Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 4 Issue 3 (2012.), Pages 18-28

PERIODIC SOLUTIONS OF DELAYED DIFFERENCE EQUATIONS

(COMMUNICATED BY PROFESSOR CLAUDIO CUEVAS)

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Abstract. In this article, existence of multiple positive T-periodic solutions for the first order delay difference equation of the form

$$\Delta x(n) = a(n)g(x(n))x(n) - \lambda f(n, x(n - \tau(n)))$$

has been studied. Leggett-Williams multiple fixed point theorem has been employed to prove the results, which are established considering different cases on functions g and f.

1. Introduction

The theory of difference equations has grown at an accelerated pace in the past decades. It now occupies a key position in applicable analysis. It is observed that there is much interest in developing theoretical analysis of functional difference equations. There are much interest in periodicity(see [1, 3, 12, 16, 17, 20, 21, 25]), asymptotic behavior [1, 4, 5, 7, 19], maximal regularity [6], invariant manifolds [18], numerical methods, etc.

This paper is concerned with the existence of multiple positive periodic solutions of delay difference equation of the form

$$\Delta x(n) = a(n)g(x(n))x(n) - \lambda f(n, x(n - \tau(n))), \tag{1.1}$$

here $\Delta x(n) = x(n+1) - x(n)$, $a(n), b(n)$ and $\tau(n)$, $n \in Z$ are T-periodic positive sequences with $T \geq 1$, $f(n, x)$ is T-periodic about n and is continuous about x for each $n \in Z$, λ is a positive parameter and R denote the set of real numbers, R_+ the set of positive reals. Z is the set of integers and Z_+ the set of positive integers. Let $[a, b] = \{a, a+1, ..., b\}$ for $a < b, a, b \in Z$, $\prod_{n=a}^{b} x(n)$ denote the product of $x(n)$ from $n = a$ to $n = b$.

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[0]2000 Mathematics Subject Classification: 34K13, 34K15, 39A10, 39A12.
Keywords and phrases. Periodic Solution, positive Solution, difference Equation, fixed point.

Submitted 02 April, 2012. Published 01 September, 2012.

Eq.(1.1) is the discrete analog of functional differential equation

$$x'(t) = a(t)g(x(t))x(t) - \lambda b(t)f(t, x(h(t))), \tag{1.2}$$

where $b(t) \equiv 1$ in our case. In recent years, considerable contribution on the existence of periodic solutions of Eq.(1.2) has been done by the authors, see [2, 11, 13, 24]. In [11, 13], Graef et.al. and Wang et.al have obtained interesting results by using upper lower solution method and fixed point index theory, when $g(x(t))$ is not necessarily bounded. In this paper, an attempt has been made to study the existence of at least two or three positive T-periodic solutions of Eq.(1.1) using well known Leggett-Williams fixed point theorem ([15], see Theorem 3.3, Theorem 3.5). We have first considered the case when function $g(x)$ is bounded and then obtained sufficient conditions for the existence of periodic solutions when $g(x)$ is not bounded. To the best of our knowledge no result has been done for the Eq.(1.1). The results obtained in this article are different from previous results in the literature and generalize the result in [13] as they considered the particular function for $g(x)$.

The whole work has been divided into three sections. Section 1 is introduction. Some preliminary results are given in Section 2. In Section 3, sufficient conditions for the existence of periodic solutions of Eq.(1.1) have been discussed, moreover the obtained results are illustrated by examples.

2. Preliminaries:

For the convenience of the reader, some necessary definitions from cone theory are described here.

Definition 2.1 Let X be a Banach space over R. A nonempty closed set $K \subset X$ is called a (positive) cone if the following conditions are satisfied:
(i) if $x \in K$, then $\lambda x \in K$ for $\lambda \geq 0$;
(ii) if $x \in K$ and $-x \in K$, then $x = 0$.
Definition 2.2 An operator A is completely continuous if A is continuous and compact, i.e., A maps bounded sets into relatively compact sets.

