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# DOUBLE SINE SERIES AND HIGHER ORDER LIPSCHITZ CLASSES OF FUNCTIONS

## (COMMUNICATED BY HÜSEYIN BOR)

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Abstract. In the present paper, we generalize the double Lipschitz classes and double Zygmund calsses of functions in two variables to the so-called double higher order Lipschitz classes, and give the necessary and sufficient conditions for double sine series belonging to the generalized higher order Lipschitz classes.

## 1. Introduction

Given a double sequence $\{a_{jk}; j,k=1,2,...\}$ of nonnegative numbers satisfying

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk} < \infty, \tag{1}$$

then the following double sine series

$$f(x,y) := \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk} \sin jx \sin ky,$$

is continuous, due to uniform convergence.

Let $\omega(h,k)$ be a modulus of continuity, that is, $\omega(h,k)$ is a continuous function on the square $[0,2\pi]\times[0,2\pi]$, nondecreasing in each variable, and possessing the following properties:

$$\begin{aligned}
&\omega(0,0)=0,\\
&\omega(t_1+t_2,t_3) \le \omega(t_1,t_3)+\omega(t_2,t_3),\\
&\omega(t_1,t_2+t_3) \le \omega(t_1,t_2)+\omega(t_1,t_3).
\end{aligned}$$

Yu ([3]) introduced the following classes of functions:

$$HH^{\omega} := \{f(x,y) : \|f(x,y)-f(x+h,y)-f(x,y+k)+f(x+h,y+k)\| = O(\omega(h,k)), h,k>0\}.$$

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When $\omega(u,v) = u^{\alpha}v^{\beta},\ 0 < \alpha, \beta \leq 1$, then $HH^{\omega}$ becomes the well known double Lipschitz class $Lip(\alpha,\beta)$. Yu ([3]) investigated the necessary and sufficient conditions for the double trigonometric series belonging to $HH^{\omega}$. In fact, some of his results can be read as follows:

**Theorem 1.** *If*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} ija_{ij} = O\left(mn\omega\left(\frac{1}{m},\frac{1}{n}\right)\right),$$

$$\sum_{i=1}^{m}\sum_{j=n}^{\infty} ia_{ij} = O\left(m\omega\left(\frac{1}{m},\frac{1}{n}\right)\right),$$

$$\sum_{i=m}^{\infty}\sum_{j=1}^{n} ja_{ij} = O\left(n\omega\left(\frac{1}{m},\frac{1}{n}\right)\right),$$

$$\sum_{i=m}^{\infty}\sum_{j=n}^{\infty} a_{ij} = O\left(\omega\left(\frac{1}{m},\frac{1}{n}\right)\right),$$

*for* $m,n = 1,2,\cdots$, *then* $f(x,y) \in HH^{\omega}$.

**Theorem 2.** *If* $f(x,y) \in HH^{\omega}$, then

$$\sum_{i=1}^{m}\sum_{j=1}^{n} ija_{ij} = O\left(mn\omega\left(\frac{1}{m},\frac{1}{n}\right)\right).$$

**Theorem 3.** *If* $\left\{\omega\left(\frac{1}{m},\frac{1}{n}\right)\right\}$ *satisfies the following conditions*

$$\sum_{i=m}^{\infty} i^{-1}\omega\left(\frac{1}{i},\frac{1}{n}\right) = O\left(\omega\left(\frac{1}{m},\frac{1}{n}\right)\right),$$

$$\sum_{j=n}^{\infty} j^{-1}\omega\left(\frac{1}{m},\frac{1}{j}\right) = O\left(\omega\left(\frac{1}{m},\frac{1}{n}\right)\right),$$

$$\sum_{i=m}^{\infty}\sum_{j=n}^{\infty} i^{-1}j^{-1}\omega\left(\frac{1}{i},\frac{1}{j}\right) = O\left(\omega\left(\frac{1}{m},\frac{1}{n}\right)\right),$$

*for all* $m,n = 1,2,\cdots$, *then* $f(x,y) \in HH^{\omega}$ *if and only if*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} ija_{ij} = O\left(mn\omega\left(\frac{1}{m},\frac{1}{n}\right)\right).$$

For any $f(x,y) \in C(T^2)$, and $r,s = 1,2,...$, the $(r,s)$-th difference of $f(x,y)$ at $x$ with stepsize $h$ and at $y$ with stepsize $k$, is defined by

$$\triangle^{r,s}(f;x,y;h,k) := \sum_{\mu=0}^{r}\sum_{\gamma=0}^{s}(-1)^{r+s-\mu-\gamma}\binom{r}{\mu}\binom{s}{\gamma}f(x+\mu h, y+\gamma k).$$

