mathfrak{E}_i^{(\alpha-1)}(\mu y;\;\lambda;a,b,c)\right]\frac{t^n}{n!}.$$

Since $\lambda\mathfrak{B}_n(x+1;\lambda)-\mathfrak{B}_n(x;\lambda)=nx^{n-1}$ (see [14]) then

$$\mathfrak{B}_n(0;\lambda)+\lambda\mathfrak{B}_n(1;\lambda)=\delta_{n,1}+2\mathfrak{B}_n(0;\lambda). \tag{3.15}$$

Making use of this last relation (3.15) involving the Apostol-Bernoulli polynomials and numbers and equating the coefficients of $\frac{t^n}{n!}$ in (3.14) and (3.15), we obtain (3.13). □

Substituting $a=1$, $b=c=\mathrm{e}$ in Theorem 6 furnishes the following corollary.

Corollary 3.7. *For $n\in\mathbb{N}_0$, $\mu,\nu\in\mathbb{N}$, $\alpha\geq 1$ and for $\lambda\in\mathbb{C}$. Let $\delta_{i,j}$ denotes the Kronecker delta defined by $\delta_{i,i}=1$ and $\delta_{i,j}=0$ for $i\neq j$. If μ is odd and ν is even then the following identity holds true:*

$$-\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{\left[\delta_{n+1-k,1}+2\mathfrak{B}_{n+1-k}(0;\lambda)\right]}{n+1}\mu^{n-k}\sum_{i=0}^{k}\binom{k}{i}\mu^i\nu^{n-i}\mathfrak{E}_{k-i}^{(\alpha)}(\mu x;\lambda)$$

$$\times\sum_{j=0}^{i}\binom{i}{j}T_{i-j}^{(1)}(\nu;\lambda)\mathfrak{E}_j^{(\alpha-1)}(\nu y;\lambda)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} \nu^{k} \mathfrak{E}_{n-k}^{(\alpha)}(\nu x;\ \lambda) \sum_{i=0}^{k} \binom{k}{i} T_{k-i}^{(1)}(\mu;\lambda) \mathfrak{E}_{i}^{(\alpha-1)}(\mu y;\ \lambda). \tag{3.16}$$

Letting $\lambda = 1$ in (3.16), we get the result obtained by Liu and Wang in [6, Theorem 2.4].

Corollary 3.8. *For $n \in \mathbb{N}_0$, $\mu, \nu \in \mathbb{N}$, $\alpha \geq 1$. If μ is odd and ν is even then the following identity holds true:*

$$-2 \sum_{\substack{k=0 \\ k \neq n}}^{n+1} \binom{n+1}{k} \frac{B_{n+1-k}(0)}{n+1} \mu^{n-k} \sum_{i=0}^{k} \binom{k}{i} \mu^{i} \nu^{n-i} E_{k-i}^{(\alpha)}(\mu x) \sum_{j=0}^{i} \binom{i}{j} T_{i-j}^{(1)}(\nu) E_{j}^{(\alpha-1)}(\nu y)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} \nu^{k} E_{n-k}^{(\alpha)}(\nu x) \sum_{i=0}^{k} \binom{k}{i} T_{k-i}^{(1)}(\mu) E_{i}^{(\alpha-1)}(\mu y). \tag{3.17}$$

We can proceed similarly to Theorem 3.6, but using this time (3.12) to establish the next result.

Theorem 3.9. *For $n \geq 1$, $\mu, \nu \in \mathbb{N}$, $\alpha \geq 1$, a, b, $c \in \mathbb{R}^{+}$, $(a \neq b)$ and for $\lambda \in \mathbb{C}$. If μ is even and ν is odd then the following identity holds true:*

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} \mathfrak{E}_{n-k}(0;\lambda) \mu^{n-k} \sum_{i=0}^{k} \binom{k}{i} \mu^{i} \nu^{n-i} \mathfrak{E}_{k-i}^{(\alpha)}\left(\mu x - \frac{\mu \ln a}{\nu \ln c}; \lambda; a, b, c\right) \\
&\quad \times \sum_{j=0}^{i} \binom{i}{j} \left[\ln\left(\frac{b}{a}\right)\right]^{n-k+i-j} T_{i-j}^{(1)}(\nu;\lambda) \mathfrak{E}_{j}^{(\alpha-1)}(\nu y; \lambda; a, b, c) \\
&\quad - \sum_{k=0}^{n} \binom{n}{k} \nu^{n-k} \mu^{k} \mathfrak{E}_{n-k}^{(\alpha)}\left(\mu x - \frac{\mu \ln a}{\nu \ln c}; \lambda; a, b, c\right) \\
&\quad \times \sum_{i=0}^{k} \binom{k}{i} \left[\ln\left(\frac{b}{a}\right)\right]^{k-i} T_{k-i}^{(1)}(\nu;\lambda) \mathfrak{E}_{i}^{(\alpha-1)}(\nu y; \lambda; a, b, c) \\
&= \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} \nu^{k} \mathfrak{E}_{n-k}^{(\alpha)}\left(\nu x - \frac{\nu \ln a}{\mu \ln c};\ \lambda; a, b, c\right) \\
&\quad \times \sum_{i=0}^{k} \binom{k}{i} \left[\ln\left(\frac{b}{a}\right)\right]^{k-i} T_{k-i}^{(1)}(\mu;\lambda) \mathfrak{E}_{i}^{(\alpha-1)}(\mu y;\ \lambda; a, b, c).
\end{aligned} \tag{3.18}$$

An interesting special case of Theorem 7 is obtained by setting $a = 1$, $b = c = \mathrm{e}$ and $\lambda = 1$ in (3.18).