The following concept will be used in the statement of the Leggett-Williams fixed point theorem. Let X be a Banach space and K be a cone in X.
A mapping ψ is said to be a concave nonnegative continuous functional on K if $\psi : K \to [0, \infty)$ is continuous and

$$\psi(\mu x + (1-\mu)y) \geq \mu\psi(x) + (1-\mu)\psi(y), \quad x, y \in K,\ \mu \in [0,1].$$

Let c_1, c_2, c_3 be positive constants. With K and X as defined above, we define $K_{c_1} = \{x \in K : \|x\| < c_1\}$, $\overline{K}_{c_1} = \{x \in K : \|x\| \leq c_1\}$, $\quad K(\psi, c_2, c_3) = \{x \in K : c_2 \leq \psi(x),\ \|x\| < c_3\}$.

Theorem 2.1. *(Leggett-Williams fixed point Theorem,(Theorem 3.5,*[15]*)): Let $c_3 > 0$ be a constant. Assume that $A : \overline{K}_{c_3} \to K$ is completely continuous, there exists a concave nonnegative functional ψ with $\psi(x) \leq \|x\|, x \in K$ and numbers c_1 and c_2 with $0 < c_1 < c_2 < c_3$ satisfying the following conditions:*
(i) $\quad \{x \in K(\psi, c_2, c_3); \psi(x) > c_2\} \neq \phi$ and $\psi(Ax) > c_2$ if $x \in K(\psi, c_2, c_3)$;
(ii) $\quad \|Ax\| < c_1$ if $x \in \overline{K}_{c_1}$

(iii) $\psi(Ax) > \frac{c_2}{c_3}\|Ax\|$ *for each* $x \in \overline{K}_{c_3}$ *with* $\|Ax\| > c_3$.
Then A *has at least two fixed points* x_1, x_2 *in* $\overline{K}_{c_3}$. *Furthermore,* $\|x_1\| \leq c_1 < \|x_2\| < c_3$.

Theorem 2.2. *(Leggett-Williams fixed point theorem,(Theorem 3.3*[15]*)):*
Let $(X, \|\cdot\|)$ *be a Banach space and* $K \subset X$ *a cone, and* c_4 *a positive constant. Suppose there exists a concave nonnegative continuous functional* ψ *on* K *with* $\psi(x) \leq \|x\|$ *for* $x \in \overline{K}_{c_4}$ *and let* $A : \overline{K}_{c_4} \to \overline{K}_{c_4}$ *be a completely continuous mapping. Assume that there are numbers* c_1, c_2, c_3, c_4 *with* $0 < c_1 < c_2 < c_3 \leq c_4$ *such that*
(i) $\{x \in K(\psi, c_2, c_3) : \psi(x) > c_2\} \neq \phi$, *and* $\psi(Ax) > c_2$ *for all* $x \in K(\psi, c_2, c_3)$;
(ii) $\|Ax\| < c_1$ *for all* $x \in \overline{K}_{c_1}$;
(iii) $\psi(Ax) > c_2$ *for all* $x \in K(\psi, c_2, c_4)$ *with* $\|Ax\| > c_3$.
Then A *has at least three fixed points* x_1, x_2, x_3 *in* $\overline{K}_{c_4}$. *Furthermore,* $\|x_1\| \leq c_1 < \|x_2\|$, *and* $\psi(x_2) < c_2 < \psi(x_3)$.

In this article, let X be the set of all bounded periodic sequences which forms a Banach space under the norm

$$\|x\| = \max_{n \in [0, T-1]} |x(n)|. \tag{2.1}$$

Define a nonnegative concave continuous functional ψ on K by

$$\psi(x) = \min_{n \in [0, T-1]} x(n). \tag{2.2}$$

3. Main Results:

In this section, we obtained sufficient conditions for the existence of at least three positive T-periodic solutions of Eq.(1.1) with the following assumptions:
(A_1) $a, \tau \in C(Z_+, R_+)$, $a(n) = a(n+T)$, $a(n) \neq 0$ and $\tau(n) = \tau(n+T)$, $n \in [0, T-1]$ where T is a positive constant, denoting common period of the system.
(A_2) $f \in C(Z \times R_+, R_+)$ is T-periodic with respect to the first variable. Also function $f(n, x)$ is nondecreasing w.r.t x.
(A_3) $g \in C(R_+, R_+)$, there exists positive constants l, m such that $0 < l \leq g(x) \leq m < \infty$ for all $x > 0$.
Now consider the Banach space as defined in (2.1). It is clear that Eq.(1.1) can be written as