Define the double higher order Lipschitz classes $\Lambda_{r,s}^{\omega}$ and the double higher order lipshcitz classes $\lambda_{r,s}^{\omega}$ as follows:

$$\Lambda_{r,s}^{\omega} := \left\{f(x,y) \in C(T^2) : ||\Delta^{r,s}(f;x,y;h,k)|| = O\left(\omega(h,k)\right), \quad h>0, k>0\right\},$$
$$\lambda_{r,s}^{\omega} := \left\{f(x,y) \in C(T^2) : ||\Delta^{r,s}(f;x,y;h,k)|| = o\left(\omega(h,k)\right), \quad h>0, k>0\right\}.$$

Clearly, if $r=s=1$, $\Lambda^{\omega}_{r,s}$ reduces to the class $HH^{\omega}$, and if $r=s=2$, $\Lambda^{\omega}_{r,s}$ reduces to the double Zygmund class $ZZ^{\omega}$, while if $r=1, s=2$, $\Lambda^{\omega}_{r,s}$ is the Lipschitz-Zygmund class $HZ^{\omega}$ (see [3] for the definitions of $ZZ^{\omega}$ and $HZ^{\omega}$, they are generalizations of the multiplicative Zygmund class $Zy\,(\alpha,\beta)$ and the multiplicative Lipschitz-Zygmund class, respectively).

Our main purpose is to generalize Theorem 1-Theorem 3 to the double higher Lipschitz classes $\Lambda^{\omega}_{r,s}$ (see Theorem A-Theorem C below). Our results also generalize some well known results considering single trigonometric series from Lipscitz class and Zygmund class to the higher order Lipschitz classes.

## 2. Main results

In what follows, we always assume that $\{a_{jk}\}$ is a double sequence of nonnegative numbers satisfying (1). We first give a sufficient condition for $f\in\Lambda^{\omega}_{r,s}$.

**Theorem A.** *If*

$$\sum_{j=1}^{m}\sum_{k=1}^{n} j^r k^s a_{jk} = O\left(m^r n^s \omega\left(\frac{1}{m},\frac{1}{n}\right)\right), \tag{2}$$

$$\sum_{j=1}^{m}\sum_{k=n+1}^{\infty} j^r a_{jk} = O\left(m^r \omega\left(\frac{1}{m},\frac{1}{n}\right)\right), \tag{3}$$

$$\sum_{j=m+1}^{\infty}\sum_{k=1}^{n} k^s a_{jk} = O\left(n^s \omega\left(\frac{1}{m},\frac{1}{n}\right)\right), \tag{4}$$

$$\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty} a_{jk} = O\left(\omega\left(\frac{1}{m},\frac{1}{n}\right)\right), \tag{5}$$

*then* $f\in\Lambda^{\omega}_{r,s}$.

We have the following necessary conditions for $f\in\Lambda^{\omega}_{r,s}$ :

**Theorem B.** *If* $f\in\Lambda^{\omega}_{r,s}$ *then*

$$\sum_{j=1}^{m}\sum_{k=1}^{n} j^{r^*} k^{s^*} a_{jk} = O\left(m^{r^*} n^{s^*} \omega\left(\frac{1}{m},\frac{1}{n}\right)\right),$$

*where* $r^* := \begin{cases} r+1 & r \text{ is even}, \\ r & r \text{ is odd}, \end{cases}$, $s^* := \begin{cases} s+1, & s \text{ is even}, \\ s, & s \text{ is odd}. \end{cases}$

If $\omega(\delta,\eta)$ satisfies some further conditions, we can obtain the necessary and sufficient conditions for $f\in\Lambda^{\omega}_{r,s}$. In fact, we have the following:

**Theorem C.** *(i). If $r$ and $s$ are all odd ,* $\omega\left(\frac{1}{m},\frac{1}{n}\right)$ *satisfies*

$$\sum_{j=m}^{\infty}\frac{1}{j}\omega\left(\frac{1}{j},\frac{1}{n}\right) = O\left(\omega\left(\frac{1}{m},\frac{1}{n}\right)\right), \tag{6}$$

$$\sum_{k=n}^{\infty}\frac{1}{k}\omega\left(\frac{1}{m},\frac{1}{k}\right) = O\left(\omega\left(\frac{1}{m},\frac{1}{n}\right)\right), \tag{7}$$

*for all* $m, n = 1, 2, ...,$ *then* $f \in \Lambda^{\omega}_{r,s}$ *if and only if (2.1) holds.*
*(ii). If* $r$ *is even and* $s$ *is odd,* $\omega\left(\frac{1}{m}, \frac{1}{n}\right)$ *satisfies (2.5),(2.6) and*

$$\sum_{j=1}^{m} j^{r-1}\omega\left(\frac{1}{j}, \frac{1}{n}\right) = O\left(m^{r}\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right), \tag{8}$$