Corollary 3.10. *For $n \geq 1$, $\mu, \nu \in \mathbb{N}$, $\alpha \geq 1$. If μ is even and ν is odd then the following identity holds true:*

$$\sum_{k=0}^{n-1} \binom{n}{k} E_{n-k}(0) \mu^{n-k} \sum_{i=0}^{k} \binom{k}{i} \mu^{i} \nu^{n-i} E_{k-i}^{(\alpha)}(\mu x) \sum_{j=0}^{i} \binom{i}{j} T_{i-j}^{(1)}(\nu) E_{j}^{(\alpha-1)}(\nu y)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} \nu^{k} E_{n-k}^{(\alpha)}(\nu x) \sum_{i=0}^{k} \binom{k}{i} T_{k-i}^{(1)}(\mu) \mathfrak{E}_{i}^{(\alpha-1)}(\mu y) \tag{3.19}$$

a result given first by Liu and Wang in [6, Theorem 2.7].

Finally, we would like to mention that many other identities can be obtained from those shown in this paper. As example, it is easy to see that the two following relationships hold between the Apostol-Bernoulli and Apostol-Euler polynomials and the Apostol-Bernoulli and Apostol-Genocchi polynomials. Respectively, we have for $n \in \mathbb{N}_0$, $\mu, \nu \in \mathbb{N}$, $\alpha \in \mathbb{N}$, a, b, $c \in \mathbb{R}^+$, $(a \neq b)$ and for $\lambda \in \mathbb{C}$,

$$\mathfrak{E}_n^{(\alpha)}(x;\ \lambda;\ a,b,c) = \frac{(-2)^\alpha n!\ \mathfrak{B}_{n+\alpha}^{(\alpha)}(x;\ -\lambda;\ a,b,c)}{(n+\alpha)!} \tag{3.20}$$

and

$$\mathfrak{B}_n^{(\alpha)}(x;\ \lambda;\ a,b,c) = \frac{\mathfrak{G}_n^{(\alpha)}(x;\ -\lambda;\ a,b,c)}{(-2)^\alpha}. \tag{3.21}$$

Let us see two examples of application of these two results. First, combining (3.20) with Theorem 4 yields

Theorem 3.11. *For $n \in \mathbb{N}_0$, $\mu, \nu \in \mathbb{N}$, $\alpha \in \mathbb{N}$, a, b, $c \in \mathbb{R}^+$, $(a \neq b)$ and for $\lambda \in \mathbb{C}$. If μ and ν have the same parity, then the following identity holds true:*

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} \nu^k \left(\ln\left(\frac{b}{a}\right)\right)^k \frac{(n-k)!\ \mathfrak{B}_{n-k+\alpha}^{(\alpha)}\left(\nu x - \frac{\alpha\nu \ln a}{\mu \ln c};\ -\lambda; a,b,c\right)}{(n-k+\alpha)!} T_k^{(\alpha)}(\mu;\lambda) \\
&= \sum_{k=0}^{n} \binom{n}{k} \nu^{n-k} \mu^k \left(\ln\left(\frac{b}{a}\right)\right)^k \frac{(n-k)!\ \mathfrak{B}_{n-k+\alpha}^{(\alpha)}\left(\mu x - \frac{\alpha\mu \ln a}{\nu \ln c};\ -\lambda; a,b,c\right)}{(n-k+\alpha)!} T_k^{(\alpha)}(\nu;\lambda).
\end{aligned} \tag{3.22}$$

Next, considering (3.21) with Theorem 1 gives

Theorem 3.12. *For all integers $\mu > 0$, $\nu > 0$, $\alpha \in \mathbb{N}$, $n \geq 0$, for a, b, $c \in \mathbb{R}^+$ $(a \neq b)$ and for $\lambda \in \mathbb{C}$, we have the following identity:*

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} \mathfrak{G}_{n-k}^{(\alpha)}\left(\nu x + \frac{(\mu-1)\nu \ln a}{\mu \ln c};\ -\lambda;\ a,b,c\right)\ \mu^{n-k}\ \nu^{k+1} \\
&\quad \times \sum_{i=0}^{k} \binom{k}{i} S_{k-i}(\mu-1,\lambda)\ \mathfrak{G}_i^{(\alpha-1)}(\mu y;\ -\lambda;\ a,b,c)\ \left[\ln\left(\frac{b}{a}\right)\right]^{k-i} \\
&= \sum_{k=0}^{n} \binom{n}{k} \mathfrak{G}_{n-k}^{(\alpha)}\left(\mu x + \frac{(\nu-1)\mu \ln a}{\nu \ln c};\ -\lambda;\ a,b,c\right)\ \nu^{n-k}\ \mu^{k+1} \\
&\quad \times \sum_{i=0}^{k} \binom{k}{i} S_{k-i}(\nu-1,\lambda)\ \mathfrak{G}_i^{(\alpha-1)}(\nu y;\ -\lambda;\ a,b,c)\ \left[\ln\left(\frac{b}{a}\right)\right]^{k-i}.
\end{aligned} \tag{3.23}$$

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