$$x(n+1) = x(n)[a(n)g(x(n)) + 1] - \lambda f(n, x(n - \tau(n))) \tag{3.1}$$

$$\Delta\left(x(s) \prod_{\theta=0}^{s-1} \frac{1}{1 + a(\theta)g(x(s))}\right) = -\prod_{\theta=0}^{s-1} \frac{1}{1 + a(\theta)g(x(s))} \lambda f(s, x(s - \tau(s)), \tag{3.2}$$

summing the above equation from $s = n$ to $n + T - 1$, we obtain

$$x(n) = \lambda \sum_{s=n}^{n+T-1} G_{l,m}(n, s) f(s, x(s - \tau(s))), \tag{3.3}$$

where $G_{l,m}(n,s)$ is defined as

$$G_{l,m}(n,s) = \frac{\prod_{\theta=s+1}^{n+T-1}(1+a(\theta)g(x(s)))}{\prod_{\theta=n}^{n+T-1}(1+a(\theta)g(x(s)))-1}, \quad n \le s \le n+T-1,$$

satisfying the property

$$0 < \frac{1}{\delta_m - 1} \le G_{l,m}(n,s) \le \frac{\delta_m}{\delta_l - 1}. \tag{3.4}$$

Denote $\delta_m = \prod_{s=0}^{T-1}(1+ma(s))$ and $\delta_l = \prod_{s=0}^{T-1}(1+la(s))$, Clearly $\frac{1}{\delta_m-1}\frac{\delta_l-1}{\delta_m} < 1$.

Then $x(n)$ is a T-periodic solution of (1.1) iff $x(n)$ is a T-periodic solution of difference equation (3.3).
Define an operator $A_\lambda : X \to X$ by

$$(A_\lambda x)(n) = \lambda \sum_{s=n}^{n+T-1} G_{l,m}(n,s) f(s, x(s-\tau(s))). \tag{3.5}$$

Using (2.1) we obtain

$$\|A_\lambda x\| \le \lambda \frac{\delta_m}{\delta_l - 1} \sum_{s=n}^{n+T-1} f(s, x(s-\tau(s)))$$

and hence

$$A_\lambda x \ge \frac{1}{\delta_m - 1} \sum_{s=n}^{n+T-1} f(s, x(s-\tau(s))) \ge \frac{\delta_l - 1}{(\delta_m - 1)\delta_m} \|A_\lambda x\|.$$

In view of the above inequality, we define a cone $K \subset X$ as

$$K = \{x \in X; x(n) > 0, n \in Z, x(n) \ge \frac{\delta_l - 1}{(\delta_m - 1)\delta_m} \|x\|\}.$$

Then $A_\lambda(K) \subset K$. The existence of a positive periodic solution of (1.1) is equivalent to the existence of a fixed point of A_λ in K. Here we use Leggett-Williams fixed point theorem, that is, Theorem 2.2 to obtain the existence of three fixed point of A_λ in K. With a small exercise, it may be proved that $A_\lambda : K \to K$ is completely continuous. The following proof is similar to the proof given in [22].
To complete the proof we first show that operator A_λ is continuous. From the assumptions (A_1)-(A_3), assume that for any $M > 0$ and $\epsilon > 0$, let $u, v \in K$, with $\|u\| \le M, \|v\| \le M$, there exists $\delta > 0$ such that $\|u - v\| < \delta$ for $s \in [0, T]$ implies $\|f(s, u(s-\tau(s))) - f(s, v(s-\tau(s)))\| < \epsilon$, we have that

$$\max_{0 \le s \le T-1} |f(s, u(s-\tau(s))) - f(s, v(s-\tau(s)))| < \frac{\epsilon}{\lambda\beta T}$$

where $\beta = \frac{\delta_m}{\delta_l - 1}$ then

$$
\begin{aligned}
|A_\lambda(u) - A_\lambda(v)| &\le \lambda \sum_{s=n}^{n+T-1} |G_{l,m}(n,s)||f(s,u) - f(s,v)|\, ds \\
&\le \lambda\beta \sum_{s=n}^{n+T-1} |f(s,u) - f(s,v)|\, ds \\
&< \epsilon.
\end{aligned}
$$

Hence A_λ is continuous. Next, to prove that A_λ is completely continuous operator, we show that A_λ maps bounded subset into compact set. Let M be given, $E = \{u \in K, ||u|| < M\}$ and $G = \{A_\lambda u : u \in E)\}$ then E is a subset of Banach space X, equivalent to the space R, which is closed and bounded, therefore compact. Since continuous image of compact set is compact. This shows $A_\lambda : K \to K$ is completely continuous operator.