*then* $f \in \Lambda^{\omega}_{r,s}$ *if and only if (2.3) holds.*
*(iii). If* $r$ *is odd and* $s$ *is even, and* $\omega\left(\frac{1}{m}, \frac{1}{n}\right)$ *satisfies (2.5),(2.6) and*

$$\sum_{k=1}^{n} k^{s-1}\omega\left(\frac{1}{m}, \frac{1}{j}\right) = O\left(n^{s}\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right), \tag{9}$$

*then* $f \in \Lambda^{\omega}_{r,s}$ *if and only if (2.2) holds.*
*(iv). If* $r$ *and* $s$ *are all even ,* $\omega\left(\frac{1}{m}, \frac{1}{n}\right)$ *satisfies (2.5)-(2.8), then* $f \in \Lambda^{\omega}_{r,s}$ *if and only if (2.4) holds.*

Now, we give some useful corollaries of Theorem C.

**Corollary A.** *Assume that there are* $\mu_1, \nu_1$ $(\mu_1, \nu_1 > 0)$ *such that* $\left\{m^{\mu_1}\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right\}$ *and* $\left\{n^{\nu_1}\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right\}$ *are almost decreasing on* $m$ *and* $n$ *respectively, then*
*(i). If* $r$ *and* $s$ *are all odd,* $f \in \Lambda^{\omega}_{r,s}$ *if and only if (2.1) holds.*
*(ii). If* $r$ *is even and* $s$ *is odd, and there is a* $\mu_2$ $(0 < \mu_2 < r)$ *such that* $\left\{m^{\mu_2}\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right\}$ *is almost increasing on* $m$ *, then* $f \in \Lambda^{\omega}_{r,s}$ *if and only if (2.3) holds.*
*(iii). If* $r$ *is odd and* $s$ *is even, and there is a* $\nu_2$ $(0 < \nu_2 < s)$ *such that* $\left\{n^{\nu_2}\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right\}$ *is almost increasing on* $n$*, then* $f \in \Lambda^{\omega}_{r,s}$ *if and only if (2.2) holds.*
*(iv). If* $r$ *and* $s$ *are all even , and there are* $\mu_3, \nu_3$ $(0 < \mu_3 < r, 0 < \nu_3 < s)$ *such that* $\left\{m^{\mu_3}\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right\}$ *and* $\left\{n^{\nu_3}\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right\}$ *are almost increasing on* $m$ *and* $n$ *respectively, then* $f \in \Lambda^{\omega}_{r,s}$ *if and only if (2.4) holds.*

**Corollary B.** *(i). If* $r$ *and* $s$ *are all odd, and* $\omega(\delta, \eta) = \delta^{\alpha}\eta^{\beta}$ $(0 < \alpha \leq r, 0 < \beta \leq s)$*, then* $f \in \Lambda^{\omega}_{r,s}$ *if and only if*

$$\sum_{j=1}^{m}\sum_{k=1}^{n} j^{r}k^{s}a_{jk} = O\left(m^{r-\alpha}n^{s-\beta}\right).$$

*(ii). If* $r$ *is even and* $s$ *is odd, and* $\omega(\delta, \eta) = \delta^{\alpha}\eta^{\beta}$ $(0 < \alpha < r, 0 < \beta \leq s)$*, then* $f \in \Lambda^{\omega}_{r,s}$ *if and only if*

$$\sum_{j=m}^{\infty}\sum_{k=1}^{n} k^{s}a_{jk} = O\left(m^{-\alpha}n^{s-\beta}\right).$$

*(iii). If* $r$ *is odd and* $s$ *is even, and* $\omega(\delta, \eta) = \delta^{\alpha}\eta^{\beta}$ $(0 < \alpha \leq r, 0 < \beta < s)$*, then* $f \in \Lambda^{\omega}_{r,s}$ *if and only if*

$$\sum_{j=1}^{m}\sum_{k=n}^{\infty} j^{r}a_{jk} = O\left(m^{r-\alpha}n^{-\beta}\right).$$

*(iv). If $r$ and $s$ are all even, and $\omega(\delta,\eta)=\delta^\alpha\eta^\beta(0<\alpha<r, 0<\beta<s)$, then $f\in\Lambda^\omega_{r,s}$ if and only if*

$$\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}a_{jk}=O\left(m^{-\alpha}n^{-\beta}\right).$$

**Remark.** When '$O$' is replaced by '$o$', and $\Lambda^\omega_{r,s}$ is replaced by $\lambda^\omega_{r,s}$ , the corresponding results still hold. When $r=s=2$, our results also generalize the corresponding results in [4].

## 3. Auxiliary results

**Lemma 1.** *When $r=2m, m=1,2...$, we have*

$$\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}\sin k(x+jh)=2^m(\cos kh-1)^m\sin k(x+mh). \tag{10}$$

*When $r=2m-1, m=1,2...$, we have*

$$\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}\sin k(x+jh)=2^m(\cos kh-1)^{m-1}\cos k(x+mh-\frac{h}{2})\sin\frac{kh}{2}. \tag{11}$$

*Proof.* First, we have (Móricz ([2])) for $m=1,2,...,$ $t\in R$, that

$$S_{2m-1}:=\sum_{j=0}^{2m-1}(-1)^j\binom{2m-1}{j}e^{i(m-j)t}=2^{m-1}(\cos t-1)^{m-1}(e^{it}-1), \tag{12}$$

and

$$S_{2m}:=\sum_{j=0}^{2m}(-1)^j\binom{2m}{j}e^{i(m-j)t}=2^m(\cos t-1)^m. \tag{13}$$

When $r=2m$, by (3.4), we have

$$\begin{aligned}\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}\sin k(x+jh)&=Im\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}e^{ik(x+jh)}\\&=Im\left(\sum_{j=0}^{2m}(-1)^j\binom{2m}{j}e^{i(m-j)(-kh)}e^{i(mkh+kx)}\right)\\&=2^m(\cos kh-1)^m Im\left(e^{i(mkh+kx)}\right)\\&=2^m(\cos kh-1)^m\sin k(x+mh),\end{aligned}$$

which prove (3.1).

When $r = 2m-1$, by (3.3), we have

$$\begin{aligned}\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}\sin k(x+jh) &= Im\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}e^{ik(x+jh)}\\ &= Im\left(\sum_{j=0}^{2m-1}(-1)^j\binom{2m-1}{j}e^{i(m-j)(-kh)}e^{i(mkh+kx)}\right)\\ &= 2^{m-1}(\cos kh-1)^{m-1}Im\left((e^{i(-kh)}-1)e^{i(kx+mh)}\right)\\ &= 2^{m-1}(\cos kh-1)^{m-1}Im\left(e^{i(kx-kh+mkh)}-e^{i(kx+mkh)}\right)\\ &= 2^{m-1}(\cos kh-1)^{m-1}\left(\cos(kx-kh+mkh)-\cos(kx+mkh)\right)\\ &= 2^{m}(\cos kh-1)^{m-1}\cos k(x+mh-\frac{h}{2})\sin\frac{kh}{2},\end{aligned}$$

which prove (3.2). □

**Lemma 2.** *If $\omega\left(\frac{1}{m},\frac{1}{n}\right)$ satisfies (6), (7), then for any $\delta \geq r, \eta \geq s$,*

$$\sum_{j=1}^{m}\sum_{k=1}^{n} i^{\delta}j^{\eta}a_{jk} = O\left(m^{\delta}n^{\eta}\omega\left(\frac{1}{m},\frac{1}{n}\right)\right) \tag{14}$$

*implies*

$$\sum_{j=1}^{m}\sum_{k=n+1}^{\infty} j^{\delta}a_{jk} = O\left(m^{\delta}\omega\left(\frac{1}{m},\frac{1}{n}\right)\right), \tag{15}$$

$$\sum_{j=m+1}^{\infty}\sum_{k=1}^{n} k^{\eta}a_{jk} = O\left(n^{s}\omega\left(\frac{1}{m},\frac{1}{n}\right)\right), \tag{16}$$

$$\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty} a_{jk} = O\left(\omega\left(\frac{1}{m},\frac{1}{n}\right)\right). \tag{17}$$

Lemma 2 can be proved in a way similar to that of Lemma 3.1 in [4].

**Lemma 3.** *If $\omega\left(\frac{1}{m},\frac{1}{n}\right)$ satisfies (8) and (9), then, for any $\delta \geq r, \eta \geq s$, (3.8) implies (3.5)-(3.7).*

*Proof.* Let $M$ and $N$ be integers for which $1 \leq m < M, 1 \leq n < N$, by Abel's transformation, we conclude that

$$\begin{aligned}\sum_{i=1}^{m} i^{\delta}\sum_{j=1}^{n} j^{\eta}a_{ij} &= \sum_{i=1}^{m} i^{\delta}\sum_{j_1=1}^{n}(j_1^{\eta}-(j_1-1)^{\eta})\sum_{j=j_1}^{N}a_{ij} - n^{\eta}\sum_{j=n+1}^{N}a_{ij}\\ &\leq \sum_{j_1=1}^{n}\eta j_1^{\eta-1}\sum_{j=j_1}^{N}\sum_{i=1}^{m} i^{\delta}a_{ij}\\ &= \sum_{j_1=1}^{n}\eta j_1^{\eta-1}\sum_{j=j_1}^{N}\left(\sum_{i_1=1}^{m}(i_1^{\delta}-(i_1-1)^{\delta})\sum_{i=i_1}^{M}a_{ij} - m^{\delta}\sum_{i=m+1}^{M}a_{ij}\right)\\ &\leq \sum_{j_1=1}^{n}\eta j_1^{\eta-1}\sum_{i_1=1}^{m}\delta i_1^{\delta-1}\sum_{j=j_1}^{N}\sum_{i=i_1}^{M}a_{ij}.\end{aligned}$$