Theorem 3.1. *Let* (A_1)-(A_3) *hold. Further, suppose that there are positive constants* c_1, c_2 *and* c_4 *with* $0 < c_1 < c_2 < c_4$ *such that*

(H_1) $\quad \frac{\max_{s\in[0,T-1]} f(s,c_1)\delta_m}{(\delta_l - 1)c_1} < \frac{\max_{s\in[0,T-1]} f(s,c_4)\delta_m}{(\delta_l - 1)c_4} < \frac{\min_{s\in[0,T-1]} f(s,c_2)}{(\delta_m - 1)c_2}$.

Then for $\lambda \in (\frac{(\delta_m - 1)c_2}{T \min_{s\in[0,T-1]} f(s,c_2)}, \frac{(\delta_l - 1)c_4}{T\delta_m \max_{s\in[0,T-1]} f(s,c_4)}]$, *Eq.(1.1) has at least three positive* T*-periodic solutions.*

Proof: For $x \in \overline{K}_{c_4}$, we have

$$
\begin{aligned}
||A_\lambda x|| &= \max_{0\le n\le T-1} \lambda \sum_{s=n}^{n+T-1} G_{l,m}(n,s) f(s, x(s - \tau(s))) \\
&\le \frac{\delta_m}{\delta_l - 1} \lambda \sum_{s=n}^{n+T-1} f(s, x(s - \tau(s))) \\
&\le \frac{\delta_m}{\delta_l - 1} \lambda T \max_{s\in[0,T-1]} f(s, c_4) \le c_4.
\end{aligned}
$$

Now take $c_3 = \frac{\delta_m(\delta_m - 1)c_2}{\delta_l - 1}$ and $c_0(n) = c_0 = \frac{c_2 + c_3}{2}$, then $c_0 \in \{x : x \in K(\psi, c_2, c_3), \psi(x) > c_2\}$, where $\psi(x)$ is defined as in Eq.(2.2). Then for $x \in K(\psi, c_2, c_3)$, we obtain

$$
\begin{aligned}
\psi(A_\lambda x) &= \min_{0\le n\le T-1} \lambda \sum_{s=n}^{n+T-1} G_{l,m}(n,s) f(s, x(s - \tau(s))) \\
&\ge \frac{1}{\delta_m - 1} \lambda T \min_{s\in[0,T-1]} f(s, c_2) \\
&> c_2.
\end{aligned}
$$

Further for $x \in \overline{K}_{c_1}$, and using (H_1) we have

$$\begin{aligned}
||A_\lambda x|| &= \max_{0\le n\le T-1} \lambda \sum_{s=n}^{n+T-1} G_{l,m}(n,s) f(s, x(s-\tau(s))) \\
&\le \frac{\delta_m}{\delta_l - 1}\lambda \sum_{s=n}^{n+T-1} f(s, x(s-\tau(s))) \\
&\le \frac{\delta_m}{\delta_l - 1}\lambda T \max_{s\in[0,T-1]} f(s, c_1) \\
&\le \frac{(\delta_l - 1)c_4}{\delta_m T \max_{s\in[0,T-1]} f(s,c_4)} \frac{\delta_m}{(\delta_l - 1)} T \max_{s\in[0,T-1]} f(s, c_1) < c_1.
\end{aligned}$$

Finally, for $x \in K(\psi, c_2, c_4)$ with $||A_\lambda x|| > c_3$, we have

$$||A_\lambda x|| \le \frac{\delta_m}{\delta_l - 1}\lambda \sum_{s=n}^{n+T-1} f(s, x(s-\tau(s)))$$

and

$$\begin{aligned}
\psi(A_\lambda x) &\ge \frac{1}{\delta_m - 1}\lambda T \sum_{s=n}^{n+T-1} f(s, x(s-\tau(s))) \\
&> \frac{1}{\delta_m - 1} \frac{\delta_l - 1}{\delta_m} ||A_\lambda x|| > c_2.
\end{aligned}$$

Since all the conditions of Theorem 2.2 are satisfied, therefore Eq.(1.1) has at least three positive T-periodic solutions.