Letting $M$ and $N$ tend to $\infty$, by (3.8), we have

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=1}^{n} i^{\delta}j^{\eta}a_{ij} &= O\left(\sum_{i_1=1}^{m}\sum_{j_1=1}^{n} i_1^{\delta-1}j_1^{\eta-1}\omega(\frac{1}{i_1},\frac{1}{j_1})\right)\\ &= O\left(m^{\delta-r}n^{\eta-s}\sum_{i_1=1}^{m}\sum_{j_1=1}^{n} i_1^{r-1}j_1^{s-1}\omega(\frac{1}{i_1},\frac{1}{j_1})\right)\\ &= O\left(m^{\delta}n^{\eta-s}\sum_{j_1=1}^{n} j_1^{s}\omega(\frac{1}{i_1},\frac{1}{j_1})\right)\\ &= O\left(m^{\delta}n^{\eta}\omega(\frac{1}{i_1},\frac{1}{j_1})\right),\end{aligned}$$

which proves (3.5).
By Abel's transformation again, we have

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=n}^{\infty} i^{\delta}a_{ij} &\le \sum_{j=n}^{\infty}\left(\sum_{i_1=1}^{m}(i_1^{\delta}-(i_1-1)^{\delta})\sum_{i=i_1}^{M}a_{ij} - m^{\delta}\sum_{i=m+1}^{M}a_{ij}\right)\\ &\le \sum_{i_1=1}^{m}\delta i_1^{\delta-1}\sum_{i=i_1}^{M}\sum_{j=n}^{\infty}a_{ij}.\end{aligned}$$

Letting $M$ tend to $\infty$, by (2.7), we have

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=n}^{\infty} i^{\delta}a_{ij} &= O\left(m^{\delta-r}\sum_{i_1=1}^{m} i_1^{r-1}\omega(\frac{1}{i_1},\frac{1}{n})\right)\\ &= O\left(m^{\delta}\omega(\frac{1}{m},\frac{1}{n})\right).\end{aligned}$$

which proves (3.6).
In a similar way to the proof of (3.6), we have (3.7). □

Analogue to Lemma 3, we have the following lemmas.

**Lemma 4.** *If $\omega\left(\frac{1}{m},\frac{1}{n}\right)$ satisfies (6) and (9), then, for any $\delta \ge r, \eta \ge s$ (15) implies (14), (16) and (17).*

**Lemma 5.** *If $\omega\left(\frac{1}{m},\frac{1}{n}\right)$ satisfies (7), (8), then, for any $\delta \ge r, \eta \ge s$, (16) implies (14), (15) and (17).*

**Remark 2.** When '$O$' is replaced by '$o$', the corresponding results of Lemma 2-Lemma 5 still hold.

## 4. Proof of the Theorem

**Proof of Theorem A.** Write $m := \left[\frac{1}{\delta}\right], n := \left[\frac{1}{\eta}\right]$ for given $\delta > 0, \eta > 0$. Direct calculations yield that

$$
\begin{aligned}
|\Delta^{r,s}(f;x,y;\delta,\eta)| &= \left|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk} \sum_{\mu=0}^{r}(-1)^{r-\mu}\binom{r}{\mu}\sin j(x+\mu\delta)\sum_{\gamma=0}^{s}(-1)^{s-\gamma}\binom{s}{\gamma}\sin k(y+\gamma\eta)\right| \\
&\le \left|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk} e^{ijx}\left(1-e^{ij\delta}\right)^r e^{iky}\left(1-e^{ik\eta}\right)^s\right| \\
&\le 2^{r+s}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk}\left|\sin\frac{j\delta}{2}\right|^r\left|\sin\frac{k\eta}{2}\right|^s \\
&\le 2^{r+s}\left\{\sum_{j=1}^{m}\sum_{k=1}^{n} + \sum_{j=1}^{m}\sum_{k=n+1}^{\infty} + \sum_{j=m+1}^{\infty}\sum_{k=1}^{n} + \sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}\right\} a_{jk}\left|\sin\frac{j\delta}{2}\right|^r\left|\sin\frac{k\eta}{2}\right|^s \\
&=: S_1+S_2+S_3+S_4.
\end{aligned}
$$

By (2), we have

$$S_1 \le \delta^r\eta^s\sum_{j=1}^{m}\sum_{k=1}^{n} j^r k^s a_{jk} = O\left(\omega(\delta,\eta)\right).$$

By (3) and (4), we have

$$S_2 \le 2^s\delta^r\sum_{j=1}^{m}\sum_{k=n+1}^{\infty} j^r a_{jk} = O\left(\omega(\delta,\eta)\right),$$

and

$$S_3 \le 2^r\eta^s\sum_{j=m+1}^{\infty}\sum_{k=1}^{n} k^s a_{jk} = O\left(\omega(\delta,\eta)\right),$$

respectively. Finally, by (5), we have

$$S_4 \le 2^{r+s}\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty} a_{jk} = O\left(\omega(\delta,\eta)\right).$$

Combining all the above estimates, the proof of Theorem A is complete.