Corollary 3.1. *Let* (A_1),(A_3) *and* (H_1) *hold. Also suppose that* $\limsup_{x\to\infty} \frac{f(n,x)}{x} = 0$ *and* $\limsup_{x\to 0} \frac{f(n,x)}{x} = 0$.
Then for $\lambda \in (\frac{(\delta_m - 1)c_2}{T \min_{s\in[0,T-1]} f(s,c_2)}, \frac{(\delta_l - 1)c_4}{T\delta_m \max_{s\in[0,T-1]} f(s,c_4)}]$, *Eq.(1.1) has at least three positive* T*-periodic solutions.*

In the following, Theorem 3.1 is applied to the single species population model exhibiting the allee effect proposed by Gopalsamy and Ladas[10], which is the discrete analog of proposed model.

Example 3.1. *consider the equation*

$$\Delta x(n) = x(n)[a(n) + b(n)x(n-\tau(n)) - c(n)x^2(n-\tau(n))], \tag{3.6}$$

where $a(n)$, $b(n)$, $c(n)$ *and* $\tau(n)$ *are positive integers.*

Eq.(3.6) can be rewritten as

$$\Delta x(n) = a(n)x - f(n,x),$$

where $f(n,x) = (c(n)x^2 - b(n)x)x(n)$ and λ, g=1. Then applying Theorem 3.1 we have the following result:

Theorem 3.2. *Assume that there are positive constants* $0 < c_1 < c_2 < c_4$ *such that*
$\frac{\max_{n\in[0,T-1]}[c(n)c_1^2 - b(n)c_1]}{\delta-1} < \frac{\max_{n\in[0,T-1]}[c(n)c_4^2 - b(n)c_4]}{\delta-1} < \frac{\min_{n\in[0,T-1]}[c(n)c_2^2 - b(n)c_2]}{\delta-1}$.

Then Eq.(3.6) has at least three positive T-periodic solutions for
$\frac{(\delta-1)}{\min_{n\in[0,T-1]}[c(n)c_2^2 - b(n)c_2]} < T < \frac{(\delta-1)}{\delta \max_{n\in[0,T-1]}[c(n)c_4^2 - b(n)c_4]}$,
where $\delta = \prod_{n=0}^{T-1}(1 + a(n))$.

From the assumption used in above Corollary 3.1, it is easily seen that the choice of constant c_4 used in Theorem 2.2 (for the existence of three periodic solutions) lead the function f to be unimodel, Now the point to be noted that this kind of functions exclude many important class of growth functions arising in various mathematical models such as: The logistic equation with several delays ([14]), Richards single species growth model ([14]), Michaelis-Menton type single species growth model ([14, 23]).
In view of this, we make another assumption on f:

(A_2') $f \in C(Z \times R_+, R_+)$ is T-periodic with respect to the first variable. Also function $f(n,x)$ is nondecreasing with respect to x and not bounded.
Then using Theorem 2.1 we have the following result:

Theorem 3.3. *Let (A_1), (A_3) and (A_2') hold. Further assume that there are constants $0 < c_1 < c_2$ s.t. for $\lambda \in \left(\frac{(\delta_m-1)c_2}{\sum_{s=0}^{T-1} f(s,c_2)}, \frac{(\delta_l-1)c_1}{\delta_m \sum_{s=0}^{T-1} f(s,c_1)}\right)$, Eq.(1.1) has at least two positive T-periodic solutions.*

Proof: Let $c_3 = \frac{\delta_m(\delta_m-1)c_2}{\delta_l-1}$ and $c_0(s) = c_0 = \frac{c_2+c_3}{2}$, then for $x \in K(\psi, c_2, c_3)$, we obtain

$$\begin{aligned}
\psi(A_\lambda x) &= \min_{0<n<T-1} \lambda \sum_{s=n}^{n+T-1} G_{l,m}(n,s) f(s, x(s-\tau(s))) \\
&\geq \frac{1}{\delta_m - 1} \lambda \sum_{s=0}^{T-1} f(s, c_2) \\
&> c_2.
\end{aligned}$$