**Proof of Theorem B.** We prove the result by considering the following many cases.

**Case 1.** $r$ and $s$ are both odd, say $r = 2m_0-1$ for some $m_0 = 1,2,...$, $s = 2n_0-1$ for some $n_0 = 1,2,....$ Since $f \in \Lambda^{\omega}_{r,s}$, by (3.2), there exists a constant $C$ such that

$$
\begin{aligned}
|\Delta^{r,s}(f;x,y;\delta,\eta)| =& 2^{m_0+n_0}\left|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk}\left(1-\cos j\delta\right)^{m_0-1}\left(1-\cos k\eta\right)^{n_0-1}\right| \times \\
&\left|\sin\left(\frac{j\delta}{2}\right)\cos j\left(x+\left(m_0-\frac{1}{2}\right)\delta\right)\sin\left(\frac{k\eta}{2}\right)\cos k\left(y+\left(n_0-\frac{1}{2}\right)\eta\right)\right| \\
\le& C\omega(\delta,\eta), \qquad \delta>0, \eta>0.
\end{aligned}
$$

Noting that $f$ is uniformly convergent (due to (1)), we can integrate both sides of the above inequality with respects to $x$ on $(-m_0\delta, -\left(m_0-\frac{1}{2}\right)\delta)$ and $y$ on $(-n_0\eta, -\left(n_0-\frac{1}{2}\right)\eta)$ to obtain that

$$\begin{aligned}
2^{2(m_0+n_0)}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{a_{jk}}{jk}\sin^{2m_0}\frac{j\delta}{2}\sin^{2n_0}\frac{k\eta}{2}=&\left|\int_{-m_0\delta}^{-\left(m_0-\frac{1}{2}\right)\delta}\int_{-n_0\eta}^{-\left(n_0-\frac{1}{2}\right)\eta}\Delta^{r,s}(f;x,y;\delta,\eta)dxdy\right|\\
\leq&\int_{-m_0\delta}^{-\left(m_0-\frac{1}{2}\right)\delta}\int_{-n_0\eta}^{-\left(n_0-\frac{1}{2}\right)\eta}\left|\Delta^{r,s}(f;x,y;\delta,\eta)\right|dxdy\\
\leq&\, C\int_{-m_0\delta}^{-\left(m_0-\frac{1}{2}\right)\delta}\int_{-n_0\eta}^{-n_0\eta+\eta}\omega(\delta,\eta)dxdy\\
\leq&\, C\delta\eta\omega(\delta,\eta),\qquad \delta>0,\eta>0. \qquad (18)
\end{aligned}$$

By using the well known inequality

$$\sin t\geq\frac{2t}{\pi},\qquad 0\leq t\leq\frac{\pi}{2}, \tag{19}$$

and (18), we obtain

$$2^{2(m_0+n_0)}\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{a_{jk}}{jk}\left(\frac{j\delta}{2}\right)^{2m_0}\left(\frac{k\eta}{2}\right)^{2n_0}\leq C\delta\eta\omega(\delta,\eta),\qquad \delta>0,\eta>0,$$

where $m:=\left[\frac{1}{\delta}\right], n:=\left[\frac{1}{\eta}\right]$. Hence,

$$\sum_{j=1}^{m}\sum_{k=1}^{n}j^rk^sa_{jk}=O\left(m^rn^s\omega\left(\frac{1}{m},\frac{1}{n}\right)\right),$$

which proves Theorem B in the case when $r$ and $s$ are all odd.

**Case 2.** $r$ is odd, $s$ is even, say $r=2m_0-1$ for some $m_0=1,2,...$, $s=2n_0$ for some $n_0=1,2,....$ Since $f\in\Lambda^{\omega}_{r,s}$, by Lemma 1, there exists a constant $C$ such that

$$\begin{aligned}
\left|\Delta^{r,s}(f;x,y;\delta,\eta)\right|=&2^{m_0+n_0}\left|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}a_{jk}\left(1-\cos j\delta\right)^{m_0-1}\left(1-\cos k\eta\right)^{n_0}\right.\\
&\left.\sin\left(\frac{j\delta}{2}\right)\cos j\left(x+\left(m_0-\frac{1}{2}\right)\delta\right)\sin k(y+n_0\eta)\right|\\
\leq& C\omega(\delta,\eta),\qquad \delta>0,\eta>0.
\end{aligned}$$

By integrating both sides of the above inequality with respects to $x$ on $(-m_0\delta, -\left(m_0-\frac{1}{2}\right)\delta)$ and $y$ on $(-n_0\eta, -n_0\eta+\eta)$, we have

$$\begin{aligned}
2^{2(m_0+n_0)}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{a_{jk}}{jk}\sin^{2m_0}\frac{j\delta}{2}\sin^{2n_0+2}\frac{k\eta}{2}=&\left|\int_{-m_0\delta}^{-\left(m_0-\frac{1}{2}\right)\delta}\int_{-n_0\eta}^{-n_0\eta+\eta}\Delta^{r,s}(f;x,y;\delta,\eta)dxdy\right|\\
\leq&\int_{-m_0\delta}^{-\left(m_0-\frac{1}{2}\right)\delta}\int_{-n_0\eta}^{-n_0\eta+\eta}\left|\Delta^{r,s}(f;x,y;\delta,\eta)\right|dxdy\\
\leq&\int_{-m_0\delta}^{-\left(m_0-\frac{1}{2}\right)\delta}\int_{-n_0\eta}^{-n_0\eta+\eta}\omega(\delta,\eta)dxdy\\
\leq&\, C\delta\eta\omega(\delta,\eta),\qquad \delta>0,\eta>0. \qquad (20)
\end{aligned}$$

By (4.2) and (20), we have

$$2^{2(m_0+n_0)}\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{a_{jk}}{jk}\left(\frac{j\delta}{2}\right)^{2m_0}\left(\frac{k\eta}{2}\right)^{2n_0+2}\leq C\delta\eta\omega(\delta,\eta),\qquad \delta>0,\eta>0,$$

where $m:=\left[\frac{1}{\delta}\right], n:=\left[\frac{1}{\eta}\right]$. Hence,

$$\sum_{j=1}^{m}\sum_{k=1}^{n}j^rk^{s+1}a_{jk}=O\left(m^rn^{s+1}\omega\left(\frac{1}{m},\frac{1}{n}\right)\right),$$

which proves Theorem B in the case when $r$ is odd and $s$ is even.

**Case 3.** $r$ is even, $s$ is odd. By similar discussion to Case 2, we see that Theorem B holds in this case.

**Case 4.** $r$ and $s$ are both even, say $r=2m_0$ for some $m_0=1,2,...$, $s=2n_0$ for some $n_0=1,2,....$ Since $f\in\Lambda^{\omega}_{r,s}$, there exists a constant $C$ such that

$$\begin{aligned}|\Delta^{r,s}(f;x,y;\delta,\eta)|=&2^{m_0+n_0}\left|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}a_{jk}\left(1-\cos j\delta\right)^{m_0}\left(1-\cos k\eta\right)^{n_0}\sin j(x+m_0\delta)\sin k(y+n_0\eta)\right|\\=&2^{2(m_0+n_0)}\left|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}a_{jk}\sin^{2m_0}\frac{j\delta}{2}\sin^{2n_0}\frac{k\eta}{2}\sin j(x+m_0\delta)\sin k(y+n_0\eta)\right|\\\leq&C\omega(\delta,\eta),\qquad \delta>0,\eta>0.\end{aligned}$$

By integrating both sides of the above inequality with respects to $x$ on $(-m_0\delta,-m_0\delta+\delta)$ and $y$ on $(-n_0\eta,-n_0\eta+\eta)$, we have

$$\begin{aligned}2^{2(m_0+n_0)}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{a_{jk}}{jk}\sin^{2m_0+2}\frac{j\delta}{2}\sin^{2n_0+2}\frac{k\eta}{2}=&\left|\int_{-m_0\delta}^{-m_0\delta+\delta}\int_{-n_0\eta}^{-n_0\eta+\eta}\Delta^{r,s}(f;x,y;\delta,\eta)dxdy\right|\\\leq&\int_{-m_0\delta}^{-m_0\delta+\delta}\int_{-n_0\eta}^{-n_0\eta+\eta}|\Delta^{r,s}(f;x,y;\delta,\eta)|\,dxdy\\\leq&C\int_{-m_0\delta}^{-m_0\delta+\delta}\int_{-n_0\eta}^{-n_0\eta+\eta}\omega(\delta,\eta)dxdy\\\leq&C\delta\eta\omega(\delta,\eta),\qquad \delta>0,\eta>0.\end{aligned}\tag{21}$$

By (4.2) and (21), we have

$$2^{2(m_0+n_0)}\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{a_{jk}}{jk}\left(\frac{j\delta}{2}\right)^{2m_0+2}\left(\frac{k\eta}{2}\right)^{2n_0+2}\leq C\delta\eta\omega(\delta,\eta),\qquad \delta>0,\eta>0,$$

where $m:=\left[\frac{1}{\delta}\right], n:=\left[\frac{1}{\eta}\right]$. Hence,

$$\sum_{j=1}^{m}\sum_{k=1}^{n}j^{r+1}k^{s+1}a_{jk}=O\left(m^{r+1}n^{s+1}\omega\left(\frac{1}{m},\frac{1}{n}\right)\right),$$

which proves Theorem B in the case when $r$ and $s$ are both even.