Further, for $x \in \overline{K}_{c_1}$, we find

$$\begin{aligned}
||A_\lambda x|| &= \max_{0\leq n\leq T-1} \lambda \sum_{s=n}^{n+T-1} G_{l,m}(n,s) f(s, x(s-\tau(s))) \\
&\leq \frac{\delta_m}{\delta_l - 1} \lambda \sum_{s=0}^{T-1} f(s, c_1) \\
&< c_1.
\end{aligned}$$

Finally, for $x \in K(\psi, c_2, c_4)$ with $||A_\lambda x|| > c_3$, we have

$$||A_\lambda x|| \leq \frac{\delta_m}{\delta_l - 1} \lambda \sum_{s=n}^{n+T-1} f(s, x(s-\tau(s)))$$

and

$$\begin{aligned}\psi(A_\lambda x) &\geq \frac{1}{\delta_m - 1}\lambda T\sum_{s=n}^{n+T-1} f(s, x(s-\tau(s)))\\ &> \frac{1}{\delta_m - 1}\,\frac{\delta_l - 1}{\delta_m}||A_\lambda x|| > c_2.\end{aligned}$$

This shows that Eq.(1.1) has at least two positive T-periodic solutions.

Next, we study the existence of periodic solutions of Eq.(1.1) when the function $g(x)$ in Eq.(1.1) is not bounded. This case was first considered by Jin[13] to obtain the existence of one periodic solution of functional differential equation (1.2). In [11], Graef et.al. used upper lower solution method to study the existence of multiple periodic solutions of Eq.(1.2). Now using this assumption we obtained the result different from those in literature. Since $g(x) \to \infty$ as $x \to \infty$ then there exists a constant $m_1 > 0$ s.t. $g(x) = g_{m_1}(x)$ for all $0 \leq x < m_1$ and $g(x) = g_{m_1}(m_1)$ for all $x \geq m_1$.
In view of this we make the following assumption:

(A_3') There exists a constant $m_1 > 0$, s.t. $g_{m_1}(0) \leq g(x) \leq g_{m_1}(m_1)$ for $0 < ||x|| \leq m_1$.
Now inequality (3.4) will take the form

$$0 < \frac{1}{\prod_{s=0}^{T-1}(1 + g_{m_1}(m_1)a(s)) - 1} \leq G_{m_1,m_{1_0}}(n, s) \leq \frac{\prod_{s=0}^{T-1}(1 + g_{m_1}(m_1)a(s))}{\prod_{s=0}^{T-1}(1 + g_{m_1}(0)a(s)) - 1}.$$

Denote $\delta_{m_1} = \prod_{s=0}^{T-1}(1 + g_{m_1}(m_1)a(s))$ and $\delta_{m_{1_0}} = \prod_{s=0}^{T-1}(1 + g_{m_1}(0)a(s))$, Clearly $\frac{1}{\delta_{m_1}-1}\frac{\delta_{m_{1_0}}-1}{\delta_{m_1}} < 1$, then

$$\frac{1}{\delta_{m_1} - 1} \leq G_{m_1,m_{1_0}}(n, s) \leq \frac{\delta_{m_1}}{\delta_{m_{1_0}} - 1}, \qquad n \leq s \leq n + T - 1. \tag{3.7}$$

Next, we state the following theorem using (3.7).

Theorem 3.4. *Let $(A_1), (A_2')$ and (A_3') hold. Suppose that there are constants $0 < c_1 < c_2$, then for*
$\lambda \in \left(\frac{(\delta_{m_1}-1)m_1}{\sum_{s=0}^{T-1} f(s,m_1)}, \frac{(\delta_{m_{1_0}}-1)c_1}{\delta_{m_1}\sum_{s=0}^{T-1} f(s,c_1)}\right),$ *for each* $m_1 > 0$,
Eq.(1.1) has at least two positive T-periodic solutions.