We prove Theorem B by combining the results of Case 1-Case 4.

**Proof of Theorem C.** (i) The necessity follows from Theorem B, while the sufficiency follows from Theorem A and Lemma 2 (with $\delta = r, \eta = s$).
(ii) The necessity follows from Theorem B and Lemma 2 (with $\delta = r+1, \eta = s$), while the sufficiency follows from Theorem A and Lemma 5 (with $\delta = r, \eta = s$).
(iii) The necessity follows from Theorem B and Lemma 2 (with $\delta = r, \eta = s+1$), while the sufficiency follows from Theorem A and Lemma 4 (with $\delta = r, \eta = s$).
(iv) The necessity follows from Theorem B and Lemma 2 (with $\delta = r+1, \eta = s+1$), while the sufficiency follows from Theorem A and Lemma 3 (with $\delta = r, \eta = s$).

**Proof of Corollary A.** (i) If there are $\mu_1, \nu_1$ $(\mu_1, \nu_1 > 0)$ such that $\left\{m^{\mu_1}\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right\}$ and $\left\{n^{\nu_1}\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right\}$ are almost decreasing on $m$ and $n$ respectively, then

$$\begin{aligned}\sum_{j=m}^{\infty} j^{-1}\omega\left(\frac{1}{j}, \frac{1}{n}\right) &= \sum_{j=m}^{\infty} j^{-1-\mu_1}\left(j^{\mu_2}\omega\left(\frac{1}{j}, \frac{1}{n}\right)\right)\\ &= O\left(m^{\mu_1}\omega\left(\frac{1}{m}, \frac{1}{n}\right)\sum_{j=1}^{m} j^{-1-\mu_1}\right)\\ &= O\left(\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right),\end{aligned}$$

which implies (6).

Similarly, we have (7).

Therefore, the result follows from (i) of Theorem C.
(ii) If there is $\mu_2$ $(0 < \mu_2 < r)$ such that $\left\{m^{\mu_2}\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right\}$ is almost increasing on $m$, then

$$\begin{aligned}\sum_{j=1}^{m} j^{r-1}\omega\left(\frac{1}{j}, \frac{1}{n}\right) &= \sum_{j=1}^{m} j^{r-1-\mu_2}\left(j^{\mu_2}\omega\left(\frac{1}{j}, \frac{1}{n}\right)\right)\\ &= O\left(m^{\mu_2}\omega\left(\frac{1}{m}, \frac{1}{n}\right)\sum_{j=1}^{m} j^{r-1-\mu_2}\right)\\ &= O\left(m^{r}\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right).\end{aligned}$$

Thus, the result follows from (ii) of Theorem C.

Similarly, (iii) and (iv) of Corollary A follow from (iii) and (iv) of Theorem C, respectively.

**Proof of Corollary B.** Set

$$\omega(u, v) = u^{\alpha}v^{\beta}, \ \alpha, \beta > 0.$$

Then $\omega(u, v)$ satisfies the conditions of Theorem C under assumptions of Corollary B on the parameters $\alpha, \beta$. Therefore, Corollary B follows from Theorem C immediately.

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## References

[1] P. R. Boas, Fourier series with positive coefficients, J. Math. Anal. Appl., 17(1967), 463-483.
[2] F. Móricz, Higher order Lipschitz classes of functions and absolutely convergent Fourier series, Acta Math. Hungar., 120(2008), 355-366.
[3] D. S. Yu, Double trigonometric series with positive coefficients, Anal. Math., 35(2009), 149-167.
[4] G. R. Qian and D. S. Yu, Double trigonometric series and Zygmund classes of functions with two variables, Bull. Math. Anal. Appl., 3(2011), 201-212.
[5] G. G. Lorentz, Fourier-koeffizienten und funktionenklassen, Math. Z., 51(1948), 135-149.
[6] V. Fülöp, Double cosine series with nonnegative coefficients, Acta Sci. Math. (Szeged), 70(2004), 91-100.
[7] V. Fülöp, Double sine and cosine-sine series with nonnegative coefficients, Acta Sci. Math.(Szeged), 70(2004), 101-116.
[8] J. Németh. Notes on Fourier series with nonnegatives coefficients, Acta Sci. Math.(Szeged), 55(1991), 83-93.

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