Proof: Let $c_2 = m_1$ and $c_3 = \frac{\delta_{m_1}(\delta_{m_1}-1)m_1}{\delta_{m_{1_0}}-1}$. Then proceeding as in the lines of Theorem 3.3, for $x \in K(\psi, m_1, \frac{\delta_{m_1}(\delta_{m_1}-1)m_1}{\delta_{m_{1_0}}-1})$, we find

$$\begin{aligned}\psi(A_\lambda x) &= \min_{0\leq n\leq T-1}\lambda\sum_{s=n}^{n+T-1} G_{m_1,m_{1_0}}(n, s)f(s, x(s-\tau(s)))\\ &\geq \frac{1}{\delta_{m_1} - 1}\lambda\sum_{s=0}^{T-1} f(s, m_1)\\ &> m_1 = c_2.\end{aligned}$$

Further, for $x \in \overline{K}_{c_1}$, we have

$$\begin{aligned}||A_\lambda x|| &= \max_{0\le n\le T-1} \lambda \sum_{s=n}^{n+T-1} G_{m_1,m_{1_0}}(n,s) f(s, x(s-\tau(s))) \\ &\le \frac{\delta_{m_1}}{\delta_{m_{1_0}}-1} \lambda \sum_{s=0}^{T-1} f(s, ||x||) \\ &< \frac{\delta_{m_1}}{\delta_{m_{1_0}}-1} \frac{(\delta_{m_{1_0}}-1)c_1}{\delta_{m_1}\sum_{s=0}^{T-1} f(s,c_1)} \sum_{s=0}^{T-1} f(s,c_1) \\ &< c_1.\end{aligned}$$

Last hypothesis is easy to proof. Hence by Theorem 2.1, Eq.(1.1) has at least two positive T-periodic solutions.

Remark 3.1. *Now, we give an example to illustrate the Theorem 3.4. This example was considered by Graef et al.* [11] *(in continuous case) to find the existence of atleast one positive periodic solution. In the following, for the same example using Theorem 3.4, the existence of at least two positive periodic solutions has been obtained. Hence Theorem 3.4 improves the result in* [11].

Example 3.2. *Consider the equation*

$$\Delta x(n) = e^{x(n)} x(n) - \lambda(x^3(n - \cos n\pi) + 1), \tag{3.8}$$

here $a(n) = 1$, $\tau(n) = \cos n\pi$, $f(n,x) = x^3 + 1$, $g(x) = e^x$ and $T = 2$. We see that A_1, A_2', and A_3' hold. $\delta_{m_1} = 1 + e^{m_1}$ and $\delta_{m_{1_0}} = 2$. Then for each $m_1 > 0$ there are constants $0 < c_1 < c_2$ s.t. for $\lambda \in (\frac{e^{m_1} m_1}{2(m_1^3+1)}, \frac{c_1}{4(c_1^3+1)})$, by Theorem 3.4, (2.2) has at least two positive 2-periodic solutions.

Remark 3.2. *As the existence of positive periodic solutions of (1.1) is regarded, it is found from the previous sections that some similar results can be derived for functional difference equation of the form*

$$\Delta x(n) = -a(n) g(x(n)) x(n) + \lambda f(n, x(n - \tau(n))), \tag{3.9}$$

we see that (3.9) is equivalent to the summation series

$$x(n) = \sum_{s=n}^{n+T-1} G_{l,m}(n,s) f(s, x(h(s))),$$

where

$$G_{l,m}(n,s) = \frac{\prod_{\theta=s+1}^{n+T-1}(1 - a(\theta)g(x(\theta)))}{1 - \prod_{\theta=0}^{T-1}(1 - a(\theta))}, \quad n \le s \le n + T - 1$$

is the Green's kernel satisfying the property

$$0 < \frac{\prod_{\theta=0}^{T-1}(1 - ma(\theta))}{1 - \prod_{\theta=0}^{T-1}(1 - ma(\theta))} \le G_{l,m}(n,s) \le \frac{1}{1 - \prod_{\theta=0}^{T-1}(1 - la(\theta))}.$$

Acknowledgements. This work is supported by National Board for Higher Mathematics (DAE) Govt. of India, under sponsored research scheme, vide grant no. 2/40(19)/2010/NBHM-R&D-II/, Dated July 28, 2010. The author is very grateful to the referee whose thoughtful annotation and valuable suggestions lead to improve original manuscript in the present form.

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